A Central Limit Theorem Applicable to Robust Regression Estimators

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Consider a general linear model, $Y_i = x_i'\beta + R_i$, with $R_1, \ldots, R_n$ i.i.d., $\beta \in \mathbb{R}^p$, and $\{x_1, \ldots, x_n\}$ behaving like a random sample from a distribution in $\mathbb{R}^p$. Let $\hat{\beta}$ be a robust $M$-estimator of $\beta$. To obtain an asymptotic normal approximation for the distribution of $\hat{\beta}$ requires a Central Limit Theorem for $W_n = \sum y_i \psi(R_i)$, where $y_i = (X'X)^{-1}x_i$. When $p \to \infty$, previous results require $p^5/n \to 0$, but here a strong normal approximation for the distribution of $W_n$ in $\mathbb{R}^p$ is provided under the condition $(p \log n)^{3/2}/n \to 0$.

1. INTRODUCTION

The Central Limit Theorem provides a normal approximation to the distribution of sums of (nearly) independent and identically distributed (i.i.d.) random quantities. For random vectors in $\mathbb{R}^p$ the discrepancy between the true distribution of normalized means and the normal approximation depends on the dimension, $p$. Prior to the author's work [6] the best results yielded errors of order $(p^5/n)^{1/2}$. For example, Senatov [9] provided uniform bounds for the normal approximation of the form $c(p/\sqrt{n}) \cdot E \|X\|^3$ (which is $O(p^5/n)^{1/2}$ since $E \|X\|^3$ is generally of order $p^{3/2}$). In [6] the author replaced $E \|X\|^3$ by $\sup\{E |t'X|^3 : \|t\| \leq 1\}$ which is bounded in many cases (independent of $p$); thus bounding the error in normal approximation for expectations of smooth functions by a term of order $p/\sqrt{n}$. Furthermore, an example was presented showing that this order cannot be improved upon in complete generality. However, in many statistical applications even this rate is

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unreasonably slow. For example, in a regression problem with 5 independent variables, a statistician might consider \( n = 100 \) adequately large to employ a normal approximation. But if a quadratic model is used, \( p = 21 \) and \( p/\sqrt{n} \) is not small. Fortunately, in such regression models (and, perhaps, other special cases) it is possible to obtain a better rate. In particular, a result is given here providing a Central Limit Theorem applicable to robust M-estimators of regression parameters and requiring only that \( (p \log n)^{3/2}/n \to 0 \) (condition (N)). This result is applied by the author [7] to obtain a normal approximation for \( (\|\hat{\theta}\|^2 - p)/\sqrt{2p} \), where \( \theta \) is the normalized M-estimator of a \( p \)-dimensional linear regression parameter (under the condition \( (p \log p)^{3/2}/n \to 0 \)). As noted in [7], the present result can also provide asymptotic distributional approximation for a wide class of smooth functions of such M-estimators of regression parameters. Other work on the asymptotics of M-estimators of regression parameters is given by the author [5], Huber [2, 3], and Yohai and Maronna [10]. These results include norm consistency in \( \mathbb{R}^p \) (the author [5] gives reasonable conditions under which \( p \log p/n \to 0 \) is sufficient for this) and asymptotic normality for linear functions of the estimators (essentially requiring \( p^{5/2}/n \to 0 \)). These previous results do not include a uniform normal approximation in \( \mathbb{R}^p \), which follows from the results here.

To fix notation, consider the linear model

\[
y_i = x_i' \beta + R_i, \tag{1.1}
\]

where \( R_1, \ldots, R_n \) are i.i.d. errors, \( \beta \in \mathbb{R}^p \), and \( x_1, \ldots, x_n \) are vectors in \( \mathbb{R}^p \) with \( x_i \) forming the \( i \)-th row of the design matrix, \( X \). Let \( \psi \) be an odd function and define the M-estimator, \( \hat{\beta} \), to be a solution of

\[
0 = \sum_{i=1}^n x_i' \psi(y_i - x_i' \beta).
\tag{1.2}
\]

In [7] the author provides an approximation for \( \hat{\beta} \),

\[
\hat{\beta} = \sum_{i=1}^n y_i \psi(R_i) + \Delta_n = \sum_{i=1}^n W_i + \Delta_n, \tag{1.3}
\]

where \( y_i = (X'X)^{-1/2} x_i \) and where \( \Delta_n \to^p 0 \) if condition (N) holds. Since \( W_i \) depend only on a one-dimensional function of \( R_i \), sufficiently strong conditions can be found so that condition (N) is sufficient to obtain a normal approximation for \( \sum W_i \) (so that (1.3) provides a normal approximation for \( \hat{\beta} \)). Note, however, that condition (N) is not sufficient for all linear models: in [5] the author shows that if the design matrix corresponds to a one-way layout then \( p^2/n \to 0 \) is required to approximate the distribution of \( \| \hat{\beta} - \beta \| \) (using the normal approximation for \( \hat{\beta} \)). Thus, the conditions here
are designed to hold in certain multiple regression cases where the \( \{x_i\} \) behave like a random sample. In particular, condition (XD) below requires that the design vectors \( \{x_1, x_2, \ldots, x_n\} \) form a sample from a mixed multivariate normal distribution in \( \mathbb{R}^p \). Section 3 shows that if (XD) holds then certain conditions (X1)-(X5) will hold with probability tending to one. In Section 2, a Central Limit Theorem for \( \sum W_i \) will be derived from the following conditions on \( \psi \) together with conditions (X1)-(X5) and condition (N). This will provide a version of the Central Limit Theorem conditional on the design matrix. Since (X1)-(X5) hold in probability, an unconditional version can be established under (XD).

The conditions are as follows:

\[ (N) \quad (p^{3/2} \log^2 n)/n \to 0 \text{ and } p^2/n \to \infty. \]

**Remark.** The last part of condition (N) is needed for purely technical reasons. In fact, if \( p^2/n \to 0 \), a proof similar to that of the author [6] would provide the result of Theorem 2.1.

\[ (P) \quad \psi(R) \text{ has a finite moment generating function in a neighborhood of zero and for some constant, } d > 0, \text{ and each } n \text{ there exists a } \psi\text{-function } \psi_n \text{ satisfying the above part of } P \text{ and condition } (P'), \text{ and such that} \]

\[ |\psi(u) - \psi_n(u)| \leq \frac{1}{n^{d+1}}. \quad (1.4) \]

\[ (P') \quad \psi \text{ is an odd twice-differentiable function which is not constant on any interval, and } R \text{ has an even differentiable density, } g, \text{ satisfying} \]

\[ e^{a\psi(r)} \cdot g(r)/\psi'(r) \to 0 \quad \text{as } r \to \pm \infty \]

for any sufficiently small constant, \( a \). Also,

\[ \left| \frac{g(r)}{\psi'(r)} \right| \leq B b^d \text{ for some constants } B \text{ and } d', \]

and \( \frac{d}{dr} (g(r)/\psi'(r)) \) has at most \( M \) sign changes (where \( M \) is a constant not depending on \( n \)). \quad (1.5)

**Remark.** (1) Given \( \psi \) and \( g \), it is generally trivial to construct the function \( \psi_n \) satisfying \( P' \) (as required in (P)). In particular, since \( E \exp \{ t\psi(R) \} \) is finite (for \( t \) small), it is clear that as long as neither \( \psi(r) \) or \( g(r) \) have "spikes," \( g(r) e^{t\psi(r)} \to 0 \); and it is easy to smooth \( \psi \) so that \( P' \) holds. It is important to note that the condition that \( \psi \) be odd can be eliminated with much more tedious computations. The assumptions that
$E\psi(R) = 0$ and $E\psi^2(R) = \sigma^2 < +\infty$ can replace the assumption that $\psi$ be odd.

2. Under (P') it is clear that $\psi(R)$ has a density.

(XD) Let $(s_1, \ldots, s_n)$ be i.i.d. according to a distribution with compact support in $(0, \infty)$, and (given $\{s_1, \ldots, s_n\}$) let $(x_1, \ldots, x_n)$ be independent with $x_i \sim N_p(0, s_i I)$. Also (without loss of generality by rescaling) let $E s_i = 1$.

Section 3 provides the following result: under condition (XD), the following statements (X1)--(X5) hold with probability tending to one:

(X1) The maximum and minimum eigen values of $X'X$ satisfy (for constants $B$ and $b > 0$)

$$\lambda_{\max}(X'X) \leq Bn, \quad \lambda_{\min}(X'X) \geq bn.$$  \hfill (1.6)

(X2) Define

$$y_i = (X'X)^{-1/2} x_i \quad (i = 1, 2, \ldots, n).$$  \hfill (1.7)

Then uniformly in $i = 1, \ldots, n$ and $l = 1, \ldots, n$,

$$y'_i y_i = O(\sqrt{n} \log n/n) \quad \text{for} \quad i \neq l$$  \hfill (1.8)

and furthermore, there are values $\{s_i: i = 1, \ldots, n\}$ uniformly bounded and satisfying the appropriate conditions of Section 3 such that (uniformly in $i$),

$$\|y_i\|^2 = \frac{p}{n} s_i + O(\sqrt{p \log n/n}).$$  \hfill (1.9)

(X3) For each $u \in R^p$ define

$$J_0(u) = \# \{i = 1, \ldots, n: |y_i u| \geq \|u\|\}. \hfill (1.10)$$

Then for some constant $b$

$$\inf \{J_0(u): \|u\| \leq 1\} \geq bn.$$  \hfill (1.11)

(X4) define for constants $\overline{B}$ and $B_0$ and for $u \in R^p$,

$$\delta_n^2 = \overline{B} p^3 (\log n)^3/n^2$$

$$J = J(u) = \{i = 1, \ldots, n: (y'_i u)^2 \leq B_0 p \log^2 n/n\}.$$  \hfill (1.13)

Then for any $B$, $\overline{B}$, and $B_0$ there is $B^*$ arbitrarily large (a constant multiple of $\overline{B}$) such that
Let $V_i$ ($i = 1, \ldots, 9$) be given by (2.9) and let sets $\mathcal{U}_n$ and $\mathcal{V}_n$ be defined by (2.5). Let $U$ and $Z$ be independent $\mathcal{N}(0, I)$. Then for sufficiently large values of the constants defining $\mathcal{U}_n$ and $\mathcal{V}_n$,

$$EI(Z \in \mathcal{U}_n) I(U \in \mathcal{V}_n) \exp \left\{ -\sum_{i=1}^{9} V_i \right\} \rightarrow 1 \quad \text{as} \quad n \rightarrow \infty. \quad (1.15)$$

Note that in conditions (X4) and (X5), the notation $I(\cdots)$ refers to the indicator function.

