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# Limited validity of West and Yennie interference formula for elastic scattering of hadrons

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**Abstract**

The proof will be given that the commonly used West and Yennie integral formula for the relative phase of Coulomb and elastic hadronic amplitudes leads to a strong limitation of physical characteristics and should be abandoned. At the present it is only the eikonal model that may provide a reliable basis in analyzing corresponding data.


## 1. Introduction

The high energy elastic scattering of charged nucleons (if the spins of colliding particles are not taken into account) is being commonly described with the help of total elastic amplitude [1]

$$F^{C+N}(s,t) = F^C(s,t) + F^N(s,t)e^{i\alpha\Phi(s,t)}, \tag{1}$$

where $F^N(s,t)$ is hadronic amplitude, $F^C(s,t)$—Coulomb amplitude and $\alpha\Phi(s,t)$—relative phase; $\alpha = 1/137.036$ is fine structure constant, $s$—the square of the CMS energy and $t$—four momentum transfer squared.

While the Coulomb amplitude $F^C(s,t)$ is known from QED the hadronic (nuclear) amplitude $F^N(s,t)$ represents open question; due to the absence of any reliable theory of strong interactions at small momentum transfers its phenomenological shape is being looked for. This shape is believed to be determined from the measured elastic differential cross section data defined as

$$\frac{d\sigma(s,t)}{dt} = \frac{\pi}{sp^2}\left|F^{C+N}(s,t)\right|^2, \tag{2}$$

where $p$ is momentum value in the CMS system.

There are, however, two unknown functions in Eq. (1): $F^N(s,t)$ and $\Phi(s,t)$. Many attempts have been done to find whether it is possible to express $\Phi(s,t)$ with the help of hadronic amplitude. West and Yennie [2] derived in the case of charged point-like nucleons ($s \gg m^2$, $m$ being nucleon mass) and within the framework of one-photon exchange the formula

$$\alpha\Phi(s,t) = \mp\alpha\left[\ln\left(\frac{-t}{s}\right) - \int_{-4p^2}^{0} \frac{d\tau}{|t-\tau|}\left(1 - \frac{F^N(s,\tau)}{F^N(s,t)}\right)\right]. \tag{3}$$

The upper (lower) sign corresponds to the scattering of particles with the same (opposite) charges.

It is believed commonly that the West and Yennie integral formula (3) is quite general in the sense that it holds for any shape of $t$ dependent hadronic amplitude $F^N(s,t)$. However, in our recent paper [3] we have introduced that the integral formula (3) of West and Yennie may be regularly applied only to the elastic hadronic amplitudes $F^N(s,t)$ having the constant

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ratio between real and imaginary parts at all values of $t$. As the given statement has not been explicitly proved the corresponding reasoning will be given in the following and the consequences will be discussed.

## 2. Limitation of West and Yennie integral formula

The formula of West and Yennie is now being commonly made use of in all analyses of elastic nucleon data. And practically all quantities characterizing elastic processes have been based on its application.

The phase function $\alpha\Phi(s,t)$ is regarded always to be real, which requires to hold for any admissible $t$:

$$\int_{-4p^2}^{0} \frac{d\tau}{|t-\tau|}\Im\left(\frac{F^N(s,\tau)}{F^N(s,t)}\right) \equiv 0. \tag{4}$$

The condition (4) may be transformed to the condition

$$I(s,t) = \int_{-4p^2}^{0} \frac{d\tau}{|t-\tau|}\left[\Re F^N(s,t)\Im F^N(s,\tau) - \Re F^N(s,\tau)\Im F^N(s,t)\right] \equiv 0. \tag{5}$$

