On a generalization of a theorem of Erdős and Fuchs

Min Tang

Department of Mathematics, Anhui Normal University, Wuhu 241000, China

ARTICLE INFO

Article history:
Received 10 July 2008
Received in revised form 29 June 2009
Accepted 2 July 2009
Available online 17 July 2009

Keywords:
Erdős–Fuchstheorem
General sequences

ABSTRACT

Let $A = \{a_1, a_2, \ldots\}$ be an infinite sequence of nonnegative integers, let $k \geq 2$ be a fixed integer and denote by $r_k(A, n)$ the number of solutions of $a_1 + a_2 + \cdots + a_k \leq n$. Montgomery and Vaughan proved that $r_2(A, n) = cn + o(n^{1/4})$ cannot hold for any constant $c > 0$. In this paper, we extend this result to $k > 2$.

1. Introduction

Let $k \geq 2$ be a fixed integer and let $A = \{a_1, a_2, \ldots\}$ be an infinite sequence of nonnegative integers. We write $F(z) = \sum_{a \in A} z^a$, $A(n) = \sum_{a \in A, a \leq n}$.

For $n = 0, 1, 2 \ldots$ let $r_k(A, n)$ denote the number of solutions of

$$a_1 + a_2 + \cdots + a_k \leq n, \quad a_1 \in A, a_2 \in A, \ldots, a_k \in A.$$

In 1956, Erdős and Fuchs [1] proved the following result:

Theorem A. If $A \subset \mathbb{N}$, then

$$r_2(A, n) = cn + o(n^{1/4}(\log n)^{-1/2})$$

cannot hold for any constant $c > 0$.

Jurkat (unpublished), and later Montgomery and Vaughan [5] improved the Erdős–Fuchstheorem by eliminating the log power on the right-hand side:

Theorem B. If $A \subset \mathbb{N}$, then

$$r_2(A, n) = cn + o(n^{1/4})$$

cannot hold for any constant $c > 0$.

Already, the Erdős–Fuchstheorem has been extended in various directions. In [6], Sárközy generalized this theorem for two arbitrary sequences which are “near” in a certain sense; he proved the following theorem:

Theorem C. Let $A = \{a_1, a_2, \ldots\}$ and $B = \{b_1, b_2, \ldots\}$ be infinite sequences of integers such that $0 \leq a_1 < a_2 < \cdots$ and $0 \leq b_1 < b_2 < \cdots$. If

$$a_i - b_i = o\left(\frac{a_i^{1/2}}{\log a_i}\right),$$

E-mail address: tmzzz2000@163.com.

0012-365X/$ – see front matter © 2009 Elsevier B.V. All rights reserved.
Theorem. If \( A = \{a_1, a_2, \ldots\} \) and \( B = \{b_1, b_2, \ldots\} \) (where \( 0 \leq a_1 < a_2 < \cdots, 0 \leq b_1 < b_2 < \cdots \)) are infinite sequences of integers such that
\[
a_i - b_i = o(a_i^{1/2})
\]
and
\[
A(n) - B(n) = O(1),
\]
then
\[
|\{(i, j) : a_i + b_j \leq n\}| = cn + o(n^{1/4})
\]
cannot hold for any constant \( c > 0 \).

In 2002, Horváth \cite{Horvath2002} extended the Erdős–Fuchs theorem further by considering sums \( A^{(1)} + A^{(2)} + \cdots + A^{(k)} \). In particular, for the case \( A^{(1)} \equiv A^{(2)} \equiv \cdots \equiv A^{(k)} \), Horváth’s result implies that if \( A \subseteq \mathbb{N} \), then
\[
r_k(A, n) = cn + o\left(n^{1/4}(\log n)^{1-3k/4}\right)
\]
cannot hold for any constant \( c > 0 \).

In this paper, we obtain the following result:

Theorem. If \( A \subseteq \mathbb{N} \) and \( k > 2 \), then
\[
r_k(A, n) = cn + o(n^{1/4})
\]
cannot hold for any constant \( c > 0 \).

Throughout this paper, let \( z = re(\alpha) \), where \( e(\alpha) = e^{2\pi i\alpha} \) and \( r = 1 - \frac{1}{N} \). \( N \) is a large positive integer; \( \alpha \) is a real number.