2. THE CENTRAL LIMIT THEOREM FOR $W$

Theorem 2.1 provides a version of a normal approximation for $W$ (1.3) conditional on the design matrix. An unconditional version with $X$ distributed as described in condition (XD) is given in the corollary.

**Theorem 2.1.** Assume that conditions (N), (P), (X1), (X2), (X3), (X4), and (X5) hold. Let

$$W = \frac{1}{\sigma} \sum_{i=1}^{n} y_i \psi(R_i),$$

where $\delta^2 = \text{Var} \psi(R)$, and let $Z \sim \mathcal{N}(0, I)$. Let $\varepsilon_n = \mathcal{O}(1/n^d)$ for some $d > 0$ and let $A_n \subset \mathbb{R}^p$ be a sequence of sets such that

$$P\{ Z \in A_n(\varepsilon_n) - A_n \} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty, \quad (2.1)$$

where $A_n(\varepsilon)$ is the $\varepsilon$-neighborhood of $A_n$;

$$A_n(\varepsilon) = \{ u : \text{for some } v \in A_n, \| u - v \| \leq \varepsilon \}.$$

Assume (2.1) also holds for the complement, $A_n^c$, of $A_n$. Let $P_{X_n}$ denote the (conditional) distribution of $(R_1, \ldots, R_n)$ with $(x_1, \ldots, x_n)$ fixed. Then

$$| P_{X_n}(W \in A_n) - P\{ Z \in A_n \} | \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.$$
sequence of theorem 2.1 and the results of section 3 that \((X_1,\ldots, X_5)\) hold in probability under condition XD:

**Corollary.** Assume that conditions (N), (P), and (XD) hold. Then with \(P\) denoting the joint distribution of \((x_1,\ldots, x_n)\) (as given by (XD)) and \((R_1,\ldots, R_n)\), and with \(A_n\) as in Theorem 2.1, if \(p^2/n \to \infty\),

\[
|P\{W \in A_n\} - P\{Z \in A_n\}| \to 0 \quad \text{as} \quad n \to \infty.
\]

**Proof of Theorem 2.1.** First, note that by Lemma 2.2, we may assume that condition (P') holds, which implies that Lemma 2.1 holds. Thus it suffices to assume that Lemma 2.1 holds and to obtain the conclusion for any sequence of sets \(A_n \subset \mathbb{R}^p\). Now, use the method of associated distributions: let \(h_i(w)\) denote the density of \((1/\sigma) y_i \psi(R_i)\) (in \(\mathbb{R}^p\)) and define

\[
h_i^*(w) = \exp\{\frac{1}{\sigma} w \cdot \rho(y_i; t)\} \cdot h_i(w),
\]

where

\[
\rho(u) = \log E \exp \left\{ \frac{u}{\sigma} \psi(R) \right\}.
\]

Then if \(f_n\) denotes the density of \(W\) and \(f_n^*\) the \(n\)-fold convolution of \(h_i^*\),

\[
f_n^*(z) = \exp \left\{ t'z - \sum_{i=1}^n \rho(y_i; t) \right\} f_n(z).
\]

Thus, choosing \(t = z\) and using the inversion formula for characteristic functions (which is justified by Lemma 2.1),

\[
f_n(z) = \exp \left\{ \sum_{i=1}^n \rho(y_i; z) - \|z\|^2 \right\} \left( \frac{1}{2\pi} \right)^{p/2} e^{-\|z\|^2/2} \times \exp \left\{ \sum_{i=1}^n \left[ \rho(y_i; z + i y_i; u) - \rho(y_i; z) \right] \right\} du.
\]

Now define sets \(U_n\) and \(V_n\) in \(\mathbb{R}^p\),

\[
U_n = \left\{ z: \max_i (y_i; z)^2 \leq \frac{Bp}{n} \log n \text{ and } \|z\|^2 \leq Bp \right\}
\]

\[
V_n = \left\{ u: \max_i (y_i; u)^2 \leq \delta_n \text{ and } \|u\|^2 \leq B_p \log n \right\},
\]

where \(\delta_n = B_0 p^3 (\log n)^2 / n^2\). For \(z \in U_n\) and \(u \in V_n\), the argument of \(\rho\) is small; and, hence we can apply the expansion

\[
\rho(u) = \frac{1}{2} u^2 + bu^4 + \mathcal{O}(u^6)
\]
(which follows since \( \psi(R) \) is symmetric). Now if \( A_n \subset \mathcal{B}_n \), then for \( z \in A_n \), \( \Sigma(y'_i z)^6 \leq B p^3 \log^2 n/n^2 \rightarrow 0 \). Therefore applying Lemma 2.1 and using (2.6) yields

\[
P\{ W \in A_n \} \leq \int_{A_n} \exp\left( \frac{1}{2} \Sigma(y'_i z)^2 + b \Sigma(y'_i z)^4 - \| z \|^2 \right) \left( \frac{1}{2\pi} \right)^p \int_{\mathcal{B}_n} \exp\left\{ -i u' z + i \Sigma y'_i u(y'_i z + 4b(y'_i z)^3) - \frac{1}{2} \Sigma(y'_i u)^2 (1 + 12b(y'_i z)^2) + i K(y'_i u, y'_i z) + b \Sigma(y'_i u)^4 + o(\Sigma \sum y'_i u(y'_i z)^5 \right.
\]

\[
+ (y'_i u)^2(y'_i z)^4 + \cdots + (y'_i u)^6 \} \right\} \, du \, dz + o(1),
\]

where \( K \) is an appropriate polynomial. Using \( \Sigma(y'_i s)^2 = \| s \|^2 \),

\[
P\{ W \in A_n \} \leq (1 + o(1)) \int_{A_n} \exp\left\{ -\frac{1}{2} \| z \|^2 + b \Sigma(y'_i z)^4 \right\} \left( \frac{1}{2\pi} \right)^p \int_{\mathcal{B}_n} \exp\left\{ -\frac{1}{2} \| u \|^2 - 6b \Sigma(y'_i u)^2(y'_i z)^2 + b \Sigma(y'_i u)^4 + C \Sigma \left( | y'_i u | | y'_i z |^5 + \cdots + (y'_i u)^6 \right) \right\} \, du \, dz + o(1) \quad (2.7)
\]

for some constant \( C \) (if \( A_n \subset \mathcal{B}_n \)). Thus, (2.7) yields the following bound (with \( I \) denoting “indicator function”),

\[
P(W \in A_n) \leq (1 + o(1)) E I(Z \in A_n) I(U \in \mathcal{B}_n) \exp \left\{ \sum_{j=1}^9 V_j \right\} + o(1) \quad (2.8)
\]

where \( Z \sim \mathcal{N}(0, I) \), \( U \sim \mathcal{N}(0, I) \) (independent) and \( V_j \) are functions of \( Z \) and \( U \),

\[
V_1 = b \sum_{i=1}^n (y'_i Z)^4 - 3b \sum_{i=1}^n \| y'_i \|^4 \\
V_2 = -6b \sum_{i=1}^n (y'_i Z)^2(y'_i U)^2 + 6b \sum_{i=1}^n \| y'_i \|^4 \\
V_3 = b \sum_{i=1}^n (y'_i U)^4 - 3b \sum_{i=1}^n \| y'_i \|^4 \\
V_j = C \sum_{i=1}^n | y'_i U |^{1-3} | y'_i Z |^{9-j} \quad \text{for} \ j = 4, 5, 6, 7, 8, \text{and} \ 9
\]

(note that the \( \Sigma \| y'_i \|^4 \) terms exactly cancel).

