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On Multiplicative Bases in Commutative Semigroups

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We generalize some older results on multiplicative bases of integers to a certain class of commutative semigroups. In particular, we examine the structure of union bases of integers.

1. INTRODUCTION

In [2], P. Erdös proved the following theorem.

THEOREM 1. *Let $\mathbb{N}$ be the set of all positive integers and let $k \geq 2$ be an integer. Suppose that M is a subset of $\mathbb{N}$ such that every $x \in \mathbb{N}$ can be expressed in the form $x = m_1 \cdot m_2 \cdot \ldots \cdot m_k$, where $m_i \in M$ for every i. Then for every integer p there exists a number $x \in \mathbb{N}$ which can be expressed as a product of k numbers of M in at least p different ways.*

Erdös's proof of Theorem 1 was very complicated and had a purely number-theoretical character. Thus it provided no possibility of generalizing Theorem 1 to other multiplicative structures. However, in [6], J. Nešetřil and V. Rödl gave another proof of Theorem 1, based on the theorem of Ramsey, which was very simple and provided a straightforward possibility of generalizing to other structures.

In this paper we show some ways in which Theorem 1 can be generalized.

Nešetřil and Rödl's proof of Theorem 1 essentially uses the following property of the set P of all prime numbers.

PROPERTY (P). For every finite set $\{p_1, p_2, \ldots, p_r\} \subseteq P$ the following holds: if $p_1 \cdot p_2 \cdot \ldots \cdot p_r = x \cdot y$, where x, y are positive integers, then there exist sets $I, J \subseteq \{1, 2, \ldots, r\}$ such that $I \cup J = \{1, 2, \ldots, r\}$, $\prod_{i \in I} p_i = x$ and $\prod_{j \in J} p_j = y$.

Theorem 1 can easily be derived from Property (P) and the following lemma which is based on the theorem of Ramsey.

LEMMA 1. *Let X be a countably infinite set, $\mathcal{F}(X)$ the set of all finite subsets of X, and let $k \geq 2$ be an integer. Suppose that M is a subset of $\mathcal{F}(X)$ such that all but finitely many sets in $\mathcal{F}(X)$ are unions of k, not necessarily distinct, elements of M. Then for every integer p there exists a set $F \in \mathcal{F}(X)$ and at least p mutually different sets $\{F_1, F_2, \ldots, F_k\} \subseteq M$ such that $F = \bigcup_{i=1}^{k} F_i$ and $F_i \cap F_j = \emptyset$ for $i \neq j$.*

PROOF. The way in which to prove this lemma is described in [6, I], where the simple version of the lemma is stated (for $k = 2$). □

In [4], M. B. Nathanson strengthened Lemma 1 in the following way (see [4, Lemma]).

LEMMA 2. *Let X be a countably infinite set and let $k \geq 2$ be an integer. Suppose that $\mathcal{M} = (M_1, M_2, \ldots, M_k)$ is a collection of subsets of $\mathcal{F}(X)$ such that for all but finitely*

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many sets $F \in \mathcal{F}(X)$ *there exist two different k-tuples* $(F_1, F_2, \ldots, F_k)$ *such that* $F_i \in M_i$ *for* $i = 1, 2, \ldots, k$ *and* $F = \bigcup_{i=1}^{k} F_i$. *Then for every p there exists a set* $F \in \mathcal{F}(X)$ *and at least p different k-tuples* $(F_1, F_2, \ldots, F_k)$ *such that* $F_i \in M_i$ *for* $i = 1, 2, \ldots, k$, $F = \bigcup_{i=1}^{k} F_i$ *and* $F_i \cap F_j = \emptyset$ *for* $i \neq j$.

As a consequence of Lemma 2, Nathanson proved the following generalization of Theorem 1.

THEOREM 2 (see [4]). *Suppose that* $\mathcal{M} = (M_1, M_2, \ldots, M_k)$, $k \geq 2$, *is a collection of subsets of* $\mathbb{N}$ *such that all but finitely many numbers* $x \in \mathbb{N}$ *can be expressed in at least two different ways as a product* $m_1 \cdot m_2 \cdot \ldots \cdot m_k$, *where* $m_i \in M_i$ *for* $i = 1, 2, \ldots, k$. *Then for every p there exists* $x \in \mathbb{N}$ *which can be expressed in the form* $x = m_1 \cdot m_2 \cdot \ldots \cdot m_k$, *where* $m_i \in M_i$, *in at least p different ways.*

In a similar way, Nathanson proved the following theoem.

THEOREM 3 (see [4]). *Suppose that* $\mathcal{M} = (M_1, M_2, \ldots, M_k)$, $k \geq 2$, *is a collection of subsets of* $\mathbb{N}$ *such that all but finitely many numbers* $x \in \mathbb{N}$ *can be expressed in at least two different ways as the least common multiple* $[m_1, m_2, \ldots, m_k]$ *of numbers* $m_1, m_2, \ldots, m_k$ *where* $m_i \in M_i$ *for* $i = 1, 2, \ldots, k$. *Then for every p there exists* $x \in \mathbb{N}$ *which can be expressed in the form* $x = [m_1, m_2, \ldots, m_k]$, *where* $m_i \in M_i$, *in at least p different ways.*

In fact, Lemma 2 enables us to prove the analogue of Theorem 2 (concerning the usual multiplication of natural numbers) and of Theorem 3 (concerning the operation of least common multiple of natural numbers) also for other multiplicative structures. Now we describe a certain class of structures (commutative semigroups) to which Lemma 2 can be applied. First of all we give some definitions.

2. DEFINITIONS AND NOTATION

Card is the class of all cardinals, and we denote the cardinality of the set X by $|X|$. $\mathcal{F}(X)$ is the set of all finite subsets of the set X, and $\cup$ is the set-theoretical union. By $A \triangle B$ we denote the symmetric difference of sets A and B. Let $\sim$ be an equivalence relation on X. For $x \in X$ define $[x] = \{y \in X; y \sim x\}$ and put $X/\sim = \{[x]; x \in X\}$.

Let $S = (X, \cdot)$ be a commutative semigroup. We say that x *divides* y $(x, y \in X)$ and denote this by $x \mid y$ if there is an element $z \in X$ such that $y = x \cdot z$. Let us recall that an element $j \in X$ is called a *unit* if j divides the identity element. We say that x is associated with y (and denote this by $x \sim y$) if there exists a unit j such that $x = y \cdot j$. Clearly, $\sim$ is an equivalence relation on X. Let us remark that S may have no identity element. If this is the case, we define $\sim$ to be an identity relation; i.e. $x \sim y$ iff $x = y$.

