Journal of Pure and Applied Algebra 20 (1981) 7–12

PROJECTIVE DIAGRAMS OVER PARTIALLY ORDERED SETS ARE FREE

Michael HÖPPNER and Helmut LENZING
Fachbereich Mathematik, Universität-Gesamthochschule Paderborn, D-4790 Paderborn, Germany

A generalization of Kaplansky's theorem on projective modules over local rings is used to show that every projective diagram over a poset is free.

1. Introduction

In this note we prove that every projective diagram over a poset I with values in a module category $\mathcal{M}$ (i.e. an abelian category with coproducts and a generating set of small projectives) is free (Proposition 5). This answers a question of B. Mitchell [10, p. 882]. The two extreme cases of the problem are already known: if I has dcc and $\mathcal{M}$ is arbitrary, then the result is due to Brune [3] and Cheng/Mitchell [4, Corollary 3], for I finite the result is already in [8, Corollary IX. 7.2.]. On the other hand if I is arbitrary and $\mathcal{M}$ is the category of vector spaces over a field, it has been observed by Gabriel [5, p. 157] and Mitchell [10, Corollary 9.2.], that the result is a consequence of the well-known theorem of Kaplansky, that projective modules over a local ring are free.

As may be expected our proof of the general result will also depend vitally on Kaplansky's technique. In fact it follows by the possibility of lifting nice decompositions of a projective module modulo the (Jacobson-) radical (Theorem 1). Quite obvious, Theorem 1 generalizes a theorem of Gruson [6, Appendix] and Beck [2], asserting that for a projective R-module P every R/N-basis of P/NP may be lifted to a R-basis of P, N denoting a two-sided ideal contained in the Jacobson-radical of R.

Finally, under the additional assumption of pure projectivity, we give a partial answer to a related question of Cheng and Mitchell [4, p. 128] concerning the structure of direct summands of diagrams having a canonical form (Proposition 7).

2. Lifting direct decompositions

Let $\mathcal{M}$ be a *module category*, i.e. an abelian category with coproducts and a generating set of finitely generated projectives [9]. Assume that for each module M

of $\mathcal{M}$ we have a submodule rM contained in the Jacobson-radical rad M of M and depending functorially on M. We will refer to the functor $\hat{}:\mathcal{M}\to\mathcal{M}$, $M\mapsto\hat{M}=M/rM$, as a *reduction modulo radical*. Observe that by Nakayama's lemma, the natural map $\nu_P:P\to\hat{P}$ is a projective cover, if P is finitely generated projective.

Theorem 1. *Let* $\hat{}:\mathcal{M}\to\mathcal{M}$ *be a reduction modulo radical. Suppose* P *is a projective module,* $\hat{P}=\bigoplus_{\alpha\in I}Q_\alpha$, *and assume that each* Q_α *has a projective cover* $\varepsilon_\alpha:P_\alpha\to Q_\alpha$. *Then there is an isomorphism* $f:\bigoplus_{\alpha\in I}P_\alpha\to P$ *with* $\nu_P\circ f=\varepsilon$, $\varepsilon:\bigoplus_{\alpha\in I}P_\alpha\to\bigoplus_{\alpha\in I}Q_\alpha$ *being the obvious map with components* ε_α.

By the uniqueness of projective covers any f with $\nu_P\circ f=\varepsilon$ will be an isomorphism, if I is finite. Generally, I being infinite, such an f will only be a pure monomorphism.

The heart of the proof of Theorem 1 is the following lemma (Kaplansky [7, Lemma 1 and 2] and Gruson [6, Lemma 2 and 3]). Note that if $(M_\alpha)_{\alpha\in I}$ is any family of modules, we will use the notation $M_J=\bigoplus_{\alpha\in J}M_\alpha$ for any subset J of I. Note also, that in order to keep the exposition clear, we will take no efforts to avoid the use of elements in our proofs.

Lemma 2. *In the situation of Theorem* 1 *suppose that* U *is a finitely generated submodule of* P. *Then there is a direct decomposition* $P=F\oplus H$ *with* U *contained in* F *and* $F\cong P_J$ *for some finite subset* J *of* I. *Moreover* $\nu_P(F)=Q_J$ *and* $\nu_P(H)=Q_{I\setminus J}$.

Proof. By the projectivity of P there is an endomorphism $\pi:P\to P$ such that $\pi(P)$ is contained in a finitely generated submodule of P and $\pi(x)=x$ for all $x\in U$.

(Note, that the introduction of π allows to handle P to some extent like a finitely generated module: π behaves like the identity on P with finitely generated image. This will repair the defects, the reduction modulo radical usually offers for nonfinitely generated modules.)