Now note that if (N) holds, \( V_j \rightarrow^p 0 \) for \( j = 1, \ldots, 9 \). This can be shown for
j \geq 4$ by noting that $y'_iZ$ and $y'_iU$ are $\mathcal{N}(0, \|y_i\|^2)$ and computing $E V_j = C' \Sigma \|y_i\|^6 = O(p^3/n^2)$ (by (X2)). For $V_1$, $V_2$, and $V_3$, the variance is required: let $\bar{\mathcal{V}}_1$ denote the first term of $V_1$. Then

$$E \bar{\mathcal{V}}_1 = h \sum_{i=1}^n E(y'_iZ)^4 = 3h \sum_{i=1}^n \|y_i\|^4$$

$$E \bar{\mathcal{V}}_2 = b^2 \sum_{i=1}^n \sum_{l=1}^n E(y'_iZ)^4(y'_lZ)^4.$$ 

Writing $(y'_iZ)^4$ and $(y'_iZ)^4$ as 4-fold sums ($\Sigma y'_iZ_i$) and noting that subscripts must be equal in pairs, it is not too difficult to show that

$$E \bar{\mathcal{V}}_1 = 9b^2 (\Sigma \|y_i\|^4)^2 + O(\Sigma \Sigma (y'_iy_i)^2 \|y_i\|^4)$$

$$= \left( E \bar{\mathcal{V}}_1 \right)^2 + O\left( \frac{p^3 \log n}{n^2} \right)$$

(using (X2)). Thus, $\text{Var } V_1 \to 0$ (by (N)) and $V_1 \to^p 0$. A similar computation works for $V_2$ and $V_3$. Last, by condition (X5), $E I(Z \in \mathcal{U}_n) I(U \in \mathcal{V}_n) \exp \left\{ \Sigma_{i=1}^n V_i \right\} \to 1$. Therefore, by the extended dominated convergence theorem (Pratt [4], (2.8) yields

$$P(W \in A_n) \leq (1 + o(1)) P\{Z \in A_n, U \in \mathcal{V}_n\} + o(1) \quad (2.10)$$

for $A_n \subset \mathcal{U}_n$.

Last, by Lemma 2.3, $P(Z \in \mathcal{U}_n), P(W \in \mathcal{U}_n)$, and $P\{ U \in \mathcal{V}_n \}$ all converge to unity. Thus, for any set $A_n$,

$$P(W \in A_n) \leq (1+o(1)) P\{ Z \in A_n \} + o(1)$$

$$= P\{ Z \in A_n \} + o(1). \quad (2.11)$$

Since (2.11) also holds for $A'_n$, the Theorem follows. ♦

**Lemma 2.1.** Assume conditions (N), (P'), (X3), and (X4). Let $f_n$ be defined by (2.4), and let sets $\mathcal{U}_n$ (depending on $B$) and $\mathcal{V}_n$ (depending on $B_0$) be defined by (2.5). Then for any $B$, there is $B_0$ such that for any set $A_n \subset \mathcal{U}_n$, as $n \to \infty$

$$\int_{A_n} f_n(z) \, dz \leq (1 + o(1)) \int_{A_n} \exp \left\{ \Sigma \rho(y'_iz) - \| z \|^2 \right\} \left( \frac{1}{2\pi} \right)^p \int_{\gamma_n} e^{-iwz}$$

$$\times \exp \left\{ \Sigma [\rho(y'_iz + iy'_zu) - \rho(y'_iz)] \right\} du \, dz. \quad (2.12)$$
Proof. Note that (2.12) differs from the integral of $f_n$ (2.4) only by letting $\mathcal{V}_n$ replace $R^c$ for the inner integral. So define sets

$$V_1 = \{ u: \|u\|^2 \leq B_0 p \log n \text{ and } \max_i |y_i' u| > \delta_n \}$$

$$V_2 = \{ u: \|u\|^2 > B_0 p \log n \}.$$

Note that $R^p = \mathcal{V}_n \cup V_1 \cup V_2$. Shortly, we will show that for any $B$, there is $B_0 > B$ such that the inner integrals over $V_1$ and $V_2$ are both uniformly of order $O((2\pi)^{p/2} \exp\{ -B_0 p^2 \log^2 n/n \})$. The following argument bounding the outer integral shows that this is sufficient to obtain (2.12).

By (2.6), since $A_n \subset \mathcal{V}_n$, for some constant $\overline{B}$,

$$\rho(y'; z) = \frac{1}{2}(y'; z)^2 + O((y'; z)^4)$$

$$= \frac{1}{2}(y'; z)^2(1 + \overline{B} p \log n/n).$$

Therefore (since $\sum(y'; z)^2 = \|z\|^2$),

$$\int_{A_n} \exp\{ \sum (y'; z) - \|z\|^2 \} \, dz \leq \int_{\mathcal{V}_n} \exp\{ \frac{1}{2} \sum (y'; z)^2(1 + \overline{B} p \log n/n) - \|z\|^2 \} \, dz$$

$$\leq \int_{R^p} \exp\{ -\frac{1}{2} \|z\|^2(1 - \overline{B} p \log n/n) \} \, dz$$

$$= (2\pi)^{p/2}(1 - \overline{B} p \log n/n)^{-p/2}$$

$$\leq (2\pi)^{p/2} \exp\{ \overline{B} p^2 \log n/n \}$$

for $n$ large enough (since $\log(1 - x) \geq -2x$ for $x$ small). Thus, it remains to bound the inner integrals over $V_1$ and $V_2$ uniformly by terms of order $O((2\pi)^{p/2} \exp\{ -B^* p^2 \log^2 n/n \})$ (for $B^* > \overline{B}$).

First, note that the inner integrand is the (inverse) Fourier transform of a product of characteristic functions

$$\phi_i(u) = \exp\{ -\rho(y_i'; z) \} E \exp\{ (y_i' z - iy_i' u) \psi(R)/\sigma \}. \quad (2.14)$$

Now, for $V_1$, use the fact that $|\phi_i(u)| \leq 1$ to obtain for any subset $J \subset \{1, 2, \ldots, n\}$,

$$\left| \exp\left\{ \sum_{i=1}^n \left[ \rho(y_i' z + iy_i' u) - \rho(y_i' z) \right] \right\} \right|$$

$$\leq \exp\left\{ \Re \sum_{i \in J} \left[ \rho(y_i' z + iy_i' u) - \rho(y_i' z) \right] \right\}.$$ 

Then using condition (X4), let

$$J = \{ i: \|y_i' u\|^2 \leq B_0 p \log^2 n/n \},$$
and using (2.6) (for \( z \in \mathcal{U}_n \)) we have for some \( B_1 \),

\[
\int_{y_1} \cdots du \leq \int_{\|u\|^2 \leq B_0 \rho \log n} I(\max_i |y_i'u| > \delta_n) \exp \left\{ -\frac{1}{2} \sum_{i \in J} (y_i'u)^2 \right\} \times (1 - B_1 \rho \log^2 n/n) \left\{ \exp \left\{ -\frac{1}{2} \sum_{i \in J} (y_i'u)^2 \right\} \right\} du
\]

\[
\leq \exp \{ B_1 \rho^2 \log^2 n/n \} \int_{\|u\|^2 \leq 2B_0 \rho \log n} I(\max_i |y_i'u| \geq \frac{1}{2} \delta_n) \times \exp \left\{ -\frac{1}{2} \sum_{i \in J} (y_i'u)^2 \right\} du
\]

\[
\leq b \exp \{ B_1 \rho^2 \log^2 n/n \} (2\pi)^{n/2} \exp \{ -B^* \rho^2 \log^2 n/n \} \quad (2.15)
\]

for some constant \( b \), where (X4) is used to show that \( B_0 \) can be chosen so that \( B^* \) is larger than \( B_1 + B \) (so that the outer integral is cancelled).

For \( V_2 \), by (2.14), condition \( P' \), the fact that \( \rho(y_i'z) \) is bounded for \( z \in \mathcal{U}_n \), and integration by parts (twice),

\[
|\varphi_i(u)| \leq b \left| \int \exp \{ (y_i'z + i\gamma y_i'u) \psi(r) \} g(r) \, dr \right| = \frac{b}{|y_i'z + i\gamma y_i'u|} \left| \int (y_i'z + i\gamma y_i'u) \psi'(r) \right| \times \exp \left\{ (y_i'z + i\gamma y_i'u) \psi(r) \right\} \frac{g(r)}{\sigma} \left| \psi'(r) \right| \, dr
\]

\[
\leq \frac{b}{|y_i'z + i\gamma y_i'u|} \left| \int e^{(y_i'z)\psi'(r)/\sigma} \frac{d}{dr} \left( \frac{g(r)}{\psi'(r)} \right) \right| \, dr
\]

\[
- \frac{b}{|y_i'z|} \sum_{k=1}^M \left| \int_{a_k}^{a_{k+1}} e^{(y_i'z)\psi'(r)/\sigma} \frac{d}{dr} \left( \frac{g(r)}{\psi'(r)} \right) \, dr \right|
\]

\[
\leq \frac{b}{|y_i'z|} \sum_{k=1}^M \left( \left| y_i'z \right| \int_{a_k}^{a_{k+1}} e^{(y_i'z)\psi'(r)/\sigma} g(r) \, dr \right.
\]

\[
+ 2e^{(y_i'z)\psi'(a_k)} \left| \frac{g(a_k)}{\psi'(a_k)} \right| \right)
\]

\[
\leq \frac{Bn^d}{|y_i'z|}, \quad (2.16)
\]

where \( \{a_k\} \) are points at which \( (d/dr)(g(r)/\psi'(r)) \) changes sign (using (1.5)).
Also consider \( \varphi_o(v, w) = \exp\{-p(w) E \exp\{(w + i v) \psi(R)\}\} \) for real numbers \( v \) and \( w \). For some \( \varepsilon > 0 \) and \( |v| \geq \varepsilon, |w| < \varepsilon \), \( \Re(\partial^2/\partial v^2) \varphi_o(v, w) \) is bounded below zero. Also \( |\varphi_o(v, w)| \) is bounded below 1 for \( |v| < \varepsilon \) (uniformly on \( |w| \leq \varepsilon \)). Therefore, for \( |w| \leq \varepsilon, |\varphi_o(v, w)| \leq 1 - b\gamma_n^2 \) for \( |v| \geq \gamma_n \) for any sequence \( \gamma_n \to 0 \) (for \( n \) large enough). Thus (since \( |y_i z| \) is small on \( \mathfrak{U}_n \)), for some constant \( b \),

\[
|\varphi_i(u)| \leq 1 - b\gamma_n^2 \quad \text{for} \quad |y_i u| \geq \gamma_n.
\]