2. Lemmas

Lemma 1 \cite{Jurkat1972}. Let \( 2 < m = m(N) < N \), where \( m \) is a positive integer, and \( m \to \infty \) as \( N \to \infty \). Then
\[
\int_0^1 |1 - z|^{-\beta} \left| \frac{1 - z^m}{1 - z} \right|^2 \, d\alpha \ll \begin{cases} m^{\beta+1} & \text{if } 0 \leq \beta < 1, \\ m^2 \log N & \text{if } \beta = 1, \\ m^2 N^{\beta-1} & \text{if } 1 < \beta. \end{cases}
\]

Lemma 2 \cite{Horvath2004}. Let \( r = 1 - \frac{1}{N} \), where \( N \) is a large positive integer. Then:

(a) \( \sum_{n=0}^{\infty} m^{2n} \ll N^2 \).
(b) \( \sum_{n=0}^{\infty} (n+1)^4 r^{2n} \ll N^5 \).

3. Proof of theorem

Suppose that (1) holds. Let \( \vartheta(n) = r_k(A, n) - cn \); then for \( |z| < 1 \), we have
\[
\frac{1}{1-z} F^k(z) = \sum_{n=0}^{\infty} r_k(A, n) z^n = \frac{cz}{(1-z)^2} + \sum_{n=0}^{\infty} \vartheta(n) z^n.
\]
\[
F^k(z) = \frac{cz}{1-z} + (1-z) \sum_{n=0}^{\infty} \vartheta(n) z^n.
\] (2)

Using the idea of Jurkat, by differentiation of (2),
\[
kF^{k-1}(z) F'(z) = \frac{c}{(1-z)^2} - \sum_{n=0}^{\infty} \vartheta(n) z^n + (1-z) \sum_{n=0}^{\infty} (n+1) \vartheta(n+1) z^n.
\] (3)
Letting $\varepsilon$ be a fixed small positive number, $m = \lfloor \varepsilon N^{1/2} \rfloor$, and letting

$$J = \int_0^1 \left| kF^{k-1}(z)F'(z) \right| \cdot \left| \frac{1 - z^m}{1 - z} \right|^2 \, d\alpha,$$

$$J_1 = c \int_0^1 \frac{1}{|1 - z|^2} \cdot \left| \frac{1 - z^m}{1 - z} \right|^2 \, d\alpha,$$

$$J_2 = \int_0^1 \sum_{n=0}^{\infty} \vartheta(n)z^n \cdot \left| \frac{1 - z^m}{1 - z} \right|^2 \, d\alpha,$$

$$J_3 = \int_0^1 \left| (1 - z) \sum_{n=0}^{\infty} (n + 1)\vartheta(n + 1)z^n \right| \cdot \left| \frac{1 - z^m}{1 - z} \right|^2 \, d\alpha,$$

by (3), we have

$$J \leq J_1 + J_2 + J_3. \quad (4)$$

Note that

$$\left| \frac{1 - z^m}{1 - z} \right|^2 = \sum_{i=0}^{m-1} z^i = \sum_{i=0}^{m-1} r_i e(t\alpha) \cdot \sum_{i=0}^{m-1} r_i e(-t\alpha),$$

and $k > 2$, so

$$J > \int_0^1 \left| \frac{1}{z} \sum_{a \in A} a \cdot \sum_{a \in A} a^2 \cdot |F^{k-2}(z)| \cdot \left| \frac{1 - z^m}{1 - z} \right|^2 \, d\alpha \right|$$

$$> \frac{1}{r} \left| \int_0^1 \left( \sum_{a \in A} ar^ae(a\alpha) \cdot \sum_{a \in A} r^ae(-a\alpha) \cdot \sum_{i=0}^{m-1} r_i e(-t\alpha) \right) \cdot \left( F^{k-2}(z) \cdot \sum_{i=0}^{m-1} r_i e(t\alpha) \right) \, d\alpha \right|.$$ 

Let

$$\sum_{\mu = -\infty}^{\infty} S_\mu e(\mu \alpha) = \sum_{a \in A} ar^ae(a\alpha) \cdot \sum_{a \in A} r^ae(-a\alpha) \cdot \sum_{i=0}^{m-1} r_i e(-t\alpha),$$

$$\sum_{\gamma = 0}^{\infty} T_\gamma e(\gamma \alpha) = F^{k-2}(z) \sum_{i=0}^{m-1} r_i e(t\alpha).$$

