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Mixed variational-like inclusions and J^{η}-proximal operator equations in Banach spaces ☆

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Abstract

In this paper, we study mixed variational-like inclusions and J^{η}-proximal operator equations in Banach spaces. It is established that mixed variational-like inclusions in real Banach spaces are equivalent to fixed point problems. We also establish a relationship between mixed variational-like inclusions and J^{η}-proximal operator equations. This equivalence is used to suggest an iterative algorithm for solving J^{η}-proximal operator equations.

Keywords: Mixed variational-like inclusions; J^{η}-Proximal operator; Algorithm; J^{η}-Proximal operator equations

1. Introduction

Variational inequalities and variational inclusions are among the most interesting and important mathematical problems and have been studied intensively in the past years since they have wide applications in mechanics, physics, optimization and control, nonlinear programming, economics and transportation equilibrium and engineering sciences, etc. (see, for example, [1–7]). The resolvent operator techniques for solving variational inequalities and variational inclusions

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are interesting and important. The resolvent operator technique is used to establish an equivalence between mixed variational inequalities and resolvent equations. The resolvent equation technique is used to develop powerful and efficient numerical techniques for solving mixed variational inequalities and related optimization problems.

In this paper, we generalize the resolvent equations by introducing J^η-proximal operator equations in Banach spaces. A relationship between mixed variational-like inclusions and J^η-proximal operator equations is established. We propose an iterative algorithm for computing the approximate solutions which converge to the exact solutions of J^η-proximal operator equations.

2. Formulation and preliminaries

Throughout the paper, we assume that E is a real Banach space with its norm $\|\cdot\|$, $E^\star$ is the topological dual of E, d is the metric induced by the norm $\|\cdot\|$, $CB(E)$ (respectively, 2^E) is the family of all nonempty closed and bounded subsets (respectively, all nonempty subsets) of E, $D(\cdot,\cdot)$ is the Hausdörff metric on $CB(E)$ defined by

$$D(A,B)=\max\Big\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(A,y)\Big\},$$

where $d(x,B)=\inf_{y\in B} d(x,y)$ and $d(A,y)=\inf_{x\in A} d(x,y)$. We also assume that $\langle\cdot,\cdot\rangle$ is the duality pairing between $E^\star$ and E and $\mathcal{F}:E\to 2^{E^\star}$ is the normalized duality mapping defined by

$$\mathcal{F}(x)=\big\{f\in E^\star\colon\ \langle x,f\rangle=\|x\|\|f\| \text{ and } \|f\|=\|x\|\big\},\quad \text{for all } x\in E.$$

Definition 2.1. Let $A:E\to CB(E^\star)$ be a set-valued mapping, $J:E\to E^\star$, $\eta:E\times E\to E$, and $g:E\to E$ be three single-valued mappings.

(1) A is said to be λ_A-Lipschitz continuous with Lipschitz constant $\lambda_A\geqslant 0$ if

$$H(Ax,Ay)\ \leqslant\ \lambda_A\|x-y\|,\quad \text{for all } x,y\in E;$$

(2) J is said to be η-strongly accretive with constant $\alpha>0$ if

$$\big\langle Jx-Jy,\eta(x,y)\big\rangle\geqslant\alpha\|x-y\|^2,\quad \text{for all } x,y\in E;$$

(3) g is said to be k-strongly accretive ($k\in(0,1)$) if for any $x,y\in E$, there exists $j(x-y)\in\mathcal{F}(x-y)$ such that

$$\big\langle j(x-y),gx-gy\big\rangle\geqslant k\|x-y\|^2;$$

(4) η is said to be Lipschitz continuous with constant $\tau>0$ if

$$\big\|\eta(x,y)\big\|\leqslant\tau\|x-y\|,\quad \text{for all } x,y\in E,$$

where $\mathcal{F}:E\to 2^{E^\star}$ is normalized duality mapping.