Let $S = (X, \cdot)$ be a commutative semigroup and let $k \geq 2$ be an integer.

Define an equivalence $\sim$ on X^k as follows:

$$(x_1, x_2, \ldots, x_k) \sim (x_1', x_2', \ldots, x_k') \quad \text{iff } x_i \sim x_i' \qquad \text{for } i = 1, 2, \ldots, k.$$

Now, let $\mathcal{M} = (M_1, M_2, \ldots, M_k)$ be a k-tuple of subsets of X. For $x \in X$ denote $A_x = \{(m_1, m_2, \ldots, m_k) \in \prod_{i=1}^{k} M_i;\ x = m_1 \cdot m_2 \cdot \ldots \cdot m_k\}$ and define the functions $f_{\mathcal{M}}: X \to \text{Card}$ and $g_{\mathcal{M}}: X \to \text{Card}$ by

$$f_{\mathcal{M}}(x) = |A_x| \qquad \text{and} \qquad g_{\mathcal{M}}(x) = |A_x/\sim|.$$

Similarly, for $M \subseteq X$ put $B_x = \{\langle m_1, m_2, \ldots, m_k\rangle \subseteq M;\ x = m_1 \cdot m_2 \cdot \ldots \cdot m_k\}$, where $\langle m_1, m_2, \ldots, m_k\rangle$ denotes the collection of elements $m_1, m_2, \ldots, m_k$ of M (not necessarily distinct). We define functions $f_{M,k}$ and $g_{M,k}$ by

$$f_{M,k}(x) = |B_x| \quad \text{and} \quad g_{M,k}(x) = |B_x/\sim|.$$

For brevity, we denote the function $f_{M,2}$ by f_M.

Clearly, if S has at most one unit, then $g_{\mathcal{M}} \equiv f_{\mathcal{M}}$ and $g_{M,k} \equiv f_{M,k}$.

Definition 1. We say that $\mathcal{M} = (M_1, \ldots, M_k)$ is an *asymptotic multiplicative system of order k* if $f_{\mathcal{M}}(x) \geq 1$ for all but finitely many elements $x \in X$. Similarly, $M \subseteq X$ is an *asymptotic multiplicative basis of order k* if $f_{M,k}(x) \geq 1$ for all but finitely many elements $x \in X$.

Let $S = (X, \cdot)$ be a commutative semigroup and let $F = \{x_1, x_2, \ldots, x_k\}$ be a finite subset of X. Then the product $x_1 \cdot x_2 \cdot \ldots \cdot x_k$ is denoted by $\prod F$. If S has an identity element 1, we also define $\prod \emptyset = 1$.

Definition 2. Let $S = (X, \cdot)$ be a commutative semigroup. The set $P \subseteq X$ is said to be a *prime set* if it contains no unit, if no two different elements of P are associated and if for every finite (non-empty) set $F \subseteq P$ the following condition holds: if $\prod F = x_1 \cdot x_2$ then there exist finite sets $F_1, F_2 \subseteq F$ (possibly empty) such that $F_1 \cup F_2 = F$, $x_1 \sim \prod F_1$ and $x_2 \sim \prod F_2$.

Definition 3. The commutative semigroup is said to be a *prime semigroup* if it contains an infinite prime set and if it has only finitely many units.

3. General Theorems on Multiplicative Bases

In the next theorem we show that the result stated in Theorem 2 for the semigroup $(\mathbb{N}, \cdot)$ holds for every prime commutative semigroup.

Theorem 4. *Suppose that $S = (X, \cdot)$ is a prime semigroup, $k \geq 2$, $M_1, M_2, \ldots, M_k \subseteq X$, $\mathcal{M} = (M_1, M_2, \ldots, M_k)$. If $g_{\mathcal{M}}(x) \geq 2$ for all but finitely many elements $x \in X$, then for every p there exists $x \in X$ such that $g_{\mathcal{M}}(x) > p$.*

Let us prove Theorem 4. In the proof we shall use the fact that every prime set is "productively independent" in the sense of the following proposition.

Proposition. *Let $S = (X, \cdot)$ be a commutative semigroup and $P \subseteq X$ be a prime set. Then for every two finite sets $P_1, P_2 \subseteq P$ the following condition holds: if $\prod P_1 \sim \prod P_2$ then $P_1 = P_2$.*

Proof. Let P_1, P_2 be finite subsets of P such that $\prod P_1 \sim \prod P_2$ and $P_2 \backslash P_1 \neq \emptyset$. Choose an arbitrary element $p \in P_2 \backslash P_1$. Since $p \mid \prod P_1$ and P is a prime set, there is a set $Q \subseteq P_1$ such that $p \sim \prod Q$. Clearly, $p \notin Q$ and since p is not a unit, we have that $Q \neq \emptyset$. Let q be an arbitrary element of Q. Then $q \mid p$ and therefore $q \sim p$ by the definition of the prime set. Thus $q = p$, hence $p \in Q$, a contradiction. □

Proof of Theorem 4. Denote by n the number of units in S and suppose that $n > 0$. For $x \in X$ define $[x] = \{y \in X; y \sim x\}$, and for $Y \subseteq X$ put $[Y] = \bigcup_{y \in Y} [y]$. Let $P \subseteq X$ be an infinite prime set in the semigroup S. For $i = 1, 2, \ldots, k$ define sets $M_i' \subseteq \mathcal{F}(P)$ by $M_i' = \{F \in \mathcal{F}(P); \prod F \in [M_i]\}$.

By the proposition, the mapping $F \mapsto \prod F$ from $\mathscr{F}(P)$ to X is an injection and therefore for all but finitely many sets $F \in \mathscr{F}(P)$ there exist at least two non-associated k-tuples $(m_1, m_2, \ldots, m_k)$ such that $m_i \in M_i$ and $\prod F = m_1 \cdot m_2 \cdot \ldots \cdot m_k$. Let $(m_1, m_2, \ldots, m_k)$ be such a k-tuple. Then we obtain, by the definition of the prime set, that there exist sets F_i for $i = 1, 2, \ldots, k$ such that $F = \bigcup_{i=1}^k F_i$ and $m_i \sim \prod F_i$. But then $F_i \in M_i'$, and hence the infinite set P and sets $M_i' \subseteq \mathscr{F}(P)$ fulfil the assumptions of Lemma 2. Thus for every p there exists a set $F \in \mathscr{F}(P)$ and at least $p \cdot n + 1$ different k-tuples $(F_1, F_2, \ldots, F_k)$ such that $F_i \in M_i'$, $F = \bigcup_{i=1}^k F_i$ and $F_i \cap F_j = \emptyset$ for $i \neq j$. If $(F_1, F_2, \ldots, F_k)$ is such a k-tuple then $\prod F = \prod_{i=1}^k (\prod F_i)$, where $\prod F_i \in [M_i]$. Hence there exists a unit j and a k-tuple $(m_1, m_2, \ldots, m_k) \in \prod_{i=1}^k M_i$ such that $j \cdot \prod F = \prod_{i=1}^k m_i$ and $m_i \sim \prod F_i$. This yields by the proposition that there exists a unit j such that $g_{\mathscr{M}}(j \cdot \prod F) > p$.