Now, use condition (X3): let \( J(u) = \{i: |y_i u| \geq \|u\|/\sqrt{n}\} \). Then \( \#J(u) \geq an \) (for some constant \( a \)) and we can choose \( J_\rho = \{i_1, \ldots, i_{\rho+1}\} \subset J \). Thus, for \( i \in J(u) \), if \( \|u\|^2 \geq B_0 \rho \log n, |y_i u| \geq \sqrt{B_0 (p \log n/n)^{1/2}} \). Hence, using (2.16) and (2.17), for each \( u \) such that \( \|u\|^2 \geq B_0 \rho \log n \) (with \( d = d' + \frac{1}{2} \))

\[
\left| \prod_{i=1}^n \varphi_i(u) \right| \leq \prod_{i \in J_\rho} |\varphi_i(u)| \prod_{i \in J - J_\rho} |\varphi_i(u)|
\leq \left( \frac{Bn}{\|u\|} \right)^{\rho + 1} \cdot \left( 1 - bB_0 \rho \frac{\log n}{n} \right)^{an - (p + 1)}
\]

Therefore,

\[
\left| \int \prod_{i=1}^n \varphi_i(u) du \right|
\leq \left| \int_{\|u\|^2 \geq B_0 \rho \log n} \frac{du}{\|u\|} \left( p + 1 \right) \exp\{B' \rho \log n - B_0 b^* p \rho \log n\}
\leq \exp\{B'' \rho \log n - B_0 b^* p \rho \log n\},
\]

where \( B', B'', \) and \( b^* \) are constants. Again \( B_0 \) can be chosen so that (2.18) dominates the outer integral, and the proof is complete.

**Lemma 2.2.** As in condition (P), suppose \( \psi \) and \( \psi_o \) are two functions such that \( |\psi(r) - \psi_o(r)| \leq 1/n^{d+1} \) for some \( d > 0 \). Assume condition (X2) holds and suppose \( p/n \to 0 \). Define \( W = (1/\sigma) \sum_{i=1}^n y_i \psi(R_i) \) and \( W_0 = (1/\sigma) \sum_{i=1}^n y_i \psi_0(R_i) \). Let \( Z \sim \mathcal{N}(0, I) \) and assume

\[
P\{W \in B_n\} - P\{Z \in B_n\} \to 0 \quad \text{as} \quad n \to \infty
\]

for any sequence of sets \( B_n \subset R^p \). Let \( A_n \subset R^p \) be such that

\[
P\{Z \in A_n(1/n^d) - A_n\} \to 0 \quad \text{as} \quad n \to 0
\]
and similarly for the complement \( A_n^c \), where \( A_n(\varepsilon) = \{ u \in \mathbb{R}^p : \text{for some } v \in A_n, \| u - v \| \leq \varepsilon \} \). Then

\[
| P\{ W_0 \in A_n \} - P\{ Z \in A_n \} | \to 0 \quad \text{as } n \to \infty. \tag{2.19}
\]

Proof. First, note that (since \( p/n \to 0 \) and \( \| y_i \|^2 = O(p/n) \)),

\[
\left\| \sum_{i=1}^n y_i \psi(R_i) - \sum_{i=1}^n y_i \psi_0(R_i) \right\| = \left\| \sum_{i=1}^n y_i (\psi(R_i) - \psi_0(R_i)) \right\|
\leq \sum_{i=1}^n \| y_i \| \| \psi(R_i) - \psi_0(R_i) \|
= O\left( n \cdot \left( \frac{p}{n} \right)^{1/2} \cdot \frac{1}{n^d+1} \right) = o\left( \frac{1}{n^d} \right).
\]

Therefore, with \( c = 1/n^d \), for \( n \) large enough,

\[
P\{ W_0 \in A_n \} \leq P\{ W \in A_n(\varepsilon) \} \leq o(1) + P\{ Z \in A_n(\varepsilon) \}
\leq o(1) + P\{ Z \in A_n \}.
\]

Since the same inequality holds for \( P\{ W_0 \in A_n^c \} \), Eq. (2.19) follows.

**Lemma 2.3.** Assume conditions (P'), (X2), and (N). Let \( Z \) and \( U \) be independent \( \mathcal{N}_p(0, I) \) and let \( W = (1/\sigma) \sum y_i \psi(R_i) \). Let \( \mathcal{U}_n \) and \( \mathcal{V}_n \) be defined by Eq. (2.5). Then for \( B \) and \( B_0 \) large enough in the definition of \( \mathcal{U}_n \) and \( \mathcal{V}_n \), \( P\{ Z \in \mathcal{U}_n \} \), \( P\{ U \in \mathcal{V}_n \} \), and \( P\{ W \in \mathcal{V}_n \} \) all converge to unity as \( n \to \infty \).

Proof. First, note that for \( n \) large, \( \mathcal{U}_n \subset \mathcal{V}_n \); so \( P\{ U \in \mathcal{V}_n \} \geq P\{ Z \in \mathcal{U}_n \} \) and the former probability need not be computed. For the latter probability, note that \( y_i'Z/\| y_i \| \sim \mathcal{N}(0, 1) \). Using (X2), \( \| y_i \|^2 \leq B_0p/n \) for some \( B_0 \). Hence

\[
P\{ \max(y_i'Z)^2 > Bp \log n/n \} \leq n P\left\{ \frac{y_i'Z}{\| y_i \|} > \left( \frac{B}{B_0} \log n \right)^{1/2} \right\}
\leq n \exp \left\{ - \frac{1}{2} \frac{B}{B_0} \log n \right\} \to 0 \tag{2.20}
\]

if \( B > 2B_0 \) and \( n \) is large enough (since \( 1 - \Phi(x) \leq \exp\{-\frac{1}{2}x^2\} \) for \( x > 1 \)). Also,

\[
P\{ \| Z \|^2 > Bp \} = P\{ x_{p}^2 > Bp \} \to 0 \quad \text{as } p \to \infty \tag{2.21}
\]

if \( B > 1 \). Hence, from (2.20) and (2.21), \( P\{ Z \notin \mathcal{U}_n \} \to 0 \) as \( n \to \infty \).
Now, \( \sigma y_i' W = \sum_{i=1}^{n} (y_i' y_i) \psi(R_i) \) and (since (P), (X2), and (N) hold)

\[
P\{\max_i |y_i' W| > Bp \log n/n \} \to 0 \quad \text{as } n \to \infty
\]

for some \( B > 0 \) by Lemma 3.3 of Portnoy [7]. Also as in Eq. (3.17) of [7], \( \|W\|^2 = o_n(p) \); and, hence \( P\{W \notin \mathcal{U}_n\} \to 0 \) as \( n \to \infty \) if \( B \) is large enough.

3. THE CONDITIONS ON THE DESIGN MATRIX

This section shows that conditions (X1)–(X5) given in Section 1 will hold in probability under the model postulated in condition (XD). For technical reasons, some of these results will require that \( p \to +\infty \): in particular, assume \( p^2/n \to +\infty \) (see Proposition 3.2 and Lemma 3.4). First, conditions (X1) and (X3) will be considered using the argument of [5]. In particular, these conditions are of the general form: with \( c, d, \) and \( f: R^n \to R \) specified, there is a constant \( B \) such that

\[
\sup_{\|u\| \leq c_n} f(x_i' u, ..., x_i' u) \leq Bd_n. \tag{3.1}
\]

In some cases (e.g., \( \lambda_{\min}(X'X) \) in condition (X1)) the sup is over \( \|u\| = 1 \).

The basic outline of the proof that (3.1) holds in probability is as follows: First, fix \( u \) (with \( \|u\| \leq c_n \)) and show that (under condition (XD)) there is a constant \( b \) linearly increasing with \( B \) such that

\[
P\{f(x_i' u, ..., x_i' u) \geq (B - 1) d_n\} \leq \exp\{-bp \log n\}. \tag{3.2}
\]

This generally requires a large deviation result which will follow using the Markov inequality. Now cover the ball of radius \( c_n \) with cubes \( C(u_k) \) centered at \( u_k \) and having side \( 1/n^a \). It is usually easy to see that \( a \) can be chosen so that if \( u \in C(u_k) \) and \( f((x_i' u_k)) \leq (B - 1) d_n \) then \( f((x_i' u)) \leq Bd_n \).

Since \( c_n \) is bounded in all cases (say, \( c_n < c \)), the number of such cubes needed to cover the ball is \( N \leq (2c_n^a)^n \leq \exp\{ p(a + 1) \log n \} \) for \( n \) large enough. Thus, the probability that (3.1) holds is bounded by \( \exp\{ p(a + 1) \log n - bp \log n \} \to 0 \) if \( B \) is chosen by large enough so that \( b > a + 1 \).