Definition 2.2. Let $\eta:E\times E\to E$ and $\varphi:E\to R\cup\{+\infty\}$. A vector $w^\star\in E^\star$ is called an η-subgradient of φ at $x\in\operatorname{dom}\varphi$ if

$$\big\langle w^\star,\eta(y,x)\big\rangle\leqslant\varphi(y)-\varphi(x),\quad \text{for all } y\in E.$$

Each φ can be associated with the following η-subdifferential mapping $\partial_n\varphi$ defined by

$$\partial_\eta\varphi(x) = \begin{cases} \{w^\star \in E^\star\colon\ \langle w^\star, \eta(y,x)\rangle \leqslant \varphi(y) - \varphi(x),\ \text{for all } y \in E\}, & x \in \operatorname{dom}\varphi, \\ \emptyset, & x \notin \operatorname{dom}\varphi. \end{cases}$$

Definition 2.3. Let E be a Banach space with the dual space $E^\star$, $\varphi : E \to R \cup \{+\infty\}$ be a proper, η-subdifferentiable (may not be convex) functional, $\eta : E \times E \to E$ and $J : E \to E^\star$ be the mappings. If for any given point $x^\star \in E^\star$ and $\rho > 0$, there is a unique point $x \in E$ satisfying

$$\big\langle Jx - x^\star, \eta(y,x)\big\rangle + \rho\varphi(y) - \rho\varphi(x) \geqslant 0, \quad \text{for all } y \in E,$$

then the mapping $x^\star \to x$, denoted by $J_\rho^{\partial_\eta\varphi}(x^\star)$ is said to be J^η-proximal mapping of φ. We have $x^\star - Jx \in \rho\partial_\eta\varphi(x)$, it follows that

$$J_\rho^{\partial_\eta\varphi}(x^\star) = (J + \rho\partial_\eta\varphi)^{-1}(x^\star).$$

Definition 2.4. A functional $f : E \times E \to R \cup \{+\infty\}$ is said to be 0-diagonally quasi-concave (in short 0-DQCV) in y, if for any finite subset $\{x_1, \ldots, x_n\} \subset E$ and for any $y = \sum_{i=1}^n \lambda_i x_i$ with $\lambda_i \geqslant 0$ and $\sum_{i=1}^n \lambda_i = 1$,

$$\min_{1 \leqslant i \leqslant n} f(x_i, y) \leqslant 0.$$

Given single-valued mappings $P, f, h : E \to E^\star$, $g : E \to E$, $\eta : E \times E \to E$ and multivalued mappings $M, S, T : E \to CB(E^\star)$. Let $\varphi : E \to R \cup \{+\infty\}$ is a lower-semicontinuous functional on E (may not be convex) satisfying $g(x) \cap \operatorname{dom}(\partial_\eta\varphi) \neq \varphi$, where $\partial_\eta\varphi$ is η-subdifferential of φ. We consider the following mixed variational-like inclusion (MVLIP): Find $x \in E$, $u \in M(x)$, $v \in S(x)$ and $w \in T(x)$ such that $g(x) \in \operatorname{dom}(\partial_\eta\varphi)$ and

$$\text{(MVLIP)} \quad \big\langle P(u) - \big(f(v) - h(w)\big), \eta\big(y, g(x)\big)\big\rangle \geqslant \varphi\big(g(x)\big) - \phi(y), \quad \text{for all } y \in E. \tag{2.1}$$

We present some special cases of (MVLIP) to show that (MVLIP) is more general and unifying one.

Special cases.

(i) If $E = H$ is a Hilbert space and $P \equiv 0$, f, g, h, M are identity mappings and S and T are single-valued mappings, then (MVLIP) reduces to the problem of finding $x \in H$ such that $x \in \operatorname{dom}\varphi$ and

$$\big\langle T(x) - S(x), \eta(y,x)\big\rangle \geqslant \varphi(x) - \varphi(y), \quad \text{for all } y \in H. \tag{2.2}$$

It is considered and studied by Lee, Ansari and Yao [8].