Introducing the phase $\zeta(t)$ and the modulus $|F^N(t)|$ of the complex hadronic amplitude (the dependence on the fixed $s$ being depressed in the following) by

$$F^N(s,t) = i\left|F^N(t)\right|e^{-i\zeta(t)}, \tag{6}$$

it is possible to write

$$I_1(t) \equiv I_2(t) \tag{7}$$

where

$$I_1(t) = \int_{-4p^2}^{t} d\tau\, f(t,\tau), \qquad I_2(t) = \int_{t}^{0} d\tau\, f(t,\tau) \tag{8}$$

and

$$f(t,\tau) = \begin{cases} \frac{\sin[\zeta(t)-\zeta(\tau)]}{t-\tau}|F^N(\tau)| & \text{for } \tau \neq t, \\ [\zeta(\tau)]'|F^N(\tau)| & \text{for } \tau = t, \end{cases} \tag{9}$$

where the factor $\sin[\zeta(t)-\zeta(\tau)]/(t-\tau)$ is symmetrical in both the variables $t,\tau \in [-4p^2,0]$. The function $f(t,\tau)$ is continuous and bounded if $\zeta(t)$ is continuous and its derivatives are bounded for any $t \in [-4p^2,0]$. Similar properties may be assumed for the modulus $|F^N(t)|$ that is non-zero in the whole interval $[-4p^2,0]$ with the only exception at $t=-4p^2$. It holds

$$\lim_{\tau\to t}\frac{\sin[\zeta(t)-\zeta(\tau)]}{t-\tau} = \left[\zeta(t)\right]' \tag{10}$$

and both $I_1(t)$ and $I_2(t)$ are proper integrals [4].

It is then possible to write

$$\begin{aligned}\left[I_1(t)\right]' &= \int_{-4p^2}^{t} d\tau \frac{\partial}{\partial t} f(t,\tau) + f(t,t) \\ &= \int_{-4p^2}^{t} d\tau\, g(t,\tau) + f(t,t), \\ \left[I_2(t)\right]' &= \int_{t}^{0} d\tau \frac{\partial}{\partial t} f(t,\tau) - f(t,t) \\ &= \int_{t}^{0} d\tau\, g(t,\tau) - f(t,t)\end{aligned} \tag{11}$$

where

$$g(t,\tau) = \frac{\partial}{\partial t} f(t,\tau) = \begin{cases} \frac{\cos[\zeta(t)-\zeta(\tau)][\zeta(t)]'(t-\tau)-\sin[\zeta(t)-\zeta(\tau)]}{(t-\tau)^2}|F^N(\tau)| \\ \quad \text{for } t \neq \tau, \\ \frac{1}{2}[\zeta(t)]''|F^N(t)| \quad \text{for } t = \tau. \end{cases} \tag{12}$$

Eq. (7) passes now to the form

$$\int_{-4p^2}^{t} d\tau\, g(t,\tau) - \int_{t}^{0} d\tau\, g(t,\tau) + 2f(t,t) \equiv 0 \tag{13}$$

which holds for each $t \in [-4p^2,0]$. Both the integrals in Eq. (13) are proper integrals similarly as in Eq. (7) (due to the assumed finite value of $\zeta(t)''$—see Eq. (12)). All higher derivatives of $I_1(t)$ and $I_2(t)$ (if they exist) can be derived in a similar way. It is evident that they are continuous and bounded, too. From Eq. (7) it can be easily derived that they should fulfill similar condition, i.e.,

$$I_1^{(n)}(t) \equiv I_2^{(n)}(t). \tag{14}$$

It may be shown that all equations (7), (13) and (14) are fulfilled if

$$\zeta(t) = \zeta(\tau) \equiv \text{const}. \tag{15}$$

And we should ask whether this solution is unique. The question may be answered with the help of the following theorem.

## 3. Uniqueness of the solution

**Theorem.** *Let $\zeta(t)$ be continuous function on the closed interval $J=[-a,0]$, $a>0$; let $F(t)$ be continuous function and non-zero with exception of end points defined also on $J$. Suppose that for all $t$ and $\tau$ from $J$ it holds*

$$\max_t \zeta(t) - \zeta(\tau) < \pi. \tag{16}$$

*If for each* $t \in J$,

$$\int_{-a}^{t} d\tau \frac{\sin[\zeta(t)-\zeta(\tau)]}{t-\tau} F(\tau) - \int_{t}^{0} d\tau \frac{\sin[\zeta(t)-\zeta(\tau)]}{t-\tau} F(\tau) \equiv 0, \tag{17}$$

*then the function* $\zeta(t)$ *is a constant function on* $J$.