It is obvious that all the coefficients $S_\mu$, $T_\gamma$ are nonnegative, so

$$J > \left| \int_0^1 \sum_{\mu = -\infty}^{\infty} S_\mu e(\mu \alpha) \cdot \sum_{\gamma = 0}^{\infty} T_\gamma e(\gamma \alpha) \, d\alpha \right|$$

$$= \sum_{\mu + \gamma = 0} S_\mu T_\gamma$$

$$\geq \sum_{\frac{m}{4} \leq \gamma < m/2} S_\gamma T_\gamma. \quad (5)$$

If $\frac{m}{4} \leq \gamma < \frac{m}{2} < N$, noting that $r^N = (1 - \frac{1}{N})^N \to \frac{1}{e}$, we have

$$T_\gamma = \sum_{a_3 + \cdots + a_k = \gamma} r^{a_3 + \cdots + a_k + t\gamma} \geq r^N \sum_{0 \leq t \leq m/2} 1$$

$$\geq \sum_{a_3 + \cdots + a_k \leq m/4} 1 \geq \left( \sum_{a_3 \leq \frac{m}{4(k-2)}} 1 \right)^{k-2}.$$ 

and

$$\left( \sum_{a_3 \leq \frac{m}{4(k-2)}} 1 \right)^k = \left( A \left( \frac{m}{4(k-2)} \right) \right)^k \geq \sum_{a_3 + \cdots + a_k \leq \frac{m}{4(k-2)}} 1 = r_k A \left( \frac{m}{4(k-2)} \right).$$
By the indirect assumption, \( r_k \left( A, \frac{m}{4(k-2)} \right) \sim cm \); thus if \( \frac{m}{4} \leq \gamma \leq \frac{m}{2} \), then
\[
T_\gamma \gg \frac{m^{k-2}}{k^k}.
\] (6)

Note that
\[
r_k(A, N) = \sum_{a_1 + \ldots + a_k \leq N} \sum_{\gamma_j \leq N} 1 \leq A^k(N) \leq \sum_{\gamma_j \leq N} \sum_{a_1 + \ldots + a_k \leq N} 1 = r_k(A, kN),
\]
and thus, by the indirect assumption, there exist positive numbers \( c_1 \) and \( c_2 \) such that for all sufficiently large \( N \), \( c_1 N^{1/k} \leq A(N) \leq c_2 N^{1/k} \). And
\[
\sum_{\mu=-\infty}^{\infty} S_\mu e(\mu\alpha) = \sum_{a_j \in A} \sum_{a_j \in A} \sum_{t=0}^{m-1} a_i r^{a_i + a_j} e((a_i - a_j - t)\alpha) = \sum_{a_i, a_j \in A} a_i r^{a_i + a_j} e(\mu\alpha).
\]

Therefore, if \( \frac{m}{4} \leq \gamma \leq \frac{m}{2} \), we have
\[
S_{-\gamma} = \sum_{a_i, a_j \in A} a_i r^{a_i + a_j} \geq \sum_{a_i \in A} a_i r^{2a_i + \gamma}
\]
\[
> r^{3N} \sum_{a_i \in A} 1 \sum_{\gamma_j \leq N} 1 \left( \left[ \frac{c_1 N^{1/k}}{2^{k-2} c_2} \right]^2 \right)
\]
\[
= N \left( A(N) - A \left( \left[ \frac{c_1 N^{1/k}}{2^{k-2} c_2} \right] \right) \right)
\]
\[
\geq N \left( c_1 N^{1/k} - c_2 \left( \frac{c_1 N^{1/k}}{2^{k-2} c_2} \right)^{1/k} \right) = \frac{1}{2} c_1 N^{1+1/k} \gg N^{1+1/k}.
\] (7)

By (5)–(7), we have
\[
J \gg m \cdot m^{k-2} \cdot N^{1+1/k} = m^{2-\frac{k}{k}} \cdot N^{1+1/k}.
\] (8)

Now we estimate \( J_1, J_2, J_3 \).