(ii) If $E = H$ is a Hilbert space and $\varphi \equiv \delta_k$, the indicator function of the closed convex set in H defined by

$$\delta_k(x) = \begin{cases} 0, & x \in K, \\ +\infty, & \text{otherwise}, \end{cases}$$

f, h, g and M are identity mappings and $\eta(y, x) = y - P(x)$, then (MVLIP) becomes the problem of finding $x \in K$, $v \in S(x)$, $w \in T(x)$ such that

$$\big\langle P(x) - (v - w), y - P(x)\big\rangle \geqslant 0, \quad \text{for all } y \in K. \tag{2.3}$$

Such a problem is considered and studied by Verma [11].

It is now clear that for a suitable choices of the maps involved in the formulation of (MVLIP), we can derive many known variational inclusion considered and studied in the literature.

Theorem 2.1. [1] *Let E be a reflexive Banach space with the dual space $E^\star$ and $\varphi : E \to R \cup \{+\infty\}$ be a lower-semicontinuous, η-subdifferentiable, proper functional which may not be convex. Let $J : E \to E^\star$ be η-strongly accretive with constant $\alpha > 0$. Let $\eta : E \times E \to E$ be Lipschitz continuous with constant $\tau > 0$ such that $\eta(x, y) = -\eta(y, x)$, for all $x, y \in E$ and for any $x \in E$, the function $h(y, x) = \langle x^\star - Jx, \eta(y, x)\rangle$ is 0-DQCV in y. Then for any $\rho > 0$, and for any $x^\star \in E^\star$, there exists a unique $x \in E$ such that*

$$\big\langle Jx - x^\star, \eta(y, x)\big\rangle + \rho\varphi(y) - \rho\varphi(x) \geqslant 0, \quad \textit{for all } y \in E.$$

That is, $x = J_\rho^{\partial_\eta \varphi}(x^\star)$ and so the J^η-proximal mapping of φ is well defined and τ/α-Lipschitz continuous.

The following example shows the existence of the mapping $\eta : E \times E \to E$ satisfying all conditions in Theorem 2.1.

Example 2.1. Let $E = R$ be real line and $\eta : R \times R \to R$ be defined by

$$\eta(x, y) = \begin{cases} 2x - 2y & \text{if } |xy| < 1/4, \\ 8|xy|(x - y) & \text{if } 1/4 \leqslant |xy| < 1/2, \\ 4(x - y) & \text{if } 1/2 \leqslant |xy|. \end{cases}$$

Then it is easy to see that:

(1) $\langle \eta(x, y), x - y\rangle \geqslant 2|x - y|^2$ for all $x, y \in R$, i.e. η is 2-strongly monotone;
(2) $\eta(x, y) = -\eta(y, x)$ for all $x, y \in R$;
(3) $|\eta(x, y)| \leqslant 4|x - y|$ for all $x, y \in R$, i.e. η is 4-Lipschitz continuous;
(4) for any $x \in R$, the function $h(y, u) = \langle x - u, \eta(y, u)\rangle = (x - u)\eta(y, u)$ is 0-DQCV in y.

If it is false, then there exists a finite set $\{y_1, \ldots, y_n\}$ and $u_0 = \sum_{i=1}^n \lambda_i y_i$ with $\lambda_i \geqslant 0$ and $\sum_{i=1}^n \lambda_i = 1$ such that for each $i = 1, \ldots, n$,

$$0 < h(y_i, u_0) = \begin{cases} (x - u_0)(2y_i - 2u_0) & \text{if } |y_i u_0| < 1/4, \\ (x - u_0)8|y_i u_0|(y_i - u_0) & \text{if } 1/4 \leqslant |y_i u_0| < 1/2, \\ 4(x - u_0)(y_i - u_0) & \text{if } 1/2 \leqslant |y_i u_0|. \end{cases}$$

It follows that $(x - u_0)(2y_i - 2u_0) > 0$ for each $i = 1, 2, \ldots, n$, and hence we have

$$0 < \sum_{i=1}^n \lambda_i (x - u_0)(2y_i - 2u_0) = (x - u_0)(2u_0 - 2u_0) = 0,$$

which is not possible. Hence $h(y, u)$ is 0-DQCV in y. Therefore, η satisfies all assumptions in Theorem 2.1.