For (X1) the proof is quite simple if condition (N) holds since

\[
\lambda_{\max}(X'X) = \sup_{\|u\| = 1} \sum_{i=1}^{n} (x_i' u)^2. \tag{3.4}
\]

Under condition (XD), \( \sum(x_i' u)^2 \) is bounded above by a constant times a \( \chi^2_n \)
random variable, and it is easy to see that (for fixed $u$) there are $B$ and $b$ such that $P\{\Sigma(x_i'u)^2 \geq Bn\} \leq \exp\{-bn\}$ (from which (3.2) follows if $N$ holds). The remainder of the proof easily follows the above outline. Condition (X3) also follows easily (if $N$ holds): for fixed $u$, $\# \{i: |x_i'|u| \geq |u|\} = J(u)$ is a binomial random variable. Hence, by a large deviation result for the binomial, $P\{J(u) < Bn\} \leq \exp\{-bn\}$ (for some $B$ and $b$), and again the proof is straightforward. Thus, the following result has been proven:

**Proposition 3.1.** Under condition (XD), conditions (X1) and (X3) hold with probability tending to one.

The following result is needed for condition (X2):

**Proposition 3.2.** Suppose that $p/\log n \to +\infty$. Under condition (XD) there are constants $B > 0$ and $\delta > 0$ such that with probability at least $1 - \exp\{-\delta p\}$ (for $n$ large enough)

$$\max_i \|x_i\|^2 \leq Bp. \quad (3.5)$$

**Proof.** First, for each $i,$

$$\|x_i\|^2 \leq s_i \|z_i\|^2 \leq B_0 \|z_i\|^2,$$

where $z_i \sim \mathcal{N}_p(0, I);$ or $\|z_i\|^2 \sim \chi_p^2.$ But by the Markov inequality,

$$P\{\chi_p^2 \geq 2p\} \leq e^{-2pt} \frac{1}{(1-2t)p/2} = \exp\{-2pt - \frac{1}{2}p \log(1-2t)\} \leq e^{-0.15p}$$

(for $t = \frac{1}{4}$). Therefore

$$P\{\max_i \|x_i\|^2 \geq 2B_0 p\} \leq ne^{-0.15p} = \exp\left\{-p \left(0.15 - \frac{\log n}{p}\right)\right\} \leq e^{-\delta p}$$

for $\delta < 0.15.$ \[\]

**Lemma 3.1.** Assume condition (XD) and suppose $p/\log n \to +\infty.$ Let $y_i = (X'X)^{-1/2}x_i$ then there is a constant $B > 0$ such that

$$\max_{i \neq j} |y_i'y_j| \leq \frac{B(p \log n)^{1/2}}{n}$$

in probability. So the first part of condition (X2) holds in probability.
Proof. Note that \( y'_i y_i = x'_i (X'X)^{-1} x_i \). Let

\[
T_i = \sum_{k \neq i} (x_k x'_k)
\]

(i.e., \( T_i \) is like \( X'X \) based on a sample of size \( n - 1 \), and does not depend on \( x_i \) or \( s_i \)). Then

\[
(X'X)^{-1} = (T_i + x'_i x_i)^{-1} = T_i^{-1} - \frac{T_i^{-1} x_i x'_i T_i^{-1}}{1 + x'_i T_i^{-1} x_i}.
\]

Therefore

\[
y'_i y_i - x'_i T_i^{-1} x_i \left( 1 - \frac{x'_i T_i^{-1} x_i}{1 + x'_i T_i^{-1} x_i} \right) = \frac{x'_i T_i^{-1} x_i}{1 + x'_i T_i^{-1} x_i},
\]

\[
|y'_i y_i| \leq |x'_i T_i^{-1} x_i|.
\]

Now, since \( s_i, T_i, \) and \( x_i \) are independent of \( x_i \), given \( \{s_i, x_i, T_i\} \), \( x'_i T_i^{-1} x_i \sim \mathcal{N}(0, s_i x'_i T_i^{-2} x_i) \). Thus, using the normal bound for normal tail probabilities,

\[
P\{ |x'_i T_i^{-1} x_i| \geq (6 \log n) s_i x'_i T_i^{-2} x_i \}^{1/2} |s_i, x_i, T_i\} \leq e^{-3 \log n}.
\]

But, by condition (X1) and Proposition 3.2,

\[
|x'_i T_i^{-2} x_i| \leq \|x_i\|^2 \lambda_{\text{max}}(T_i^{-2}) \leq \|x_i\|^2 / \lambda_{\text{min}}(T_i) \leq \frac{B_0 p}{(n - 1)^{1/2}}
\]

with probability \( 1 - q_i \) satisfying \( n^2 q_i \to 0 \). Thus, integrating (3.9) over the set where (3.10) holds, with probability \( 1 - q_i^* \) satisfying \( n^2 q_i^* \to 0 \),

\[
|x_i T_i^{-1} x_i| \leq \frac{B(p \log n)^{1/2}}{n},
\]

from which Lemma 3.1 follows by (3.8).
As a consequence, with probability tending to 1,
\[ \sum_{i=1}^{n} \| y_i \|^4 = \left( E \sigma^2 \right)^2 \frac{p^2}{n} + o(1). \] (3.12)

Proof. As in (3.8),
\[ \| y_i \|^2 = \frac{x_i^T T_i^{-1} x_i}{1 + x_i^T T_i^{-1} x_i} \sim \frac{x_i^T (Z_i^T D_i Z_i)^{-1} x_i}{1 + x_i^T (Z_i^T D_i Z_i)^{-1} x_i}, \] (3.13)
where \( T_i = \sum_{i \neq i} x_i x_i^T \), and \( Z_i \) has \( (n-1) \) independent unit normal rows and \( D_i \) omits \( s_i \). Using a standard development of Hotelling’s \( T^2 \), let \( H \) be orthogonal with first row \( \frac{x_i}{\| x_i \|} \).

Then
\[ x_i^T T_i^{-1} x_i = (H x_i)^T \left\{ (Z_i^T H)^T D_i (Z_i^T H) \right\}^{-1} (H x_i) \]
\[ = \| x_i \|^2 \left\{ (Z_i^T H)^T D_i (Z_i^T H) \right\}_{11}^{-1} \]
\[ \sim \| x_i \|^2 \left( Z_i^T D_i Z_i \right)_{11}^{-1} \] (3.14)
(since \( Z_i H \sim Z_i \)). First note that \( \| x_i \|^2 = s_i \| z_i \|^2 \) and \( \| z_i \|^2 \sim \chi_p^2 \). Hence, for \( t \) small, using the Markov inequality (as in Proposition 3.2),
\[ P\left\{ \| z_i \|^2 \geq p + \sqrt{p} \log n \right\} \leq \exp \left\{ -t(p + \sqrt{p} \log n) - \frac{p}{2} \log(1 - 2t) \right\} \]
\[ \leq \exp \left\{ -tp - t \sqrt{p} \log n + tp + (1 + \epsilon) p t^2 \right\} \]
\[ \leq \exp \left\{ -\delta(\log n)^2 \right\} \]
for some \( \delta > 0 \) if \( t = \delta(\log n)\sqrt{p} \). Since a similar inequality holds for \( P\left\{ \| z_i \|^2 \leq p - \sqrt{p} \log n \right\} \), with probability at least \( 1 - q_i \), where \( n q_i \to 0 \).

Now partitioning \( (Z_i^T D_i Z_i) \) (in (3.14)),
\[ (Z_i^T D_i Z_i)_{11}^{-1} = \frac{1}{a - b'C^{-1}b}, \] (3.16)
where \( C \) is the lower \( (p-1) \times (p-1) \) corner and (with \( J = \{ k: k \neq i, k \neq 1 \} \))
\[ a = (Z_i^T D_i Z_i)_{11} = \Sigma_j z_i^2 s_i \]
\[ b_j = \Sigma_j z_i^2 z_i^2 s_i = z_i^2 s_i = z_i^2 D_i z_1 \quad (j = 1, \ldots, p - 1), \] (3.17)
where \( z_i \) denotes the \( j \)th column of \( Z_i \).
First, consider \( a \) in (3.17): as in (3.15), given \( \{s_i\} \) for \( t \) small enough,

\[
P[a \leq n - \sqrt{n \log n} \{s_i\}] 
\leq \exp\{t(n - \sqrt{n \log n}) - \frac{1}{2} \sum_j \log(1 + 2t s_i)\} 
\leq \exp\{t(n - \sqrt{n \log n}) - \sum_j s_i t + t^2 \sum_j s_i^2\}.
\tag{3.18}
\]

Now, since \( s_i \) is bounded and \( \mathbb{E} s_i = 1 \), for \( t \) small \( \mathbb{E} e^{-s_i t} \leq \exp\{-t + \alpha t^2\} \) (for some \( \alpha > 0 \)).