**Proof.** Let us assume that the function $\zeta$ is not constant. Let us define $t_{\max}$ as $\zeta(t_{\max}) = \max_t \zeta(t)$. If there are more such points we can take any of them. Let $M$ be the set of all the numbers from $J$ such that $\zeta(t) < \zeta(t_{\max})$. Owing to the assumption that the function $\zeta$ is continuous and non-constant the Lebesgue measure of the set $M$, i.e., $\mu(M)$, is positive.

Let us define further

$$L_1(t) = \int_{-a}^{t} d\tau \frac{\sin[\zeta(t)-\zeta(\tau)]}{t-\tau} F(\tau)$$

and

$$L_2(t) = \int_{t}^{0} d\tau \frac{\sin[\zeta(t)-\zeta(\tau)]}{t-\tau} F(\tau).$$

Let us introduce now $L(t_{\max}) = L_1(t_{\max}) - L_2(t_{\max})$. For $-a \leqslant \tau \leqslant t_{\max}$ it holds $\zeta(t_{\max}) - \zeta(\tau) \geqslant 0$ and due to the validity of condition (16) we obtain $\sin[\zeta(t_{\max}) - \zeta(\tau)] \geqslant 0$; as $t_{\max} - \tau \geqslant 0$ and we can assume the function $F(\tau)$ to be positive it holds in the corresponding interval $L_1(t_{\max}) \geqslant 0$.

Similarly also for $L_2(t_{\max})$: for $t_{\max} \leqslant \tau \leqslant 0$ it holds $\zeta(t_{\max}) - \zeta(\tau) \geqslant 0$. Due to (16) it holds also $\sin[\zeta(t_{\max}) - \zeta(\tau)] \geqslant 0$. Owing to $t_{\max} - \tau \leqslant 0$, one obtains

$$\frac{\sin[\zeta(t_{\max}) - \zeta(\tau)]}{t_{\max} - \tau} \leqslant 0$$

and therefore $L_2(t_{\max}) \leqslant 0$. Consequently it holds $L(t_{\max}) \geqslant 0$ and as the set $M$ has a positive measure we obtain $L(t_{\max}) > 0$, which contradicts the requirement (17). And the theorem is proved. □

The theorem corresponds fully to our problem if we put $-a = -4p^2$; also the condition (16) required to be valid for hadronic phase is fulfilled practically in all phenomenological models. Thus the function $\zeta(t) = \text{const}$ represents the unique possibility for the $t$ dependence of $\zeta(t)$ function, if the relative phase between the Coulomb and hadronic amplitudes (given by integral formula of West and Yennie (3)) is to be real quantity, as commonly required.

## 4. Eikonal model

It follows from the preceding results that the West and Yennie formula cannot be brought to agreement with experimental data. And a new more suitable approach is to be looked for.

Such an approach may be seen in the eikonal model as the elastic scattering amplitude may be expressed as the Fourier–Bessel transformation of elastic eikonal $\delta(s,b)$,

$$F(s, q^2 = -t) = \frac{s}{4\pi i} \int_{\Omega_b} d^2b\, e^{i\vec{q}\vec{b}} \left[e^{2i\delta(s,b)} - 1\right] \tag{18}$$

where $\Omega_b$ is the two-dimensional Euclidean space of the impact parameter $\vec{b}$. Mathematically consistent formulation of Fourier–Bessel transformation requires the function $F(s,t)$ to be defined also in the region of unphysical $t$ values as analytical continuation from the region of physical $t$ values in agreement with formula (18) (for detail see [5,6]). Then Eq. (18) is valid at any $s$ and $t$.

Due to the additivity of corresponding potentials the Coulomb and hadronic interactions may be characterized by the total eikonal $\delta^{C+N}(s,b)$ being the sum of both the Coulomb $\delta^C(s,b)$ and hadronic $\delta^N(s,b)$ eikonals at the same value of impact parameter $b$ [6,7]:

$$\delta^{C+N}(s,b) = \delta^C(s,b) + \delta^N(s,b). \tag{19}$$

The total elastic scattering amplitude can be then written as [7,8]

$$\begin{aligned} F^{C+N}(s,t) &= F^C(s,t) + F^N(s,t) \\ &\quad + \frac{i}{\pi s} \int_{\Omega_{q'}} d^2q'\, F^C(s, q'^2) F^N(s, [\vec{q} - \vec{q}']^2). \end{aligned} \tag{20}$$