Proposition 2.1. [10] *Let E be a real Banach space and $\mathcal{F}: E \to 2^{E^\star}$ be the normalized duality mapping. Then, for any $x, y \in E$,*

$$\|x+y\|^2 \leqslant \|x\|^2 + 2\langle y, j(x+y)\rangle, \quad \textit{for all } j(x+y) \in \mathcal{F}(x+y).$$

In connection with (MVLIP), we consider the following J^η-proximal operator equation problem (J^η-POEP):

$$(J^\eta\text{-POEP}) \quad \begin{cases} \text{Find } z \in E^\star,\ x \in E,\ u \in M(x),\ v \in S(x) \text{ and } w \in T(x) \text{ such that} \\ \quad [P(u) - (f(v) - h(w))] + \rho^{-1} R_\rho^{\partial_\eta \varphi}(z) = 0, \end{cases} \tag{2.4}$$

where $\rho > 0$ is a constant, $R_\rho^{\partial_\eta\varphi} = I - J[J_\rho^{\partial_\eta\varphi}(z)]$, where $J[J_\rho^{\partial_\eta\varphi}(z)] = [J(J_\rho^{\partial_\eta\varphi})](z)$ and I is the identity mapping. Equation (2.4) is called J^η-proximal operator equation.

3. An iterative algorithm and convergence result

We mention the following equivalence between (MVLIP) and a fixed point problem which can be easily proved by using Definition 2.3.

Lemma 3.1. *Let (x, u, v, w), where $x \in E$, $u \in M(x)$, $v \in S(x)$ and $w \in T(x)$, is a solution of* (MVLIP) *if and only if it is a solution of the following equation*:

$$g(x) = J_\rho^{\partial_\eta\varphi}\big\{J\big(g(x)\big) - \rho\big[P(u) - \big(f(v) - h(w)\big)\big]\big\}. \tag{3.1}$$

Now, we show that the (MVLIP) is equivalent to the (J^η-POEP).

Lemma 3.2. *The* (MVLIP) *has a solution (x, u, v, w) with $x \in E$, $u \in M(x)$, $v \in S(x)$ and $w \in T(x)$, if and only if* (J^η-POEP) *has a solution (z, x, u, v, w) with $z \in E^\star$, $x \in E$, $u \in M(x)$, $v \in S(x)$ and $w \in T(x)$, where*

$$g(x) = J_\rho^{\partial_\eta\varphi}(z) \tag{3.2}$$

and

$$z = J\big(g(x)\big) - \rho\big[P(u) - \big(f(v) - h(w)\big)\big].$$

Proof. Let (x, u, v, w) be a solution of (MVLIP). Then by Lemma 3.1, it is a solution of the following equation:

$$g(x) = J_\rho^{\partial_\eta\varphi}\big\{J\big(g(x)\big) - \rho\big[P(u) - \big(f(v) - h(w)\big)\big]\big\}$$

using the fact $R_\rho^{\partial_\eta\varphi} = (I - J[J_\rho^{\partial_\eta\varphi}])$, and Eq. (3.1), we have

$$\begin{aligned} &R_\rho^{\partial_\eta\varphi}\big[J\big(g(x)\big) - \rho\big[P(u) - \big(f(v) - h(w)\big)\big]\big] \\ &\quad = J\big(g(x)\big) - \rho\big[P(u) - \big(f(v) - h(w)\big)\big] \\ &\qquad - J\big[J_\rho^{\partial_\eta\varphi}\big\{J\big(g(x)\big) - \rho\big[P(u) - \big(f(v) - h(w)\big)\big]\big\}\big] \\ &\quad = J\big(g(x)\big) - \rho\big[P(u) - \big(f(v) - h(w)\big)\big] - J\big(g(x)\big) \\ &\quad = -\rho\big[P(u) - \big(f(v) - h(w)\big)\big] \end{aligned}$$

which implies that

$$\big[P(u) - \big(f(v) - h(w)\big)\big] + \rho^{-1} R_\rho^{\partial_\eta \varphi}(z) = 0$$

with $z = J(g(x)) - \rho[P(u) - (f(v) - h(w))]$, i.e. (z, u, v, w) is a solution of (J^η-POEP).