Thus, taking expectations in (3.18),

\[
P[a \leq n - \sqrt{n \log n}] \leq \exp\{ - \sqrt{n \log n} t + (\alpha + \max_i \{s_i^2\}) nt^2\} 
\leq \exp\{-\delta (\log n)^2\}
\]

for some \( \delta > 0 \) (if \( t = \delta' \log n/\sqrt{n} \)). A similar inequality holds for \( P(a \geq n + \sqrt{n \log n}) \). Hence, with probability at least \( 1 - q_1 \), where \( n q_1 \to 0 \)

\[
a = n \left( 1 + \mathcal{O}\left( \frac{\log n}{n} \right) \right) \quad \text{(uniformly in } l) \tag{3.19}\]

Now consider \( b' C^{-1} b \leq \|b\|^2 \lambda_{\max}(C^{-1}) = \|b\|^2 (\lambda_{\min}(C))^{-1} \). For \( \|b\|^2 \), from (3.17) given \( \{s_i, \tilde{s}_i, \frac{1}{n}\}, b_j \sim \mathcal{N}(0, \sigma^2) \), where \( \sigma^2 = \sum z_{n+j+1}^2 \sum_i s_i^2 \leq \max_i \{s_i^2\} \cdot \tilde{\chi}_{n-1}^2 \leq Bn \)

(with probability at least \( 1 - \exp\{-\delta n\} \), from Proposition 3.2) Thus (with probability at least \( 1 - 1/n^3 \) \( \|b_i\| \leq (Bn \log n)^{1/2} \) and \( \|b\|^2 \leq Bpn \log n \) (for some \( B > 0 \)). Last, by condition (X1), for appropriate matrices \( \tilde{Z} \) and \( \tilde{D} \),

\[
\lambda_{\min}(C) = \lambda_{\min}(\tilde{Z}' \tilde{D} \tilde{Z}) \geq \lambda_{\min}(\tilde{Z}' \tilde{Z}) \min_i \{s_i\} \geq bn
\]

with probability as above. Therefore, uniformly in \( l \)

\[
b' C^{-1} b \leq B^* p \log n \tag{3.20}
\]

with probability at least \( 1 - q_1 \), where \( n q_1 \to 0 \). Thus from (3.16), (3.19), and (3.20), with probability at least \( 1 - q_1 \), uniformly in \( l \),

\[
(Z_i' D_i Z_i)^{-1} = \frac{1}{n} \left( 1 + \mathcal{O}\left( \frac{\log n}{\sqrt{n}} + \frac{p \log n}{n} \right) \right). \tag{3.21}
\]

Hence, from (3.14), (3.15), and (3.21), with probability at least \( 1 - q_1 \).
\[
 x_i T_i^{-1} x_i = \frac{p}{n} \left(1 + O\left( \frac{\log n}{\sqrt{p}} \right) + \frac{\log n}{\sqrt{n}} \right)
\]
\[
 = \frac{p}{n} \left(1 + O\left( \frac{\log n}{\sqrt{p}} \right) \left(1 + \left( \frac{p}{n} \right)^{1/2} + \frac{p^{3/2}}{n} \right)\right)
\]
\[
 = \frac{p}{n} \left(1 + O\left( \frac{\log n}{\sqrt{p}} \right) \right).
\]

Thus, from (3.13), since \( nq \to 0 \), (3.11) holds.

Last, from (3.11) (since \( \Sigma s_i^2 \leq Bn \)), in probability
\[
\sum_{i=1}^{n} \| y_i \|^4 = \sum_{i=1}^{n} s_i^2 \frac{p^2}{n} \left(1 + O\left( \frac{\log n}{\sqrt{p}} \right) \right) = \frac{n^{p/2}}{p} \sum_{i=1}^{n} s_i^2 + O\left( \frac{p^{3/2} \log n}{n} \right).
\]

Thus, since \( \Sigma s_i^2 = nE s^2 + O_p(\sqrt{n}) \), (3.12) follows.

**Proposition 3.3.** Under condition (XD), if \( p^3(\log n)^2/n^2 \to 0 \) and \( p/\log n \to +\infty \), with probability tending to 1,
\[
\det(X'X)^{1/2} = (1 + o(1)) n^{p/2} \exp \left\{ - \frac{E s^2}{4} \frac{p}{n} + \frac{p}{2} (s - 1) \right\}
\]
\[
 = o(1) n^{p/2} \exp \{ ap^2 \log n/n \} \quad (3.22)
\]

for some constant \( a > 0 \) (where the second equality holds if \( p^2/n \to +\infty \)).

**Proof.** First, note that with \( Z \) as defined in the proof of Lemma 3.2 (i.i.d. \( N(0, I) \) rows),
\[
\det(X'X) = \det(Z'Z) \det(I + W'DW), \quad (3.23)
\]
where \( W = (Z'Z)^{-1/2} Z \) and \( D = \text{diag}(s_i - 1) \).

We now note that in probability
\[
\log \det(I + W'DW) = p(s - 1) - \frac{1}{4} E(s - 1)^2 \frac{p^2}{n} + o(1). \quad (3.24)
\]

This can be shown as follows: since \( W \) is distributed as \( Y \) defined with \( s_i \equiv 1 \), the proof of Lemma 3.2 of [7] yields
\[
\lambda_{\max}(W'DW) = O\left( \frac{p \log n}{n} \right)^{1/2} \quad (3.25)
\]
in probability. Thus, we have the expansion
\[
\log \det(I + W'DW) = D_1 + D_2 + D_3 + \sum_{m=4}^{\infty} D_m, \quad (3.26)
\]
where
\[
D_m = \frac{(-1)^{m+1}}{m} \text{tr}(W'DW)^m \quad (m = 1, 2, 3, \ldots).
\]

Now by (3.25), the error term in (3.26) is bounded in probability by
\[
Bp \sum_{m=4}^{\infty} \frac{1}{m} \left( \frac{\lambda_{\max}(W'DW)}{n} \right)^{m/2} = O \left( \frac{p^3 \log^2 n}{n^2} \right) \to 0.
\]

For $D_2$, conditioning on $W$ and letting $\tau^2 = E(s - 1)^2$,
\[
E D_2 = \sum_i \sum_{i \neq i} (w_i'w_i)^2 E(s_i - 1)(s_i - 1) = E(s - 1)^2 \sum \|w_i\|^4
\]
\[
\text{Var } D_2 = 2\tau^4 \sum \sum_{i \neq i} (w_i'w_i)^4 + (E(s - 1)^4 - \tau^4) \sum \|w_i\|^8
\]
\[
= O \left( n^2 \cdot \frac{p^2 \log^2 n}{n^4} \right) + O \left( n \cdot p^4 \right) \to 0
\]
(where Lemmas 3.1 and 3.2 have been used). Hence, in probability $D_2 = E(s - 1)^2 p^2/n + O(1)$ by Lemma 3.2. Using similar computations, $D_3 = O(1)$ and it remains to consider $D_1$. Again by Lemma 3.2, $\|w_i\|^2 = p/n + A_i$, where $A_i = O(\sqrt{p \log n/n})$ (uniformly in $i$ and independently of $\{s_i\}$). Hence, conditioning on $A_i$,
\[
D_1 = \sum_{i=1}^{n} \|w_i\|^2 (s_i - 1) = p(\bar{s} - 1) + \sum_{i=1}^{n} A_i (s_i - 1)
\]
\[
= p(\bar{s} - 1) + O \left( \sum_{i=1}^{n} A_i^2 \right)^{1/2}
\]
\[
= p(\bar{s} - 1) + O \left( \frac{p \log n}{n} \right) = p(\bar{s} - 1) + O(1)
\]
(in probability). Thus, (3.24) holds; and (in probability),
\[
det(I + W'DW)^{1/2} = \exp \left( \frac{P}{2} (\bar{s} - 1) - \frac{1}{4} (Es^2 - 1) \frac{p^2}{n} \right) (1 + O(1)). \quad (3.27)
\]
Last, to bound $\det(Z'Z)$, note that

$$\det Z'Z = \prod_{i=1}^{p} U_i = \exp \left\{ \sum_{i=1}^{p} \log U_i \right\}, \tag{3.28}$$

where $U_i$ are independent, $U_i \sim \chi^2_{n-i+1}$ $(i = 1, \ldots, p)$. Now define

$$R_i(t) = \log E e^{t \log U_i} = \log E U_i^t = t \log 2 + \log \Gamma \left( \frac{n-i+1}{2} + t \right) - \log \Gamma \left( \frac{n-i+1}{2} \right).$$

Thus, using standard expansions for the gamma and digamma functions (see Abramowitz and Stegun [1, pp. 257–59]),

$$E \log U_i - R_i(0) = \log(n - i + 1) - \frac{1}{n-i+1} + O \left( \frac{1}{n^2} \right),$$

$$\text{Var}(\log U_i) = \frac{2}{n+i-1} + O \left( \frac{1}{n^2} \right)$$

uniformly in $i$. Therefore

$$E \sum_{i=1}^{p} \log U_i = p \log n + \sum_{i=1}^{p} \log \left( 1 - \frac{i-1}{n} \right) + O \left( \frac{p}{n} \right)$$

$$= p \log n - \sum_{i=1}^{p} \left( \frac{i-1}{n} + O \left( \frac{p^2}{n^2} \right) \right) + O \left( \frac{p}{n} \right)$$

$$= p \log n - \frac{p^2}{2n} + O \left( \frac{p^3}{n^2} \right) + O \left( \frac{p}{n} \right)$$

$$\text{Var} \left( \sum_{i=1}^{p} \log U_i \right) = O \left( \frac{p}{n} \right) \to 0.$$  

Hence $\sum_{i=1}^{p} \log U_i = p \log n - p^2/2n + o(1)$ in probability. Thus, from (3.28),

$$\det(Z'Z)^{1/2} = \left( 1 + o(1) \right) n^{p/2} \exp \left\{ - \frac{p^2}{4n} \right\}. \tag{3.29}$$

Therefore, the first part of (3.22) follows from (3.27) and (3.29). The last part of (3.22) follows from the fact that $\bar{s} - 1 = O_p(1/\sqrt{n})$; and, thus, $\frac{1}{2} p (\bar{s} - 1) = O((p \log n) / \sqrt{n}) = O(p^2 \log n/n)$ if $p^2/n \to \infty$ with probability tending to 1.
Lemma 3.4. Assume condition (XD) holds, $p^3(\log n)^2/n^2 \to 0$ and $p^2/n \to +\infty$. Then condition (X4) holds in probability.