Therefore $\hat{\pi}(\hat{P})\subset Q_J$ for some finite subset J of I. P and P_J being projective, we may lift the injection $i:Q_J\to\hat{P}$ and the projection $p:\hat{P}\to Q_J$ associated with the decomposition $\hat{P}=Q_J\oplus Q_{I\setminus J}$, thus obtaining morphisms $f:P_J\to P$ and $g:P\to P_J$ with $\varepsilon_J\circ(g\circ f)=\varepsilon_J$. By the uniqueness of the projective cover $\varepsilon_J:P_J\to Q_J$, the morphism $g\circ f$ is an isomorphism, thus $P=F'\oplus H'$ with $F'=\operatorname{Im}f\cong P_J$ and $H'=\operatorname{Ker}g$. This implies $\nu_P(F')=Q_J$ and $\nu_P(H')=Q_{I\setminus J}$.

Assume $\pi=\varphi+\psi$, with φ and ψ the components of $\pi:P\to F'\oplus H'$ considered as endomorphisms of P. From $\hat{\pi}(\hat{P})\subset Q_J$ we get $\hat{\psi}=0$ and therefore $\psi(P)\subset rP\subset\operatorname{rad}P$. Because $\psi(P)$ is further contained in a finitely generated submodule of P, $\psi(P)$ is small in P by Nakayama's lemma. It follows that $1-\psi:P\to P$ is an isomorphism (see [1, Proposition 2.7] or [11, Lemma 1]). From $\pi(x)=x$ we get $(1-\psi)(x)=\varphi(x)\in F'$ for each $x\in U$, thus $U\subset(1-\psi)^{-1}(F')=F$. Therefore $P=F\oplus H$ with $H=(1-\psi)^{-1}(H')$, and of course we have $\nu_P(F)=Q_J$ and $\nu_P(H)=Q_{I\setminus J}$, thus completing the proof.

We now return to the proof of Theorem 1. If P is countably generated, Theorem 1 follows from Lemma 2 by an obvious induction. So we have to reduce to the countably generated case. Decomposing each P_α into a direct sum of countably generated modules [7], we may assume each P_α (and Q_α) to be countably generated. Assume further, $P = \bigoplus_{\beta \in K} S_\beta$ with countably generated modules S_β, then for any $x \in P$ we can find countable subsets $L \subset K$ and $J \subset I$ with $x \in S_L$ and $\nu_P(S_L) = Q_J$ by an easy variant of Kaplansky's snaking argument. Now P is a well ordered union of direct summands F_γ where $\nu_P(F_\gamma) = Q_{K(\gamma)}$ for some $K(\gamma) \subset I$ and the quotients $F_{\gamma+1}/F_\gamma$ are countably generated. If we are only interested to know, that P and $\bigoplus_{\alpha \in I} P_\alpha$ are isomorphic, we may apply the previous remarks to $F_{\gamma+1}/F_\gamma$ and the result will follow from $P \cong \bigoplus F_{\gamma+1}/F_\gamma$ [7].

On the other hand we have to be a little bit more careful if we really want to lift the given decomposition from $\hat{P}$ to P. First note that the countable version of Theorem 1 applies to the obvious morphism $\mu : F_{\gamma+1}/F_\gamma \to Q_K$, where $K = K(\gamma+1) \backslash K(\gamma)$ is countable. Therefore, we may replace μ by the morphism $\varepsilon_K : P_K \to Q_K$. The direct decomposition $P_K = \bigoplus_{i=1}^{\infty} P_{\alpha(i)}$, where $K = \{\alpha(1), \ldots, \alpha(i), \ldots\}$, allows to refine the inclusion $F_\gamma \subset F_{\gamma+1}$ into a chain

$$F_\gamma = F_{\gamma,0} \subset F_{\gamma,1} \subset \cdots \subset F_{\gamma,i} \subset \cdots \subset F_{\gamma+1}$$

such that

$$\nu_P(F_{\gamma,i}) = Q_{K(\gamma)} \oplus Q_{\alpha(1)} \oplus \cdots \oplus Q_{\alpha(i)} \quad \text{and} \quad \bigcup_i F_{\gamma,i} = F_{\gamma+1}$$

Replacing if necessary the γ's by the (γ, i)'s in lexicographic order, we may assume that $K(\gamma+1) \backslash K(\gamma)$ consists of a single element of I. This allows to identify I with the set of all γ's. So I is well ordered and we may even assume that $K(\gamma+1) \backslash K(\gamma) = \{\gamma\}$. Now lift the inclusion $Q_\gamma \to Q_{K(\gamma+1)}$ to a morphism $s : P_\gamma \to F_{\gamma+1}$. With $H_\gamma = \operatorname{Im}(s)$ we get $\nu_P(H_\gamma) = Q_\gamma$ and $F_{\gamma+1} = F_\gamma \oplus H_\gamma$, thus $P = \bigoplus_{\gamma \in I} H_\gamma$ as desired.