Conversely, let (z, x, u, v, w) be a solution of (J^η-POEP), then

$$\rho\big[P(u) - \big(f(v) - h(w)\big)\big] = -R_\rho^{\partial_\eta \varphi}(z) = J\big[J_\rho^{\partial_\eta \varphi}(z)\big] - z. \tag{3.3}$$

From (3.2) and (3.3), we have

$$\begin{aligned}\rho\big[P(u) - \big(f(v) - h(w)\big)\big] = {} & J\big[J_\rho^{\partial_\eta \varphi}\big\{J\big(g(x)\big) - \rho\big[P(u) - \big(f(v) - h(w)\big)\big]\big\}\big] \\ & - J\big(g(x)\big) + \rho\big[P(u) - \big(f(v) - h(w)\big)\big]\end{aligned}$$

which implies that

$$J\big(g(x)\big) = J\big[J_\rho^{\partial_\eta \varphi}\big\{J\big(g(x)\big) - \rho\big[P(u) - \big(f(v) - h(w)\big)\big]\big\}\big]$$

and thus

$$g(x) = J_\rho^{\partial_\eta \varphi}\big\{J\big(g(x)\big) - \rho\big[P(u) - \big(f(v) - h(w)\big)\big]\big\},$$

i.e. (x, u, v, w) is a solution of (MVLIP). □

Alternative proof. Let

$$z = J\big(g(x)\big) - \rho\big[P(u) - \big(f(v) - h(w)\big)\big],$$

then from (3.2), we have

$$g(x) = J_\rho^{\partial_\eta \varphi}(z)$$

and

$$z = J\big[J_\rho^{\partial_\eta \varphi}(z)\big] - \rho\big[P(u) - \big(f(v) - h(w)\big)\big].$$

By using the fact that $J[J_\rho^{\partial_\eta \varphi}(z)] = [J[J_\rho^{\partial_\eta \varphi}]](z)$, it follows that

$$\big[P(u) - \big(f(v) - h(w)\big)\big] + \rho^{-1} R_\rho^{\partial_\eta \varphi}(z) = 0,$$

the required (J^η-POEP). □

We now invoke Lemmas 3.1 and 3.2 to suggest the following iterative algorithm for solving (J^η-POEP).

Algorithm 3.1. For any $z_0 \in E^\star$, $x_0 \in E$, $u_0 \in M(x_0)$, $v_0 \in S(x_0)$ and $w_0 \in T(x_0)$, from (3.2), let

$$z_1 = J\big(g(x_0)\big) - \rho\big[P(u_0) - \big(f(v_0) - h(w_0)\big)\big].$$

Take $z_1 \in E^\star$, $x_1 \in E$ such that

$$g(x_1) = J_\rho^{\partial_\eta \varphi}(z_1).$$

Since $u_0 \in M(x_0)$, $v_0 \in S(x_0)$ and $w_0 \in T(x_0)$, by Nadler's theorem [9], there exists $u_1 \in M(x_1)$, $v_1 \in S(x_1)$ and $w_1 \in T(x_1)$ such that

$$\|u_0 - u_1\| \leqslant (1+1)D\big(M(x_0), M(x_1)\big),$$
$$\|v_0 - v_1\| \leqslant (1+1)D\big(S(x_0), S(x_1)\big),$$
$$\|w_0 - w_1\| \leqslant (1+1)D\big(T(x_0), T(x_1)\big),$$

where D is Hausdörff metric on $CB(E)$. Let

$$z_2 = J\big(g(x_1)\big) - \rho\big[P(u_1) - \big((v_1) - h(w_1)\big)\big]$$

and take any $x_2 \in E$ such that

$$g(x_2) = J_\rho^{\partial_\eta \varphi}(z_2).$$

Continuing the above process inductively, we can obtain the following:

For any $z_0 \in E^\star$, $u_0 \in M(x_0)$, $v_0 \in S(x_0)$ and $w_0 \in T(x_0)$, compute the sequences $\{z_n\}$, $\{x_n\}$, $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ by iterative schemes such that

$$\text{(i)} \qquad g(x_n) = J_\rho^{\partial_\eta \varphi}(z_n); \tag{3.4}$$

$$\text{(ii)} \qquad u_n \in M(x_n), \quad \|u_n - u_{n+1}\| \leqslant \left(1 + \frac{1}{n+1}\right) D\big(M(x_n), M(x_{n+1})\big); \tag{3.5}$$

$$\text{(iii)} \qquad v_n \in S(x_n), \quad \|v_n - v_{n+1}\| \leqslant \left(1 + \frac{1}{n+1}\right) D\big(S(x_n), S(x_{n+1})\big); \tag{3.6}$$

$$\text{(iv)} \qquad w_n \in T(x_n), \quad \|w_n - w_{n+1}\| \leqslant \left(1 + \frac{1}{n+1}\right) D\big(T(x_n), T(x_{n+1})\big); \tag{3.7}$$

$$\text{(v)} \qquad z_{n+1} = J\big(g(x_n)\big) - \rho\big[P(u_n) - \big(f(v_n) - h(w_n)\big)\big], \quad n = 0, 1, 2, \ldots, \tag{3.8}$$

and $\rho > 0$ is a constant.

Theorem 3.1. *Let E be a reflexive Banach space. Let $M, S, T : E \to CB(E^\star)$ be D-Lipschitz continuous mappings with Lipschitz constants λ_M, λ_S and λ_T, respectively. Let $P, f, h : E \to E^\star$ be Lipschitz continuous mappings with Lipschitz constants λ_p, λ_f and λ_h, respectively. Let $g : E \to E^\star$ be Lipschitz continuous with Lipschitz constant λ_g and k-strongly accretive $(k \in (0,1))$ and $J : E \to E^\star$ is Lipschitz continuous with Lipschitz constant λ_j and η-strongly monotone with constant $\alpha > 0$. Assume that $\eta : E \times E \to E$ be Lipschitz continuous with Lipschitz constant $\tau > 0$ such that $\eta(x, y) = -\eta(y, x)$ for all $x, y \in E$ and for each given $x \in E$, the function $h(y, x) = \langle x^\star - Jx, \eta(y, x)\rangle$ is 0-DQCV in y. Let $\varphi : E \to R \cup \{+\infty\}$ is lower-semicontinuous, η-subdifferentiable, proper functional satisfying $g(x) \in \operatorname{dom}(\partial_\eta \varphi)$. Suppose there exists a constant $\rho > 0$ such that the following condition is satisfied*:

$$0 < \big[\lambda_j \lambda_g + \rho(\lambda_P \lambda_M + \lambda_f \lambda_S + \lambda_h \lambda_T)\big] < \frac{\alpha\sqrt{1+2k}}{\tau} \tag{3.9}$$

then there exist $z \in E^\star$, $x \in E$, $u \in M(x)$, $v \in S(x)$ and $w \in T(x)$ satisfying (J^η-POEP) and the iterative sequences $\{z_n\}$, $\{x_n\}$, $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ generated by Algorithm 3.1 converge strongly to z, x, u, v and w, respectively.