The case $n = 0$ is similar to the case $n = 1$. □

As for the previous theorem, we can deduce from Lemma 1 the following generalization of Theorem 1.

THEOREM 5. *Suppose that $S = (X, \cdot)$ is a prime semigroup, $k \geq 2$, $M \subseteq X$. If $g_{M,k}(x) \geq 1$ for all but finitely many x, then for every p there exists $x \in X$ such that $g_{M,k}(x) > p$.*

EXAMPLES. (1) The semigroups $(\mathbb{N}, \cdot)$ and $(\mathbb{N}, LCM)$, where LCM is the least common multiple, are prime semigroups. (The set of all prime numbers is an infinite prime set.)

(2) The semigroup $(\mathscr{F}(\mathbb{N}), \cup)$ of all finite subsets of $\mathbb{N}$ with the union operation is a prime semigroup. (The set of all singletons is an infinite prime set.)

(3) Let $\mathscr{K}$ be the class of all isomorphism types of finite simple graphs. The semigroup $(\mathscr{K}, \times)$, where $\times$ is the cardinal (direct) product, is a prime semigroup. (It can be shown (see [7]) that the set of all complete bipartite graphs $K_{1,p}$, where $p \geq 2$ is a prime number, is a prime set.)

4. THE SEMIGROUP $(\mathscr{F}(\mathbb{N}), \cup)$

Let $S = (X, \cdot)$ be a countable prime commutative semigroup with at most one unit and let $\mathscr{M} = (M_1, \ldots, M_k)$ be an asymptotic multiplicative system of order k in the semigroup S. Then Theorem 4 states that the following condition holds:

$$\text{(1)} \qquad \text{If } \liminf_{x \in X} f_{\mathscr{M}}(x) \geq 2 \text{ then } \limsup_{x \in X} f_{\mathscr{M}}(x) = \infty.$$

This gives no lower bound of the number $\limsup_{x \in X} f_{\mathscr{M}}(x)$ under the assumption that $\liminf_{x \in X} f_{\mathscr{M}}(x) \geq 1$ (i.e. $\mathscr{M}$ is an asymptotic multiplicative system). In particular, in [4], Nathanson showed that in the semigroup $(\mathbb{N}, \cdot)$ the condition (1), together with the obvious condition $\liminf_{x \in X} f_{\mathscr{M}}(x) \leq k$, are the only conditions that restrict the behaviour of functions $f_{\mathscr{M}}$. Thus the set $\mathscr{T}_k$ of all pairs (i, s), where $i = \liminf_{x \in X} f_{\mathscr{M}}(x)$, $s = \limsup_{x \in X} f_{\mathscr{M}}(x)$ and $\mathscr{M}$ is an asymptotic multiplicative system of order k in the semigroup $(\mathbb{N}, \cdot)$, is given by the formula

$$\mathscr{T}_k = \{(1, s); s \in \mathbb{N}\} \cup \{(i, \infty); 1 \leq i \leq k\}.$$

In the remaining part of this paper we show that $(1, 2) \notin \mathscr{T}_2$ for the semigroup $(\mathscr{F}(\mathbb{N}), \cup)$. Moreover, we give the full description of the set $\mathscr{T}_2$ for this semigroup.

First we introduce some definitions.

DEFINITION 4. Let $S = (X, \cdot)$ be a countable commutative semigroup and let $\mathcal{M} = (M_1, M_2, \ldots, M_k)$ be a k-tuple of subsets of X.

The *type* $t(\mathcal{M})$ of the system $\mathcal{M}$ in the semigroup S is the ordered pair $(i(\mathcal{M}), s(\mathcal{M}))$, where $i(\mathcal{M}) = \liminf_{x \in X} f_{\mathcal{M}}(x)$ and $s(\mathcal{M}) = \limsup_{x \in X} f_{\mathcal{M}}(x)$.

The *set of types of order* k of S is the set $\mathcal{T}_k(S) = \{t(\mathcal{M});\ \mathcal{M}$ is an asymptotic multiplicative system of order $k\}$.

Denote $\mathbb{N}^* = \mathbb{N} \cup \{\infty\}$. It can easily be seen that

$$i(\mathcal{M}) = \sup\{n \in \mathbb{N}^*; f_{\mathcal{M}}(x) \geq n \text{ for all but finitely many } x \in X\}$$

and

$$s(\mathcal{M}) = \min\{n \in \mathbb{N}^*; f_{\mathcal{M}}(x) \leq n \text{ for all but finitely many } x \in X\}.$$

Hence, $i(\mathcal{M})$ is the best asymptotic lower bound of the function $f_{\mathcal{M}}$ and $s(\mathcal{M})$ is the best asymptotic upper bound of $f_{\mathcal{M}}$.

In particular, $s(\mathcal{M}) = \infty$ iff for every p there exist (infinitely many) $x \in X$ such that $f_{\mathcal{M}}(x) > p$.

The main result of this section is the following.

THEOREM 6. $\mathcal{T}_2(\mathcal{F}(\mathbb{N}), \cup) = \{(1, s); s \in \mathbb{N} \setminus \{2\}\} \cup \{(1, \infty), (2, \infty), (3, \infty)\}$.

PROOF OF THEOREM 6. First we show that $(1, 2) \notin \mathcal{T}_2(\mathcal{F}(\mathbb{N}), \cup)$.

Suppose that $\mathcal{M} = (M_1, M_2)$ be an asymptotic multiplicative system of order 2 in the semigroup $(\mathcal{F}(\mathbb{N}), \cup)$ and denote $\limsup\{f_{\mathcal{M}}(A); A \in \mathcal{F}(\mathbb{N})\} = s$. Our purpose is to show that $s \neq 2$. Without loss of generality, we can suppose that $s < \infty$. Denote $X_1 = \{x \in \mathbb{N}; \{x\} \in M_1\}$ and $X_2 = \{x \in \mathbb{N}; \{x\} \in M_2\}$. We say that some statement about sets of some set system $\mathcal{A}$ is true for "almost every" set of $\mathcal{A}$ if it is true for all but finitely many sets of $\mathcal{A}$.