Eq. (20) shows that at difference to Eq. (1) where the sum of Coulomb and hadronic amplitudes is weighted by the phase factor multiplying the hadronic amplitude only, here a new complex function represented by a convolution integral defined over kinematically allowed region of momentum transfers $\Omega_{q'}$ is added to the sum of both t he amplitudes. Then Eq. (20) may be finally written as

$$F^{C+N}(s,t) = \pm \frac{\alpha s}{t} f_1(t) f_2(t) + F^N(s,t)\left[1 \mp i\alpha G(s,t)\right], \tag{21}$$

where

$$\begin{aligned} G(s,t) = \int_{-4p^2}^{0} dt' \bigg\{ &\ln\left(\frac{t'}{t}\right) \frac{d}{dt'}\left[f_1(t') f_2(t')\right] \\ &+ \frac{1}{2\pi}\left[\frac{F^N(s,t')}{F^N(s,t)} - 1\right] I(t,t') \bigg\}, \end{aligned} \tag{22}$$

and

$$I(t,t') = \int_0^{2\pi} d\Phi'' \frac{f_1(t'') f_2(t'')}{t''}; \tag{23}$$

here $t'' = t + t' + 2\sqrt{tt'}\cos\Phi''$. And $f_1(t)$ and $f_2(t)$ are form factors of corresponding charged nucleons.

The form of the total elastic scattering amplitude specified by Eqs. (21), (22), (23) has been used to the analysis of elastic nucleon scattering data in Ref. [8]. The value of the total cross section and $t$ dependence of the diffractive slope can be easily obtained with the help of the optical theorem and as the logarithmic derivative of the corresponding differential cross section.

## 5. Hadronic phase and its $t$ dependence

The eikonal model enables to analyze the distribution of different values of impact parameter in the different kinds of hadronic collisions. Including the unitarity condition the values of so-called root-mean-squares of impact parameter corresponding to different scattering kinds may be determined from the elastic hadronic amplitude obtained by fitting the experimental data (see Eq. (2)). As to the elastic collisions one can write [9,10]

$$\begin{aligned}\langle b^2(s)\rangle_{\text{el}} &= \langle b^2(s)\rangle_{\text{mod}} + \langle b^2(s)\rangle_{\text{ph}} \\ &= 4\frac{\int_{t_{\min}}^0 dt\,|t|(\frac{d}{dt}|F^N(s,t)|)^2}{\int_{t_{\min}}^0 dt\,|F^N(s,t)|^2} \\ &\quad + 4\frac{\int_{t_{\min}}^0 dt\,|t||F^N(s,t)|^2(\frac{d}{dt}\zeta^N(s,t))^2}{\int_{t_{\min}}^0 dt\,|F^N(s,t)|^2}\end{aligned} \tag{24}$$

where the modulus of elastic hadronic amplitude itself contributes to the first term and the phase (its derivative) is involved only in the second term. And one can distinguish between the central picture (contribution of the first term only) and peripheral picture (depending mainly on the $t$ dependence of $\zeta^N(s,t)$).

Having started from the West and Yennie formula (3) the high energy elastic hadron scattering has been interpreted usually as central; only very weak $t$ dependence of $\zeta^N(s,t)$) being allowed. This a priori limitation has brought an artificial discrepancy between the elastic and single diffractive processes, both having the same dynamical characteristics; and the latter processes being always regarded as peripheral. This discrepancy is removed when some stronger $t$ dependence of $\zeta^N(s,t)$ is admitted.

## 6. Conclusion

The strong $t$ dependence of the phase $\zeta^N(s,t)$ requires also for the ratio $\rho(s,t) = \frac{\Re F^N(s,t)}{\Im F^N(s,t)}$ of real to imaginary parts of the elastic hadronic amplitude to exhibit similar $t$ dependence. It leads to the peripheral behavior of elastic hadronic scattering as shown earlier [8].

And it is possible to conclude that to remove any a priori limitation the West and Yennie integral formula (3) must be fully abandoned in analysis of experimental elastic data. At the present it is only the eikonal model that may represent a reliable basis for solving actually the problems connected with a consistent description of the interference between the Coulomb and elastic nucleon scattering at high energies.

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