Proof. From Algorithm 3.1, we have

$$\begin{aligned}\|z_{n+1}-z_n\| &= \big\| J\big(g(x_n)\big) - \rho\big[P(u_n) - \big(f(v_n) - h(w_n)\big)\big] - \big[J\big(g(x_{n-1})\big) \\ &\quad - \rho\big[P(u_{n-1}) - \big(f(v_{n-1}) - h(w_{n-1})\big)\big]\big]\big\| \\ &\leqslant \big\| J\big(g(x_n)\big) - J\big(g(x_{n-1})\big)\big\| + \rho \big\| \big(P(u_n) - \big(f(v_n) - h(w_n)\big)\big) \\ &\quad - \big(P(u_{n-1}) - \big(f(v_{n-1}) - h(w_{n-1})\big)\big)\big\|. \end{aligned} \tag{3.10}$$

By the Lipschitz continuity of J and g, we have

$$\big\| J\big(g(x_n)\big) - J\big(g(x_{n-1})\big)\big\| \leqslant \lambda_j \big(\big\|g(x_n) - g(x_{n-1})\big\|\big) \leqslant \lambda_j\lambda_g \|x_n - x_{n-1}\|. \tag{3.11}$$

By the Lipschitz continuity of P, f, h and D-Lipschitz continuity of M, S and T, we have

$$\begin{aligned} &\big\|\big(P(u_n) - \big(f(v_n) - h(w_n)\big)\big) - \big(P(u_{n-1}) - f\big(v_{n-1} - h(w_{n-1})\big)\big)\big\| \\ &\quad \leqslant \big\|P(u_n) - P(u_{n-1})\big\| + \big\|f(v_n) - f(v_{n-1})\big\| + \big\|h(w_n) - h(w_{n-1})\big\| \\ &\quad \leqslant \lambda_P \|u_n - u_{n-1}\| + \lambda_f \|v_n - v_{n-1}\| + \lambda_h \|w_n - w_{n-1}\| \\ &\quad \leqslant \lambda_P \left(1 + \frac{1}{n}\right) D\big(M(x_n), M(x_{n-1})\big) + \lambda_f \left(1 + \frac{1}{n}\right) D\big(S(x_n), S(x_{n-1})\big) \\ &\qquad + \lambda_f \left(1 + \frac{1}{n}\right) D\big(T(x_n), T(x_{n-1})\big) \\ &\quad \leqslant \left[\lambda_P\lambda_M\left(1 + \frac{1}{n}\right) + \lambda_f\lambda_S\left(1 + \frac{1}{n}\right) + \lambda_f\lambda_T\left(1 + \frac{1}{n}\right)\right]\|x_n - x_{n-1}\|. \end{aligned} \tag{3.12}$$

Combining (3.11)–(3.12) with (3.10), we obtain

$$\begin{aligned}\|z_{n+1} - z_n\| &\leqslant \left[\lambda_j\lambda_g + \rho\left(\lambda_P\lambda_M\left(1 + \frac{1}{n}\right) + \lambda_f\lambda_S\left(1 + \frac{1}{n}\right) + \lambda_h\lambda_T\left(1 + \frac{1}{n}\right)\right)\right] \\ &\quad \times \|x_n - x_{n-1}\|. \end{aligned} \tag{3.13}$$

By using Theorem 2.1 and k-strong accretiveness of g, we have

$$\begin{aligned}\|x_n - x_{n-1}\|^2 &= \big\| J_\rho^{\partial_\eta\varphi}(z_n) - J_\rho^{\partial_\eta\varphi}(z_{n-1}) - \big[g(x_n) - x_n - \big(g(x_{n-1}) - x_{n-1}\big)\big]\big\|^2 \\ &\leqslant \big\| J_\rho^{\partial_\eta\varphi}(z_n) - J_\rho^{\partial_\eta\varphi}(z_{n-1})\big\|^2 \\ &\quad - 2\big\langle g(x_n) - x_n - \big(g(x_{n-1}) - x_{n-1}\big), j(x_n - x_{n-1})\big\rangle \\ &\leqslant \frac{\tau^2}{\alpha^2}\|z_n - z_{n-1}\|^2 - 2k\|x_n - x_{n-1}\|^2 \end{aligned}$$

which implies that

$$\|x_n - x_{n-1}\|^2 \leqslant \frac{(\tau/\alpha)^2}{1+2k}\|z_n - z_{n-1}\|^2 \tag{3.14}$$

using (3.14), (3.13) becomes

$$\|z_{n+1} - z_n\| \leqslant \frac{[\lambda_j\lambda_g + \rho(\lambda_P\lambda_M(1+\frac{1}{n}) + \lambda_f\lambda_S(1+\frac{1}{n}) + \lambda_h\lambda_T(1+\frac{1}{n}))]\tau}{\alpha\sqrt{(1+2k)}}\|z_n - z_{n-1}\|,$$

i.e.