We divide the proof into some facts.

FACT 1. *If* $s < \infty$, *then* $|M_2 \cap \mathcal{F}(X_1)| < \infty$ *(and, similarly,* $|M_1 \cap \mathcal{F}(X_2)| < \infty$*).*

PROOF. We shall use the following simple proposition. Let $k \geq 1$ be an integer. Then every infinite family $\mathcal{Y}$ of sets of size k contains an infinite family $\mathcal{T}$ such that every two sets from $\mathcal{T}$ have the same intersection.

Let us proceed to the proof of Fact 1. Clearly, $f_{\mathcal{M}}(A) \geq |A|$ for $A \in M_2 \cap \mathcal{F}(X_1)$, and hence $|A| \leq s$ for almost every set $A \in M_2 \cap \mathcal{F}(X_1)$. Furthermore, $f_{\mathcal{M}}(A) \geq 1$ for almost every set $A \in \mathcal{F}(\mathbb{N})$ and therefore exists a finite subset F of X_1 such that for every non-empty set $A \in \mathcal{F}(X_1) \setminus \mathcal{F}(F)$ the following conditions hold:

$$f_{\mathcal{M}}(A) \geq 1 \tag{1}$$

and

$$A \in M_2 \Rightarrow |A| \leq s. \tag{2}$$

Let $\emptyset \neq A \in \mathcal{F}(X_1) \setminus \mathcal{F}(F)$. Then we have $A = A_1 \cup A_2$, where $A_1 \in M_1$, $A_2 \in M_2$ and $|A_2| \leq s$. In particular:

(3) If $\emptyset \neq A \in \mathcal{F}(X_1 \setminus F)$ then there exists a set $A' \subseteq A$ such that $|A'| \leq s$ and $A \setminus A' \in M_1$.

Suppose that $|M_2 \cap \mathcal{F}(X_1)| = \infty$. Then there exists a set $F' \subseteq F$ such that the set $\mathcal{P} = \{A \in M_2 \cap \mathcal{F}(X_1); A \cap F = F'\}$ is infinite. Consider the set $\mathcal{Y} = \{A \setminus F'; A \in \mathcal{P}\}$. By (2), the size of all sets in $\mathcal{Y}$ is not greater than s. Therefore there exists an infinite set $\mathcal{T} \subseteq \mathcal{Y}$ and a set F'' such that the intersection of every pair of sets in $\mathcal{T}$ is equal to F''.

Let $F_1, F_2, \ldots$ be a sequence of pairwise distinct members from $\mathcal{T}$ different from F''. Then $F_i \backslash F''$, $i = 1, 2, \ldots$ are non-empty pairwise disjoint sets.

According to (3), for every p there exists a finite set $A_1 \in M_1$ such that A_1 contains at least p sets $F_i \backslash F''$. Furthermore, if A_1 contains $F_i \backslash F''$ then

$$A_1 \cup F' \cup F'' = A_1 \cup F' \cup F'' \cup (F_i \backslash F'') = A_1 \cup (F' \cup F_i).$$

Moreover, since $F' \cup F_i$ are pairwise distinct members of M_2, we have $f_{\mathcal{M}}(A_1 \cup F' \cup F'') \geqslant p$. Thus $\limsup\{f_{\mathcal{M}}(A);\ A \in \mathcal{F}(\mathbb{N})\} \geqslant p$ for every p, a contradiction. ∎

Furthermore, we shall often use the following immediate corollary of Fact 1.

COROLLARY. *If $s < \infty$, then there exists a finite set $F \subseteq X_1$ such that*

$$\varnothing \neq A \in \mathcal{F}(X_1 \backslash F) \Rightarrow A \in M_1.$$

FACT 2. $|X_1 \cap X_2| < \infty$.

PROOF. An immediate corollary of Fact 1. ∎

FACT 3. *If $|X_1| = \infty$, then $\varnothing \in M_2$.*

PROOF. We have $|X_1 \backslash X_2| = \infty$ by Fact 2. Thus there exists $x \in X_1 \backslash X_2$ such that $f_{\mathcal{M}}(\{x\}) \geqslant 1$. But the only possible expression of the set $\{x\}$ as a union of sets from M_1 and M_2 is $\{x\} \cup \varnothing$. We conclude that $\varnothing \in M_2$. ∎

From now we shall suppose that $s \leqslant 2$.

FACT 4. *If $|X_1| = |X_2| = \infty$, then $X_1 \cap X_2 = \varnothing$.*

PROOF. We have $\varnothing \in M_1 \cap M_2$ by Fact 3. Let (by Fact 2) $U = \{u_1, u_2, \ldots\} \subseteq X_1 \backslash X_2$ and $V = \{v_1, v_2, \ldots\} \subseteq X_2 \backslash X_1$ be infinite (disjoint) sets. Suppose that there exists $x \in X_1 \cap X_2$. Then there exists an infinite set $I \subseteq \{1, 2, \ldots\}$ such that $f_{\mathcal{M}}(\{x, u_i, v_i\}) \geqslant 1$ for $i \in I$, i.e. $\{x, u_i, v_i\} = A_1^i \cup A_2^i$, where $A_1^i \in M_1$ and $A_2^i \in M_2$. Since $A_1^i \ni x$ or $A_2^i \ni x$ for every i, we can suppose that the set $J = \{i \in I;\ A_1^i \ni x\}$ is infinite. Let $i \in J$. Then one of the following possibilities holds:

(α) $$A_1^i \ni v_i.$$

Then $f_{\mathcal{M}}(A_1^i) \geqslant 3$ because $A_1^i = A_1^i \cup \varnothing = A_1^i \cup \{x\} = A_1^i \cup \{v_i\}$.

(β) $$A_1^i = \{x, u_i\}.$$

Then we again have $f_{\mathcal{M}}(A_1^i) \geqslant 3$ because $A_1^i = A_1^i \cup \varnothing = A_1^i \cup \{x\} = \{u_i\} \cup \{x\}$.

(γ) $$A_1^i = \{x\}.$$

Then either $A_2^i = \{u_i, v_i\}$ and so $f_{\mathcal{M}}(A_2^i) \geqslant 3$ because $A_2^i = \varnothing \cup A_2^i = \{u_i\} \cup A_2^i = \{u_i\} \cup \{v_i\}$, or $A_2^i = \{x, u_i, v_i\}$ and then also $f_{\mathcal{M}}(A_2^i) \geqslant 3$ because $A_2^i = \varnothing \cup A_2^i = \{x\} \cup A_2^i = \{u_i\} \cup A_2^i$. We conclude that $s \geqslant 3$, a contradiction. ∎

FACT 5. *If $|X_1| = |X_2| = \infty$, then $(\mathcal{F}(X_1) \backslash \{\varnothing\}) \cap M_2 = \varnothing$.*

PROOF. By Fact 1 there exists a finite set $F \subseteq X_1$ such that for $\varnothing \neq A \in \mathcal{F}(X_1) \backslash \mathcal{F}(F)$ holds: $f_{\mathcal{M}}(A) \geqslant 1$ and $A \notin M_2$.