Proof. First, transform variables in the integral in (X4) (Eq. (1.14)) using $w = (X'X)^{-1/2}u$. Then $y_i'w = x_i'w$ and (by condition (X1)), for some $B_1$, if $\|u\|^2 \leq B_1p \log n$ then $\|w\|^2 \leq B_1p \log n/n$. Hence, the integral in (X4) is bounded above by

$$\det(X'X)^{1/2} \int_{\|w\|^2 \leq B_1p \log n/n} I(\max_i \|x_i'w\| > \delta_n) \times \exp\left\{-\frac{1}{2} \sum_{i \neq l} I((x_i'w)^2 \leq B_0 \frac{p}{n} \log^2 n) \times (x_i'w)^2\right\} \, dw. \quad (3.30)$$

Thus, by Proposition 3.3 (Eq. (3.22)), it suffices to show that the integral in (3.30) is $O((2\pi/n)^{p/2} \exp\{-B^*(p^2 \log^2 n/n)\}$.

Now note that since $p^2/n \to +\infty$, $\delta^2_1 > B_0p \log^2 n/n$ (at least for $n$ large enough). Hence, if $A_4$ denotes the conditional expectation given $\{s_i\}$ of the integral in (3.30),

$$M \leq E \sum_{i=1}^n \int_{\|w\|^2 \leq B_1p \log n/n} I(\|x_i'w\| > \delta_n) \times \exp\left\{-\frac{1}{2} \sum_{i \neq l} I((x_i'w)^2 \leq B_0 \frac{p}{n} \log^2 n) \times (x_i'w)^2\right\} \, dw$$

$$= \sum_{i=1}^n \int_{\|w\|^2 \leq B_1p \log n/n} P(\|x_i'w\| > \delta_n) \prod_{i \neq l} \exp\left\{-\frac{1}{2} \left((x_i'w)^2 \leq B_0 \frac{p}{n} \log^2 n \right) (x_i'w)^2\right\} \, dw. \quad (3.31)$$

Given $\{s_i\}$, $x_i'w \sim N(0, s_i, \|w\|^2)$ and $\|w\|^2 \leq B_1p \log n/n$; hence for some $\bar{B}'$ (since $s_i$ are bounded),

$$P(\|x_i'w\| > \delta_n) \leq P \left\{ |Z| \geq \left( \bar{B}' \frac{p^2}{n} \log^2 n \right)^{1/2} \right\} \leq \frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{\bar{B}' \frac{p^2}{2n} \log^2 n}{2} \right\}. \quad (3.32)$$

The expectation in (3.31) can be bounded by

$$P \left\{ \|x_i'w\| > \left( B_0 \frac{p}{n} \log^2 n \right)^{1/2} \right\} + E \exp\left\{-\frac{1}{2} (x_i'w)^2\right\} \leq \frac{2}{\sqrt{2\pi}} \exp\left\{ -\frac{B_0}{2B} \log n \right\} + (1 + s_i \|w\|^2)^{-1/2}. \quad (3.33)$$
If $B_0$ is large enough so that $B_0^2/2B > 2$, the first term is bounded by $1/n^2$ (for $n$ large). The product in (3.31) sums (3.33) over $i$, yielding the bound on $M$ (using (3.32)),

$$M \leq \sum_{i=1}^{n} \int_{\|\omega\|^2 < B \log n/n}^{*} \frac{2}{\sqrt{2\pi} n} \exp \left\{ - \frac{\tilde{B}^* p^2}{2 n} \log^2 n \right\} \times \exp \left\{ - \frac{1}{2} \sum_{i \neq i} \log(1 + s_i \|w\|^2) + \frac{1}{n} \right\} \, dw$$

$$\leq \sum_{i=1}^{n} \frac{2}{\sqrt{2\pi} n} \exp \left\{ - \frac{\tilde{B}^* p^2}{2 n} \log^2 n \right\} \times \int_{\|\omega\|^2 < B \log n/n}^{*} \exp \left\{ - \frac{n-1}{2} \tilde{s}_i \|w\|^2 + B_n \frac{p^2}{n} \right\} \, dw$$

(3.34)

for some constant $B_1$, where $\tilde{s}_i$ is the mean of $\{s_i\}$ omitting $s_i$. By the central limit theorem, $\tilde{s}_i = 1 + O_p(1/\sqrt{n})$. Thus, from (3.34)

$$M \leq \exp \left\{ - \tilde{B}^* \frac{p^2}{n} \log^2 n \right\} \left[ \exp \left\{ - \frac{n}{2} \|w\|^2 \right\} \right] \, dw$$

$$= \exp \left\{ - \tilde{B}^* \frac{p^2}{n} \log^2 n \right\} \cdot \left( \frac{2\pi}{n} \right)^{n/2}$$

(3.35)

in probability (under the distribution of $\{s_i\}$) where $\tilde{B}^*$ absorbs the other factors and is positive if $\tilde{B}^*$ is sufficiently large. Therefore, the result follows using a first moment Chebychev inequality. \[ \Box \]

**Lemma 3.5.** Assume condition (XD) holds and suppose $p^3 \log^2 n/n^2 \to 0$ while $p^2/n \to +\infty$. Then condition (X5) holds in probability.

**Proof.** Let $T_n$ denote the expectation in (X5) (Eq. (1.15)) and define

$$q_i(w, v) = -\frac{4}{5}(x'_i w)^2 - \frac{1}{2}(x'_i v)^2 + h(x'_i w)^4 - 6h(x'_i w)^2(x'_i v)^2 + b(x'_i v)^4$$

$$+ C \sum_{j=4}^{9} (x'_i v)^{j-3}(x'_i w)^{9-j}$$

$$S_n = \int_{A} \int_{A} I(\max_i (x'_i w)^2 \leq \varepsilon_n) I(\max_i (x'_i v)^2 \leq \delta_n)$$

$$\times \exp \{ \sum q_i(w, v) \} \, dv \, dw,$$  
(3.36)

where $A = \{u: u'(X'X) u \leq Bp \log n\}$, $\varepsilon_n = Bp \log n/n$ and $\delta_n$ is given by
(2.5). Then, by changing variables to \( W = (X'X)^{-1/2}Z \) and \( V = (X'X)^{-1/2}U \) in (1.15) and using Proposition 3.3,

\[
T_n = (1 + o(1)) \left( \frac{n}{2\pi} \right)^p \exp \left\{ p(\bar{s} - 1) - \frac{1}{2} Es^2 \frac{p^2}{n} \right\} S_n. \tag{3.37}
\]

By Condition (X1), if \( S_n^* \) is defined by (3.36) with \( A \) replaced by \( A^* = \{ u: ||u||^2 \leq B* p \log n/n \} \), it suffices to show that the corresponding \( T_n^* \to^p 1 \) for any sufficiently large values of \( B \) and \( B^* \).

Now let \( E_s \) denote conditional expectation given \( \{s_i\} \). Then, since \( \{s_i\} \) are conditionally independent,

\[
E_s S_n^* = \int_{A^*} \int_{A^*} \prod_{i=1}^n E_s(I((x_i'w)^2 \leq \epsilon_s) \wedge ((x_i'v)^2 \leq \delta_s)) \times \exp \{ q_s(w, v) \} \, dv \, dw
\]

\[
= \int_{A^*} \int_{A^*} \prod_{i=1}^n E_s(I((x_i'w)^2 \leq \epsilon_s, (x_i'v)^2 \leq \delta_s)) \left\{ 1 + q_s(w, v) \right\} \left[ 1 + 2(x_i'w)^4 + \frac{1}{8} (x_i'v)^4 + \cdots \right] \, dv \, dw. \tag{3.38}
\]

Now \( E_s(I(C)(x_i'u)^k) = E_s(x_i'u)^k - EI(C')(x_i'u)^k \); and with \( C = \{(x_i'w)^2 \leq \epsilon_s, (x_i'v)^2 \leq \delta_s\} \),

\[
E_s I(C')(x_i'u)^k \leq \left( E_s I^2(C') \right)^{1/2} \left\{ E_s(x_i'u)^{2k} \right\}^{1/2}
\]

\[
\leq B_0 \left( P\left( (x_i'w)^2 \geq \epsilon_s \right) + P\left( (x_i'v)^2 \geq \delta_s \right) \right)^{1/2} = o \left( \frac{1}{n} \right),
\]

where the last step involves a simple bound on normal tail probabilities. Thus, using the computations \( E_s(x_i'u)^2 = s, ||u||^2, E_s(x_i'u)^4 = 3 ||u||^4 s^2 \), and \( E_s(x_i'w)^2(x_i'v)^2 = s_t^2 ||w||^2 ||v||^2 + 2w'v)^2 \), (3.38) yields

\[
E_s S_n^* = \int_{A^*} \int_{A^*} \prod_{i=1}^n \left\{ 1 - \frac{1}{2} s_t ||w||^2 - \frac{1}{2} s_t ||v||^2 + 3s_t^2 (b + \frac{1}{8}) ||w||^4
\]

\[
+ 3s_t^2 (b + \frac{1}{8}) ||v||^4 - 6bs_t^2 (||w||^2 ||v||^2 + 2w'v)^2) + O(p/n^3)
\]

\[
+ o \left( \frac{1}{n} \right) \right\} \, dv \, dw.
\]
\[
(1 + o(1)) \int_{A^*} \int_{A^*} \exp \left\{ \sum_{i=1}^{n} \left[ -\frac{1}{2} s_i (\|w\|^2 + \|v\|^2) + (3b + 4) s_i^2 (\|w\|^4 + \|v\|^4) - 6bs_i^2 (\|w\|^2 \|v\|^2 + 2(wv)^2) \right] \right\} \, dv \, dw
\]

\[
= (1 + o(1)) \int_{A^*} \int_{A^*} \exp \left\{ -\frac{1}{2} n\tilde{s} (\|w\|^2 + \|v\|^2) + (3b + 4) \Sigma s_i^2 (\|w\|^4 + \|v\|^4) - 6b \Sigma s_i^2 (\|w\|^2 \|v\|^2 + 2(wv)^2) \right\} \, dv \, dw. \tag{3.39}
\]