By Lemma 1,
\[
J_1 \ll m^2 N.
\] (9)

By Cauchy’s inequality and Parseval’s formula,
\[
J_2 \ll m^2 \int_0^1 \left| \sum_{n=0}^{\infty} \vartheta(n) z^n \right| \, d\alpha
\]
\[
\leq m^2 \left( \int_0^1 \left| \sum_{n=0}^{\infty} \vartheta(n) z^n \right|^2 \, d\alpha \right)^{1/2}
\]
\[
= m^2 \left( \sum_{n=0}^{\infty} |\vartheta(n)|^2 r^{2n} \right)^{1/2}.
\]

By the indirect assumption, and Lemma 2(a), we have
\[
J_2 \ll m^2 \left( \sum_{n=0}^{\infty} m^{2n} \right)^{1/2} \leq m^2 (N^2)^{1/2} = m^2 N.
\] (10)
Similarly,
\[
J_3 \ll \int_0^1 \left| \sum_{n=0}^{\infty} (n+1) \vartheta(n+1)z^n \right| \frac{1 - z^m}{1-z} \, d\alpha \\
\leq \left( \int_0^1 \left| \sum_{n=0}^{\infty} (n+1) \vartheta(n+1)z^n \right|^2 \, d\alpha \right)^{1/2} \cdot \left( \int_0^1 \left| \frac{1 - z^m}{1-z} \right|^2 \, d\alpha \right)^{1/2} \\
= \left( \sum_{n=0}^{\infty} (n+1)^2 \vartheta^2(n+1)r^{2n} \right)^{1/2} \cdot \left( \sum_{j=0}^{m-1} r^{2j} \right)^{1/2} \\
\leq m^{1/2} \left( \sum_{n=0}^{\infty} (n+1)^2 \vartheta^2(n+1)r^{2n} \right)^{1/2}.
\]
Furthermore, by Cauchy’s inequality and Lemma 2(b),
\[
\sum_{n=0}^{\infty} (n+1)^2 \vartheta^2(n+1)r^{2n} \leq \left( \sum_{n=0}^{\infty} (n+1)^4r^{2n} \right)^{1/2} \left( \sum_{n=0}^{\infty} \vartheta^4(n+1)r^{2n} \right)^{1/2} \\
\ll N^{5/2} \left( \sum_{n=0}^{\infty} \vartheta^4(n+1)r^{2n} \right)^{1/2}.
\]
By the indirect assumption, for every $\varepsilon > 0$, there exists a natural number $k$ such that for all $n \geq k$, $|\vartheta(n)| \leq \varepsilon n^{1/4}$. Then for $N > N(\varepsilon)$, we have
\[
\sum_{n=0}^{\infty} \vartheta^4(n+1)r^{2n} \leq \sum_{n=0}^{k-1} \vartheta^4(n+1) + \sum_{n=k}^{\infty} \varepsilon^4(n+1)r^{2n} \\
\leq \sum_{n=0}^{k-1} \vartheta^4(n+1) + 2\varepsilon^4 \sum_{n=0}^{\infty} nr^{2n} \\
\ll \sum_{n=0}^{k-1} \vartheta^4(n+1) + 2\varepsilon^4 N^2 \\
\ll 3\varepsilon^4 N^2,
\]
and thus
\[
\sum_{n=0}^{\infty} \vartheta^4(n+1)r^{2n} = o(N^2).
\]
By (11)–(13), we have
\[
J_3 \ll m^{1/2} \left( N^{5/2} (o(N^2))^{1/2} \right) = o(m^{1/2}N^{7/2}).
\]
By (4), (8)–(10), (14),
\[
m^{2-\varepsilon^2} N^{1+\varepsilon} \ll m^2 N + o(m^{1/2}N^{7/2}).
\]
Since $m = [\varepsilon N^{1/2}]$, (15) yields
\[
\varepsilon N^2 \ll \varepsilon^2 N^2 + o(\varepsilon^{1/2} N^2)
\]
for all sufficiently large $N$; hence $\varepsilon \ll \varepsilon^2 + o(\varepsilon^{1/2})$. Thus $1 \ll \varepsilon$; but this cannot hold for sufficiently small $\varepsilon$.
This completes the proof of the theorem.

Acknowledgements

This work was supported by the Youth Foundation of Mathematical Tianyuan, Grant No. 10726074 and the National Natural Science Foundation of China, Grant No. 10901002.
References