Suppose that there exists a set $\varnothing \neq A \in \mathcal{F}(X_1) \cap M_2$ (and so $A \subseteq F$). Then $|A| \geqslant 2$ by Fact 4. Choose in A two fixed different points x, y. If $\varnothing \neq B \in \mathcal{F}(X_1 \backslash F)$ then from the choice of F and Fact 4 it follows that $B \in M_1$ and also $B \cup \{x\} \in M_1$ and $B \cup \{y\} \in M_1$. This yields that $f_{\mathcal{M}}(A \cup B) \geq 3$ and so $s \geq 3$, a contradiction. ∎

FACT 6. *If $|X_1| = |X_2| = \infty$, then $|\mathcal{F}(X_1) \setminus M_1| < \infty$ (i.e. $\mathcal{F}(X_1) \subseteq M_1$ excepting at most finitely many finite subsets of X_1).*

PROOF. By Fact 5 no non-empty subset of X_1 belongs to M_2. Thus $A \in M_1$ for every set $A \in X_1$, fulfilling the condition $f_{\mathcal{M}}(A) \geq 1$. ∎

FACT 7. *If $|X_1| = |X_2| = \infty$, then $|M_1 \setminus \mathcal{F}(X_1)| < \infty$.*

PROOF. Suppose that $|M_1 \setminus \mathcal{F}(X_1)| = \infty$.

Let $A \in M_1 \setminus \mathcal{F}(X_1)$ and $|A \cap X_2| \geq 2$. Choose $x, y \in A \cap X_2$, $x \neq y$. Then $A = A \cup \{x\} = A \cup \{y\} = A \cup \varnothing$ and $\varnothing \in M_2$ by Fact 3; hence $f_{\mathcal{M}}(A) \geq 3$. Therefore

$$|\{A \in M_1; |A \cap X_2| = 1\}| = \infty.$$

Assume that $A \in M_1$, $A \cap X_2 = \{x\}$ and $A \cap X_1 \in M_1$. Then $A = A \cup \{x\} = A \cup \varnothing = (A \cap X_1) \cup \{x\}$, and hence $f_{\mathcal{M}}(A) \geq 3$. Therefore

$$|\{A \in M_1; |A \cap X_2| = 1 \;\&\; A \cap X_1 \notin M_1\}| = \infty.$$

Since, by Fact 6, $|\mathcal{F}(X_1) \setminus M_1| < \infty$, there is a set $A_1 \in \mathcal{F}(X_1) \setminus M_1$ and an infinite set $Z \subseteq X_2$ such that $A_1 \cup \{x\} \in M_1$ for every $x \in Z$. Since, again by Fact 6, $|\mathcal{F}(X_2) \setminus M_2| < \infty$, there are infinitely many sets $B \subseteq Z$ such that $|B| \geq 3$ and $B \in M_2$. But then $f_{\mathcal{M}}(A_1 \cup B) \geq 3$ (since $A_1 \cup B = (A_1 \cup \{x\}) \cup B$ for $x \in B$) which contradicts the assumption $s \leq 2$. ∎

FACT 8. *If $|X_1| < \infty$, then $|M_1 \setminus \mathcal{F}(X_1)| < \infty$ (and so $|M_1| < \infty$).*

PROOF. According to the corollary following Fact 1, there is a finite set $F \subseteq X_2$ such that $B \in M_2$ for every finite non-empty set $B \subseteq X_2 \setminus F$.

Suppose that $|M_1| = \infty$. Since $|X_1 \cup F| < \infty$, there are a set $A \subseteq X_1 \cup F$ and infinitely many non-empty sets $B_1, B_2, \ldots$ such that $B_i \subseteq \mathbb{N} \setminus (X_1 \cup F)$ and $A \cup B_i \in M_1$ for $i = 1, 2, \ldots$. If $i, j, k \in \mathbb{N}$, $i < j < k$, then $B = B_i \cup B_j \cup B_k \subseteq X_2 \setminus F$, thus $B \in M_2$ and, moreover, $A \cup B = (A \cup B_i) \cup B = (A \cup B_j) \cup B = (A \cup B_k) \cup B$. We conclude that $f_{\mathcal{M}}(A \cup B) \geq 3$ and so $s \geq 3$, which is a contradiction. ∎

FACT 9. *If $|X_2| = \infty$, then $\mathcal{F}(X_1) = M_1$.*

PROOF. (a) We show that $\mathcal{F}(X_1) \subseteq M_1$. According to Facts 6, 7 and 8 we have that $|M_1 \triangle \mathcal{F}(X_1)| < \infty$. Furthermore, we have that $\varnothing \in M_1$, by Fact 3, and that $\{x\} \in M_1$ for all $x \in X_1$, by the definition of X_1. Suppose that $A \subseteq X_1$ and $|A| = k \geq 2$. We show by induction on k that $A \in M_1$.

Suppose that all subsets of X_1 with size less than k belong to M_1. Choose two fixed elements $x, y \in A$, $x \neq y$. Since $|M_1 \setminus \mathcal{F}(X_1)| < \infty$, there is a finite set F such that the following conditions hold:

(1) if B is finite and $B \nsubseteq F$ then $f_{\mathcal{M}}(B) \geq 1$

and

(2) if $B \in M_1 \setminus \mathcal{F}(X_1)$ then $B \subseteq F$.