Now Proposition 3.4 and some calculation yields

\[
E_{s} S_{n}^* = (1 + o(1)) \left( \frac{2\pi}{n\tilde{s}} \right)^p \exp \left\{ \frac{\Sigma s_i^2 p^2}{2\Sigma_{n}^2 n^2} \right\} \tag{3.40}
\]

(where \( o(1) \) is uniform in \( \{s_i\} \)). Since \( \tilde{s} = 1 + \mathcal{O}(1/\sqrt{n}) \) and \( \Sigma s_i^2 = n(Es^2 + o_p(1)) \) and since \( \{s_i\} \) are bounded above and below, (3.40) and (3.37) imply that \( ET_{n}^* \to 1 \) as \( n \to +\infty \).

In a similar manner, compute \( E_s S_{n}^{*2} \).

\[
E_{s} S_{n}^{*2} = \int_{A^*} \int_{A^*} \int_{A^*} \int_{A^*} \prod_{i=1}^{n} E_i \left[ 1 + q_i(w, v) + q_i(\tilde{w}, \tilde{v}) \right]
\]

\[
+ \frac{1}{4} \left[ -\frac{1}{2}(x'_i w)^2 - \frac{1}{2}(x'_i v)^2 - \frac{1}{2}(x'_i \tilde{w})^2 - \frac{1}{2}(x'_i \tilde{v})^2 \right]^2 + \cdots \right] \times dv \, dw \, d\tilde{v} \, d\tilde{w}
\]

\[
= (1 + o(1)) \int_{A^*} \int_{A^*} \int_{A^*} \int_{A^*} \exp \left\{ \sum_{i=1}^{n} \left[ E_{s} q_i(w, v) + E_{s} q_i(\tilde{w}, \tilde{v}) \right] \right. 
\]

\[
+ \frac{s_i^2}{4} (\|w\|^4 + \|\tilde{w}\|^4 + \|v\|^4 + \|\tilde{v}\|^4) \right\} \, dv \, dw \, d\tilde{v} \, d\tilde{w}, \tag{3.41}
\]

where all the cross-product terms exactly cancel. Thus, comparing (3.41) and (3.39), it is clear that \( E_s S_{n}^{*2} = (E_s S_{n}^*)^2 (1 + o(1)) \), and (again using Proposition 3.4) it can be shown that \( ET_{n}^{*2} \to 1 \) also. Therefore, \( \text{Var} \ T_{n}^* \to 0 \) and \( T_{n}^* \to^d 1 \).

**Proposition 3.4.** Let \( U \sim \mathcal{N}_p(0, I) \). Then, if \( p^{3/2}/n \to 0 \),

\[
EI(\|U\|^2 \leq Bp \log n) \exp \left\{ \frac{c}{n} \|U\|^4 \right\} = (1 + o(1)) \exp \left\{ \frac{c p^2}{n} \right\}. \tag{3.42}
\]
and uniformly in \( \|w\|_2 \leq Bp \),

\[
E I(\|U\|_2 \leq Bp \log n) \exp \left\{ \frac{c}{n} \|U\|^4 + \frac{d}{n} (w'U)^2 \right\} = (1 + o(1)) \exp \left\{ \frac{c p^2}{n} \right\}.
\]

**Proof.** First, using Markov’s inequality (as in Proposition 3.2) it can be shown that for \( \delta < \frac{1}{4} \) and \( p \) large enough

\[
P\{ \|U\|_2 \geq p + p^\alpha \} \leq \exp\{ -\delta p^{2\alpha - 1} \}
\]

for any \( \alpha \in \left( \frac{1}{2}, 1 \right) \). Now define sets

\[
A_1 = \{ p + p^{3/4} \leq \|U\|_2 \leq Bp \log n \}, \quad A_2 = \{ p + p^{5/8} \leq \|U\|_2 \leq p + p^{3/4} \}, \\
A_3 = \{ p - p^{5/8} \leq \|U\|_2 \leq p + p^{5/8} \}, \quad A_4 = \{ 0 \leq \|U\|_2 \leq p - p^{5/8} \}
\]

and let \( I_i \) be the indicator function of \( A_i \) \((i = 1, 2, 3, 4)\). Then

\[
E I_1 \exp \left( \frac{c}{n} \|U\|^4 \right) \leq \exp \left\{ \frac{c B^2 p^2 \log^2 n}{n} \right\} P(A_1) \leq \exp \left\{ c B^2 p^2 \log^2 n - \delta p^{1/2} \right\} = \exp \left\{ \frac{c p^2}{n} \right\} \exp\{ -\delta \sqrt{p(1 - B^* p^{3/2} \log^2 n/n)} \} = \exp \left\{ \frac{c p^2}{n} \right\} \cdot o(1) \quad (3.45)
\]

\[
E I_2 \exp \left( \frac{c}{n} \|U\|^4 \right) \leq \exp \left\{ c \left( p + p^{3/4} \right)^2 - \delta p^{1/4} \right\} = \exp \left\{ \frac{c p^2}{n} \right\} \exp\{ -\delta p^{1/4} \left( 1 - \frac{B^* p^{3/2}}{n} \right) \} = \exp \left\{ \frac{c p^2}{n} \right\} \cdot o(1) \quad (3.46)
\]

\[
E I_4 \exp \left( \frac{c}{n} \|U\|^4 \right) \leq \exp \left\{ \frac{c p^2}{n} \right\} P\{ \|U\|_2 \leq p - p^{5/8} \} = \exp \left\{ \frac{c p^2}{n} \right\} \cdot o(1) \quad (3.47)
\]
For the expectation over $A_3$, consider $\|U\|^2$ as a sum of $p \chi_i^2$ variables and use a local limit theorem (Richter [8]) with $\|U\|^2 = p + \sqrt{2p} x$,

$$EI_3 \exp \left\{ \frac{c}{n} \|U\|^4 \right\}$$

$$= (1 + o(1)) \int_{-\rho^{1/2}/\sqrt{2}}^{\rho^{1/2}/\sqrt{2}} \phi(x) \exp \left\{ \frac{c}{n} (p + \sqrt{2p} x)^2 \right\} dx$$

$$= (1 + o(1)) \int_{-\rho^{1/2}/\sqrt{2}}^{\rho^{1/2}/\sqrt{2}} \phi(x) \exp \left\{ \frac{2cpx^2}{n} + \frac{c \sqrt{2p}}{n} x \right\} dx \cdot e^{c\rho^2/n}$$

$$\leq (1 + o(1)) \exp \left\{ \frac{p^2}{n} \right\} \tag{3.48}$$

(using straightforward calculations for the normal integral). Combining (3.45), (3.46), (3.47), and (3.48) immediately yields (3.42).

To obtain (3.43), note that the integral can be written

$$\int_{\|u\|^2 \leq Bp} \exp \left\{ -\frac{1}{2} u^t \left( I + \frac{a}{n} ww' \right) u + \frac{c}{n} \|u\|^4 \right\} du$$

$$= \det \left( \frac{a}{n} ww' + I \right)^{-1/2}$$

$$\times \int_{\|v\|^2 \leq Bp} \exp \left\{ -\frac{1}{2} \|v\|^2 + \frac{c}{n} \left( v^t \left( \frac{a}{n} ww' + I \right)^{-1} v \right) \right\} dv. \tag{3.49}$$

Using the facts that $\det(I + bww') = (1 + b\|w\|^2)$, $(I + bww')^{-1} = (I - bww'/(1 + b\|w\|^2))$, and $1 + (a/n)\|w\|^2 = 1 + \mathcal{O}(p/n)$, (3.49) can be written

$$(1 + \mathcal{O} \left( \frac{p}{n} \right)) \ EI(\|U\|^2 \leq Bp) \exp \left\{ \frac{c}{n} \|U\|^4 \left( 1 + \mathcal{O} \left( \frac{p}{n} \right) \right) \right\}$$

$$= (1 + o(1)) \ EI(\|U\|^2 \leq Bp) \exp \left\{ \frac{c}{n} \|U\|^4 + \mathcal{O} \left( \frac{p^2}{n^2} \right) \right\}$$

$$= (1 + o(1)) \ EI(\|U\|^2 \leq Bp) \exp \left\{ \frac{c}{n} \|U\|^4 \right\}$$

and (3.43) follows from (3.42). \qed

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