$$\|z_{n+1} - z_n\| \leqslant \theta_n \|z_n - z_{n-1}\|, \tag{3.15}$$

where

$$\theta_n = \frac{[\lambda_j \lambda_g + \rho(\lambda_P \lambda_M (1 + \frac{1}{n}) + \lambda_f \lambda_S (1 + \frac{1}{n}) + \lambda_h \lambda_T (1 + \frac{1}{n}))]\tau}{\alpha\sqrt{(1+2k)}}.$$

Letting $\theta = \frac{[\lambda_j \lambda_g + \rho(\lambda_P \lambda_M + \lambda_f \lambda_S + \lambda_h \lambda_T)]\tau}{\alpha\sqrt{(1+2k)}}$, it follows that $\theta_n \to \theta$ as $n \to \infty$. From (3.9), we have $\theta < 1$, and consequently $\{z_n\}$ is a Cauchy sequence in $E^\star$. Since $E^\star$ is a Banach space, there exists $z \in E^\star$ such that $z_n \to z$ as $n \to \infty$. From (3.14), we know that the sequence $\{x_n\}$ is also a Cauchy sequence in E. Therefore, there exists $x \in E$ such that $x_n \to x$ as $n \to \infty$. Since the mappings M, S and T are D-Lipschitz continuous, it follows from (3.5)–(3.7) that $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are also Cauchy sequences, we can assume that $u_n \to u$, $v_n \to v$ and $w_n \to w$. Since J, g, P, f and h are continuous and by (v) of Algorithm 3.1, it follows that

$$\begin{aligned} z_{n+1} &= J\big(g(x_n)\big) - \rho\big[P(u_n) - \big(f(v_n) - h(w_n)\big)\big] \\ &\to z = J\big(g(x)\big) - \rho\big[P(u) - \big(f(v) - h(w)\big)\big] \quad (n \to \infty), \end{aligned} \tag{3.16}$$

$$J_\rho^{\partial_\eta \varphi}(z_n) = g(x_n) \to g(x) = J_\rho^{\partial_\eta \varphi}(z) \quad (n \to \infty). \tag{3.17}$$

By (3.16), (3.17), and Lemma 3.2, we have

$$\big[P(u) - \big(f(v) - h(w)\big)\big] + \rho^{-1}\big[I - J\big(J_\rho^{\partial_\eta \varphi}(z)\big)\big] = 0.$$

Finally, we prove that $u \in M(x)$, $v \in S(x)$ and $w \in T(x)$. In fact, since $u_n \in M(x_n)$ and

$$\begin{aligned} d\big(u_n, M(x)\big) &\leqslant \max\Big\{d\big(u_n, M(x)\big), \sup_{q_1 \in M(x)} d\big(M(x_n), q_1\big)\Big\} \\ &\leqslant \max\Big\{\sup_{q_2 \in M(x_n)} d\big(q_2, M(x)\big), \sup_{q_1 \in M(x)} d\big(M(x_n), q_1\big)\Big\} \\ &= D\big(M(x_n), M(x)\big). \end{aligned}$$

We have

$$\begin{aligned} d\big(u, M(x)\big) &\leqslant \|u - u_n\| + d\big(u_n, M(x)\big) \\ &\leqslant \|u - u_n\| + D\big(M(x_n), M(x)\big) \\ &\leqslant \|u - u_n\| + \lambda_M \|x_n - x\| \to 0 \quad \text{as } n \to \infty, \end{aligned}$$

which implies that $d(u, M(x)) = 0$. Since $M(x) \in CB(E)$, it follows that $u \in M(x)$. Similarly, we can prove that $v \in S(x)$ and $w \in T(x)$. By Lemma 3.2, the required result follows. □

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