Assume that $A \notin M_1$. We have $|X_2 \setminus X_1| = \infty$, by Fact 2, and hence $|(X_2 \setminus X_1) \setminus F| = \infty$. Let $z \in (X_2 \setminus X_1) \setminus F$. By the definition of F we have that $A \cup \{z\} = A_1 \cup A_2$, where

$A_1 \in M_1$, $A_2 \in M_2$ and $A_1 \subseteq A$. Furthermore, $A_1 \subsetneq A$ because $A \notin M_1$. Thus either

$$(\alpha) \qquad A_2 = A \cup \{z\} \in M_2$$

or

$$(\beta) \qquad A_2 = (A_2 \cap X_1) \cup \{z\}, \qquad \text{where } \emptyset \neq A_2 \cap X_1 \subsetneq A.$$

In particular, by the induction hypothesis, $A_2 \cap X_1 \in M_1$. In case (α) we have $A_2 = \emptyset \cup A_2 = \{x\} \cup A_2 = \{y\} \cup A_2$, and hence $f_{\mathcal{M}}(A_2) \geqslant 3$. Similarly, in case (β) we have $A_2 = \emptyset \cup A_2 = (A_2 \cap X_1) \cup A_2 = (A_2 \cap X_1) \cup \{z\}$, thus again $f_{\mathcal{M}}(A_2) \geqslant 3$. In both cases (α) and (β), $A_2 \cap ((X_2 \backslash X_1) \backslash F) = \{z\}$, while $|(X_2 \backslash X_1) \backslash F| = \infty$; thus there are infinitely many sets A_2 such that $f_{\mathcal{M}}(A_2) \geqslant 3$, a contradiction.

(b) We show that $\mathscr{F}(X_1) = M_1$. Suppose that there is a set $A \in M_1$ such that $A \nsubseteq X_1$ and choose an element $x \in A \backslash X_1$. Let F be the set defined in part (a) and let $z \in (X_2 \backslash X_1) \backslash F$. Then, by (1) in the definition of F, $f_{\mathcal{M}}(\{x, z\}) \geqslant 1$. Now, since $x \notin X_1$, condition (2) in the definition of F implies that $\{x, z\} \in M_2$. We show that $f_{\mathcal{M}}(A \cup \{z\}) \geqslant 3$. We distinguish two cases.

(α) Let $|A \backslash X_1| \geqslant 2$ and let x, y be two different elements from $A \backslash X_1$. Then the equalities $A \cup \{z\} = A \cup \{x, z\} = A \cup \{y, z\}$ show that $f_{\mathcal{M}}(A \cup \{z\}) \geqslant 3$.

(β) Let $A \backslash X_1 = \{x\}$. Then $A \cup \{z\} = A \cup \{x, z\} = (A \backslash \{x\}) \cup \{x, z\}$ and since $A \backslash \{x\} \in M_1$ by (a), we have again that $f_{\mathcal{M}}(A \cup \{z\}) \geqslant 3$.

Since the set $(X_2 \backslash X_1) \backslash F$ is infinite, there are infinitely many sets $A \cup \{z\}$ such that $f_{\mathcal{M}}(A \cup \{z\}) \geqslant 3$, a contradiction. ■

We complete the proof of the statement $(1,2) \notin \mathscr{T}_2(\mathscr{F}(\mathbb{N}), \cup)$ by the following lemma.

Lemma. *Suppose that* $s \leqslant 2$.
(1) *If* $|M_1| = |M_2| = \infty$, *then* $|X_1| = |X_2| = \infty$, $X_1 \cap X_2 = \emptyset$, $X_1 \cup X_2 = \mathbb{N}$, $M_1 = \mathscr{F}(X_1)$ *and* $M_2 = \mathscr{F}(X_2)$.
(2) *If* $|M_1| < \infty$, *then* $|X_1| < \infty$, $M_1 = \mathscr{F}(X_1)$ *and* $|\mathscr{F}(\mathbb{N} \backslash X_1) \triangle M_2| < \infty$.
In both cases $s = 1$.

Proof. (1) We have $|X_1| = |X_2| = \infty$ by Fact 8, $X_1 \cap X_2 = \emptyset$ by Fact 4, $M_1 = \mathscr{F}(X_1)$ and $M_2 = \mathscr{F}(X_2)$ by Fact 9, and from this it immediately follows that $X_1 \cup X_2 = \mathbb{N}$ and $s = 1$.

(2) If $|M_1| < \infty$ then $|X_1| < \infty$, and thus $|X_2| = \infty$ and $\mathscr{F}(X_1) = M_1$ by Fact 9. This implies that for every set $A \in \mathscr{F}(\mathbb{N} \backslash X_1)$ the condition $f_{\mathcal{M}}(A) \geqslant 1$ holds iff $A \in M_2$. Hence $|\mathscr{F}(\mathbb{N} \backslash X_1) \backslash M_2| < \infty$. It follows that there is a finite set $F \subseteq \mathbb{N} \backslash X_1$ such that $\mathscr{F}(\mathbb{N} \backslash X_1) \backslash M_2 \subseteq \mathscr{F}(F)$.

Now suppose that $A \in M_2 \backslash \mathscr{F}(\mathbb{N} \backslash X_1)$. Then either $A \subseteq X_1 \cup F$ or $f_{\mathcal{M}}(A) \geqslant 3$. For this, let $A \nsubseteq X_1 \cup F$. Then $A \cap (\mathbb{N} \backslash X_1) \in M_2$ by the definition of F and $\emptyset \in M_1$ by Fact 3, and hence the equations $A = \emptyset \cup A = (A \cap X_1) \cup A = (A \cap X_1) \cup (A \cap (\mathbb{N} \backslash X_1))$ demonstrates that $f_{\mathcal{M}}(A) \geqslant 3$. Since $s \leqslant 2$, it follows from the above that $|M_2 \backslash \mathscr{F}(\mathbb{N} \backslash X_1)| < \infty$. Hence there is a finite set $E \subseteq \mathbb{N} \backslash X_1$ such that $M_2 \backslash \mathscr{F}(\mathbb{N} \backslash X_1) \subseteq \mathscr{F}(X_1 \cup E)$. But then for every finite set $A \nsubseteq X_1 \cup E$ we have $f_{\mathcal{M}}(A) \leqslant 1$, and thus $s = 1$. ■

The previous lemma has an interesting corollary which gives the full characterization of asymptotic multiplicative systems $\mathcal{M} = (M_1, M_2)$ of order 2 in the semigroup $(\mathscr{F}(\mathbb{N}), \cup)$ such that $s(\mathcal{M}) = 1$ (i.e. $t(\mathcal{M}) = (1, 1)$).

COROLLARY. *Let M_1, M_2 be non-empty subsets of $\mathcal{F}(\mathbb{N})$. Then $f_{\mathcal{M}}(A) = 1$ for almost every set $A \in \mathcal{F}(\mathbb{N})$ iff either:*
(1) *there is a partition $\mathbb{N} = X_1 \cup X_2$ such that $X_1 \cap X_2 = \emptyset$, $|X_1| = |X_2| = \infty$, $M_1 = \mathcal{F}(X_1)$ and $M_2 = \mathcal{F}(X_2)$ (hence $|M_1| = |M_2| = \infty$); or*
(2) *there is a finite set $X_1 \subseteq \mathbb{N}$ such that $M_1 = \mathcal{F}(X_1)$, and $|\mathcal{F}(\mathbb{N}\backslash X_1) \triangle M_2| < \infty$ (hence $|M_1| < \infty$ and $|M_2| = \infty$); or*
(3) *there is a finite set $X_2 \subseteq \mathbb{N}$ such that $M_2 = \mathcal{F}(X_2)$ and $|\mathcal{F}(\mathbb{N}\backslash X_2) \triangle M_1| < \infty$ (hence $|M_1| = \infty$ and $|M_2| < \infty$).*

Let us continue the proof of Theorem 6. We show that for every $s \in \mathbb{N}\backslash\{2\}$ there is a system $\mathcal{M}$ of order 2 and of type $(1, s)$.