3. Applications

Recall that the left modules over a *ringoid* $\mathscr{A}$ (i.e. a small additive category) are just the covariant additive functors $M : \mathscr{A} \to \mathrm{Ab}$ where Ab denotes the category of abelian groups [9]. An $\mathscr{A}$-module isomorphic to a direct sum of representable functors $\mathscr{A}(A, -)$ is called *free*.

Theorem 1 extends the theorems of Beck [2] and Gruson [6] to modules over a ringoid. Note, that Beck's proof uses Eilenberg's trick, which is not available in this more general situation.

Proposition 3. *Let $\mathscr{A}$ be a ringoid and $\mathscr{R}$ a two-sided ideal contained in the Jacobson-radical of $\mathscr{A}$. Then a projective $\mathscr{A}$-module P is free, if and only if $P/\mathscr{R}P$ is a free $\mathscr{A}/\mathscr{R}$-module.*

Proof. By Nakayama's lemma the natural morphism

$$\mathcal{A}(A,-)\to\mathcal{A}(A,-)/\mathcal{R}(A,-)=(\mathcal{A}/\mathcal{R})(A,-)$$

is a projective cover for each $A\in\mathcal{A}$.

A ringoid $\mathcal{A}$ is called *local*, if each $A\in\mathcal{A}$ has a local endomorphism ring $\mathcal{A}(A,A)$. This implies that $\mathcal{A}/\mathcal{J}$ ($\mathcal{J}$ the Jacobson-radical of $\mathcal{A}$) is semi-simple and each simple $\mathcal{A}$-module is of the form $\mathcal{A}(A,-)/\mathcal{J}(\mathcal{A},-)$. Therefore we have Kaplansky's theorem [7], see also [5, 10]:

Proposition 4. *Let $\mathcal{A}$ be a local ringoid, then any projective $\mathcal{A}$-module is free.*

Now for any poset I and module category $\mathcal{M}$ the category $[I,\mathcal{M}]$ of diagrams over I with values in $\mathcal{M}$ is a module category, too. For any $\alpha\in I$ and module M of $\mathcal{M}$ we denote by $S_\alpha(M)$ the diagram whose restriction to $\{\beta\mid\beta\geqslant\alpha\}$ is constant with value M and which is 0 elsewhere. It is well known [9] that the projective diagrams are precisely the direct summands of diagrams $\bigoplus_{\alpha\in I}S_\alpha(Q_\alpha)$ for some projective modules Q_α of $\mathcal{M}$.

Proposition 5. *Let I be a poset and $\mathcal{M}$ a module category, then any projective diagram $P:I\to\mathcal{M}$ is free, i.e. $P\cong\bigoplus_{\alpha\in I}S_\alpha(Q_\alpha)$ for some projective modules Q_α of $\mathcal{M}$.*

Proof. For each diagram D and $\alpha\in I$ let $\hat{D}_\alpha=D_\alpha/\sum_{\beta<\alpha}\operatorname{Im}(D_\beta\to D_\alpha)$. Together with the zero maps, this defines the diagram $\hat{D}:I\to M$. Observe that any simple diagram has the form $E_\alpha(S)$ with S simple in $\mathcal{M}$ and $\alpha\in I$, where $E_\alpha(M)$ denotes the diagram with value M at α and 0 elsewhere. Thus, any morphism from D to a simple diagram factors through the natural epimorphism $\nu_D:D\to\hat{D}$. This proves that the functor $\hat{}:[I,\mathcal{M}]\to[I,\mathcal{M}]$ is a reduction modulo radical.

If P is a projective diagram, then $\hat{P}_\alpha$ is projective for each $\alpha\in I$, since the functor $[I,\mathcal{M}]\to\mathcal{M}$, $D\mapsto\hat{D}_\alpha$ is left adjoint to the exact embedding $\mathcal{M}\to[I,\mathcal{M}]$, $M\mapsto E_\alpha(M)$. Because $\hat{P}=\bigoplus_{\alpha\in I}E_\alpha(\hat{P}_\alpha)$, Proposition 5 reduces to the following lemma.