First we define $M_1 = \mathcal{F}(\mathbb{N})$ and $M_2 = \{\emptyset, C_1, \ldots, C_k\}$, where $k \geq 1$ and $|M_2| = k + 1$. Then, clearly, $\mathcal{M} = (M_1, M_2)$ is an asymptotic multiplicative system of order 2 and of type $(1, 1 + \sum_{i=1}^{k} 2^{|C_i|})$. In particular, we can construct systems of type $(1, 1 + 2k)$ for every k.

Now we construct systems $\mathcal{M} = (M_1, M_2)$ of type $(1, s)$ for every even number $s > 2$. We distinguish three cases.

(α) Type $(1, 4)$: let $A = \{a_i; i \in \mathbb{N}\}$, $B = \{b_i; i \in \mathbb{N}\}$ and $C = \{c_i; i \in \mathbb{N}\}$ be countable pairwise disjoint sets such that $\mathbb{N} = A \cup B \cup C$. Define

$$M_1 = \mathcal{F}(A) \cup \{\{a_i, b_i\}; i \in \mathbb{N}\}$$

and

$$M_2 = \mathcal{F}(B \cup C) \cup \{\{a_i, c_i\}; i \in \mathbb{N}\}.$$

It is easy to see that $f_{\mathcal{M}}(\{a_i, b_i, c_i\}) = f_{\mathcal{M}}(\{a_i, b_i\} \cup \{a_i, c_i\}) = 4$, $f_{\mathcal{M}}(\{a_i, b_i\} \cup \{a_j, c_j\}) = 2$ for $i \neq j$ and $f_{\mathcal{M}}(F) \in \{1, 3\}$ for the other sets $F \in \mathcal{F}(\mathbb{N})$. Moreover, $f_{\mathcal{M}}(F) = 1$ for infinitely many sets F. Hence, $\mathcal{M} = (M_1, M_2)$ is an asymptotic multiplicative system of type $(1, 4)$.

(β) Type $(1, 2^{p+1} + 2)$ for $p \geq 1$: more generally, we construct a system $\mathcal{M} = (M_1, M_2)$ of type $(1, 2^p + 2^q + 2)$ for $p, q \geq 1$. Let X and Y be countable disjoint sets such that $\mathbb{N} = X \cup Y$ and let $X = \bigcup_{i=0}^{\infty} A_i$ and $Y = \bigcup_{i=0}^{\infty} B_i$ be disjoint partitions of sets X and Y such that $|A_i| = p$ and $|B_i| = q$ for every i. Define

$$M_1 = \mathcal{F}(X) \cup \{A_i \cup B_i; i \in \mathbb{N}\}$$

and

$$M_2 = \mathcal{F}(Y) \cup \{A_i \cup B_i; i \in \mathbb{N}\}.$$

Then $f_{\mathcal{M}}(A_i \cup B_i) = 2^p + 2^q + 2$ for every i. Furthermore, for $\emptyset \neq F \in \mathcal{F}(\mathbb{N})$, $F \cap (A_i \cup B_i) = \emptyset$ we have

$$f_{\mathcal{M}}(A_i \cup B_i \cup F) = \begin{cases} 1 + 2^q & \text{if } F \subseteq Y \\ 1 + 2^p & \text{if } F \subseteq X \\ 3 & \text{if } F = A_j \cup B_j, \text{ where } j \neq i \\ 1 & \text{in the other cases.} \end{cases}$$

It follows that $\mathcal{M} = (M_1, M_2)$ is a system of type $(1, 2^p + 2^q + 2)$.

(γ) Type $(1, s)$, where $s > 2$ is an even number which cannot be expressed in the form $2^p + 2$ for $p \geq 1$: by the assumption, the positive integer $s - 2$ is even and is not a power of the number 2. Hence, we can write $s - 2 = 2^{p_0} + 2^{p_1} + \cdots + 2^{p_n}$, where $0 < p_0 < p_1 < \cdots < p_n$ and $n > 0$. Let X and Y be countable disjoint sets such that $X \cup Y = \mathbb{N}$. Let us denote $X = \{x_i; i \in \mathbb{N}\}$ and form a disjoint partition $Y = \bigcup_{i=0}^{\infty} A_i$ of Y such that $|A_i| = p_n$ for every i. We put $A_i = \{a_i^j; 1 \leq j \leq p_n\}$ for $i \in \mathbb{N}$ and $A_i^k = \{a_i^j; 1 \leq j \leq k\}$ for $k \in \{1, 2, \ldots, p_n\}$. Hence

$$A_i^{p_0} \subsetneq A_i^{p_1} \subsetneq \cdots \subsetneq A_i^{p_n} = A_i.$$

Now define

$$M_1 = \mathscr{F}(X) \cup \{\{x_i\} \cup A_i^{p_j};\, i \in \mathbb{N} \text{ and } 0 \leqslant j \leqslant n\}$$

and

$$M_2 = \mathscr{F}(Y) \cup \{\{x_i\} \cup A_i^{p_0};\, i \in \mathbb{N}\}.$$

It can easily be shown that for $F \in \mathscr{F}(\mathbb{N})$ the following holds:
(1) If $|F \cap X| \geqslant 3$ then $f_{\mathcal{M}}(F) \in \{1, 3\}$. Moreover, $f_{\mathcal{M}}(F) = 3$ iff $F \cap X \ni x_i$ and $F \cap Y = A_i^{p_0}$ for some $i \in \mathbb{N}$.
(2) If $|F \cap X| = 2$ then $f_{\mathcal{M}}(F) \leqslant 3$.
(3) Let us examine the case $|F \cap X| = 1$. Let $F \cap X = \{x_i\}$. Then $f_{\mathcal{M}}(F) = f + \sum_{j=0}^{n} f_j$, where f is the number of sets $B \in M_2$ such that $(F \cap X) \cup B = F$ and f_j, $0 \leqslant j \leqslant n$, is the number of sets $B \in M_2$ such that $(\{x_i\} \cup A_i^{p_j}) \cup B = F$. We can easily show that

$$f = \begin{cases} 2 & \text{if } F = \{x_i\} \cup A_i^{p_0} \\ 1 & \text{otherwise} \end{cases}$$

and, for $0 \leqslant j \leqslant n$,

$$f_j = \begin{cases} 0 & \text{if } F \not\supseteq \{x_i\} \cup A_i^{p_j} \\ 2^{p_j} & \text{if } F \supsetneqq \{x_i\} \cup A_i^{p_j} \\ 2^{p_j} + 1 & \text{if } F = \{x_i\} \cup A_i^{p_j}. \end{cases}$$

From this it follows that

$$f_{\mathcal{M}}(\{x_i\} \cup A_i^{p_0}) = 2 + 2^{p_0} + 1 = 2^{p_0} + 3$$

and

$$f_{\mathcal{M}}(F) \leqslant 1 + 2^{p_0} + 2^{p_1} + \cdots + 2^{p_n} + 1 = s \qquad \text{for } F \neq \{x_i\} \cup A_i^{p_0}.$$

Since $n > 0$, we have $\max(3, 2^{p_0} + 3) \leqslant s$. Thus, in every case of (1)–(3) we have $f_{\mathcal{M}}(F) \leqslant s$ and, moreover, by (1), $f_{\mathcal{M}}(F) = 1$ for infinitely many sets $F \in \mathscr{F}(\mathbb{N})$. Furthermore, since $n > 0$, we have $\{x_i\} \cup A_i^{p_n} = \{x_i\} \cup A_i \neq \{x_i\} \cup A_i^{p_0}$ for every $i \in \mathbb{N}$ and thus $f_{\mathcal{M}}(\{x_i\} \cup A_i^{p_n}) = 1 + 2^{p_0} + 2^{p_1} + \cdots + 2^{p_{n-1}} + (2^{p_n} + 1) = s$. Hence $\mathcal{M} = (M_1, M_2)$ is a system of type $(1, s)$.

To complete the proof of Theorem 6 it is sufficient to construct asymptotic multiplicative systems of types $(1, \infty)$, $(2, \infty)$ and $(3, \infty)$. Indeed, systems of the other types are excluded according to Theorem 4 and because $f_{\mathcal{M}}(\{x\}) \leqslant 3$ for all $x \in \mathbb{N}$ and for every asymptotic multiplicative system $\mathcal{M} = (M_1, M_2)$ in the semigroup $(\mathscr{F}(\mathbb{N}), \cup)$.

The construction can be performed, for example, as follows:

Type $(1, \infty)$:
Define $M_1 = \mathscr{F}(\mathbb{N})$ and $M_2 = \mathscr{F}(\mathbb{N}) \setminus \{A \subseteq \mathbb{N};\, |A| = 1\}$.
Then $f_{\mathcal{M}}(A) \geqslant 1$ for every $A \in \mathscr{F}(\mathbb{N})$ and $f_{\mathcal{M}}(A) = 1$ iff $|A| \leqslant 1$.

Type $(2, \infty)$:
Define $M_1 = M_2 = \mathscr{F}(\mathbb{N}) \setminus \{A \subseteq \mathbb{N};\, |A| = 2\}$.
Then $f_{\mathcal{M}}(\varnothing) = 1$, $f_{\mathcal{M}}(A) = 2$ for every two-element set A and $f_{\mathcal{M}}(A) \geqslant 3$ for the other sets $A \in \mathscr{F}(\mathbb{N})$.

Type $(3, \infty)$:
Define $M_1 = M_2 = \mathscr{F}(\mathbb{N})$.
Then $f_{\mathcal{M}}(\varnothing) = 1$ and $f_{\mathcal{M}}(A) \geqslant 3$ for $A \neq \varnothing$.

In every case, $f_{\mathcal{M}}(A)$ is arbitrarily large provided that A is sufficiently large, and thus the required systems are constructed, and the proof of Theorem 6 is completed. □

REMARK. Multiplicative bases in the semigroup $(\mathcal{F}(\mathbb{N}), \cup)$ are usually called union bases. Union bases have also been studied by Deza and Erdös [1], Grekos [3] and Nathanson [5].

2. AN OPEN QUESTION

Note that the additive version of Theorem 1 is open even for $k = 2$. This is an old problem of P. Erdös, as follows.

PROBLEM. Let M be a set of positive integers with the property that for every positive integer n there are $x, y \in M$ such that $n = x + y$. Is it true that for every positive integer p there exists a positive integer n such that n can be expressed as the sum of two numbers of M in at least p different ways?

The previous problem concerns the semigroup $(\mathbb{N}, +)$, where $\mathbb{N}$ is the set of all *positive* integers. Let us notice that there is no similar problem for the semigroup $(\mathbb{Z}, +)$, where $\mathbb{Z}$ is the set of all integers, because the following proposition holds (see [8]):

Let $G = (X, \cdot)$ be a countable abelian group such that every equation $x^k = a$, where $a \in X$ and $k \in \{2, 3\}$, has only finitely many solutions. Then for every function $f: X \to \mathbb{N}$ there is a set $M \subseteq X$ such that $f_{M,2} \equiv f$.

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REFERENCES

1. M. Deza and P. Erdös, Extension de quelques théorèmes sur les densités de séries d éléments de $\mathbb{N}$ à des séries de sous-ensembles finis de $\mathbb{N}$, *Discr. Math.*, **12** (1975), 295–308.
2. P. Erdös, On the multiplicative representation of integers, *Israel J. Math.*, **2** (1964), 251–261.
3. G. P. Grekos, Nonexistence of maximal asymptotic union nonbases, *Discr. Math.*, **33** (1981), 267–270.
4. M. B. Nathanson, Multiplicative representation of integers, *Israel J. Math.*, **57** (1987), 129–136.
5. M. B. Nathanson, Oscillations of bases in number theory and combinatorics, in: *Number Theory Day*, M. B. Nathanson, (ed.), Lecture Notes in Math., vol. 626, Springer-Verlag, Berlin, 1977, pp. 217–231.
6. J. Nešetřil and V. Rödl, Two proofs in combinatorial number theory, *Proc. Am. Math. Soc.*, **93** (1985), 185–188.
7. V. Puš, Combinatorial properties of products of graphs, *Czech. Math. J.*, **41** (1991), 269–277.
8. V. Puš, On multiplicative bases in Abelian groups, *Czech. Math. J.*, **41** (1991), 282–287.
9. R. L. Graham, *Rudiments of Ramsey Theory*, Am. Math. Soc., Providence, Rhode Island, 1981.

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