Lemma 6. *If Q is projective in $\mathcal{M}$, then the natural morphism $\nu:S_\alpha(Q)\to E_\alpha(Q)$ with $\nu_\alpha=1_Q$ is a projective cover.*

Proof. Since $S_\alpha(Q)$ is projective, we have to show that $K=\operatorname{Ker}\nu$ is small in $S_\alpha(Q)$. Note that $K_\alpha=0$, so if $U+K=S_\alpha(Q)$, then $U_\alpha=Q$. This implies $U_\beta=Q$ for each $\beta\geqslant\alpha$, i.e. $U=S_\alpha(Q)$ thus completing the proof of Lemma 6 and Proposition 5.

Remark. If I has dcc, then $\hat{D}=0$ implies $D=0$ for any diagram D. Thus for any projective P the natural morphism $\nu_P : P \to \hat{P}$ is a projective cover, therefore Proposition 5 can be proved along the lines of Bass's theory of modules over perfect rings [1], thus avoiding the use of Kaplansky's theorem. For the details we refer to [3].

Recall that a module M of $\mathcal{M}$ is called *pure-projective* if it is a direct summand of a direct sum of finitely-presented modules. Of course, any projective module is pure-projective.

Proposition 7. *Let I be a poset and $\mathcal{M}$ be a module category. If M_α is a pure-projective module for each $\alpha \in I$ and if P is a direct summand of $\bigoplus_{\alpha \in I} S_\alpha(M_\alpha)$, then $P \cong \bigoplus_{\alpha \in I} S_\alpha(N_\alpha)$ for some pure-projective modules N_α of $\mathcal{M}$.*

Proof. Let $\mathscr{C}$ be the full subcategory of finitely presented modules in $\mathcal{M}$. The functor

$$H : \mathcal{M} \to \text{Mod-}\mathscr{C}, \quad M \mapsto (-, M)|\mathscr{C}$$

is a full embedding of $\mathcal{M}$ into the category of right modules over $\mathscr{C}$. H commutes with arbitrary direct sums and induces an equivalence from the pure-projective modules of $\mathcal{M}$ to the projective $\mathscr{C}$-modules. Now consider

$$H_* : [I, \mathcal{M}] \to [I, \text{Mod-}\mathscr{C}], \quad D \mapsto H \circ D.$$

Being a direct summand of $H_*(\bigoplus_{\alpha \in I} S_\alpha(M_\alpha)) = \bigoplus_{\alpha \in I} S_\alpha(HM_\alpha)$ the diagram $H_*(P)$ is projective, thus $H_*(P) \cong \bigoplus_{\alpha \in I} S_\alpha(Q_\alpha)$ for some projective $\mathscr{C}$-modules Q_α (Proposition 5). Since $Q_\alpha = H(N_\alpha)$ for some pure-projective module N_α of $\mathcal{M}$, we get $H_*(P) \cong H_*(\bigoplus_{\alpha \in I} S_\alpha(N_\alpha))$ which implies $P \cong \bigoplus_{\alpha \in I} S_\alpha(N_\alpha)$.

Note that Proposition 7 remains true without the hypothesis on pure-projectivity if I has dcc [4]. We do not know whether the result remains valid without any restrictions on I or the M_α's.

References

[1] H. Bass, Finitistic dimension and a homological generalization of semi-primary rings, Trans. Amer. Math. Soc. 95 (1960) 466–488.
[2] I. Beck, Projective and free modules, Math. Z. 129 (1972) 231–234.
[3] H. Brune, Einige globale Dimensionen bei geordneten Mengen und die Exaktheit der Funtoren $\varinjlim$ und $\varprojlim$, Dissertation, Gesamthochschule Paderborn (1977).
[4] Ch. Cheng and B. Mitchell, DCC posets of cohomological dimension one, J. Pure Appl. Algebra 13 (1978) 125–137.
[5] P. Gabriel, Représentations indécomposables, in: Séminaire Bourbaki, Lecture Notes in Math. 431 (Springer-Verlag, Berlin, 1975) 143–168.
[6] D. Lazard, Liberté des gros modules projectifs, J. Algebra 31 (1974) 437–451.
[7] I. Kaplansky, Projective modules, Ann. Math. 68 (1958) 372–377.
[8] B. Mitchell, Theory of Categories (Academic Press, New York, 1965).

[9] B. Mitchell, Rings with several objects, Advances Math. 8 (1972) 1–161.
[10] B. Mitchell, Some applications of module theory to functor categories, Bull. Amer. Math. Soc. 84 (1978) 867–885.
[11] R. Ware and J. Zelmanowitz, The Jacobson radical of the endomorphism ring of a projective module, Proc. Amer. Math. Soc. 26 (1970) 15–20.