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On a Mutation Problem for Oriented Matroids

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For uniform oriented matroids $\mathcal{M}$ with n elements, there is in the realizable case a sharp lower bound $L_r(n)$ for the number $\text{mut}(\mathcal{M})$ of mutations of $\mathcal{M}: L_r(n) = n \leq \text{mut}(\mathcal{M})$, see Shannon [17]. Finding a sharp lower bound $L(n) \leq \text{mut}(\mathcal{M})$ in the non-realizable case is an open problem for rank $d \geq 4$. Las Vergnas [9] conjectured that $1 \leq L(n)$. We study in this article the rank 4 case. Richter-Gebert [11] showed that $L(4k) \leq 3k+1$ for $k \geq 2$. We confirm Las Vergnas' conjecture for $n < 13$. We show that $L(7k+c) \leq 5k+c$ for all integers $k \geq 0$ and $c \geq 4$, and we provide a 17 element example with a mutation free element.

1. INTRODUCTION

We assume the reader is familiar with basic concepts of the theory of oriented matroids [1]. The Folkman–Lawrence representation theorem provides a bijective map between reorientation classes of uniform oriented matroids of rank $d+1$ and equivalence classes (with respect to homeomorphic transformations of the projective d-space P^d) of non-degenerated pseudo-hyperplane arrangements. We make the general assumption in this article that we study the uniform rank 4 case, i.e., in the pseudoplane arrangements not more than three pseudoplanes meet in a point of the projective 3-space P^3. For the set of arrangements in P^3 with n pseudoplanes, or n planes, we denote by $L(n)$, or $L_r(n)$, the minimal number of its simplicial cells, respectively. We recall Shannon's result, $L_r(n) = n$, with a short proof which works in the same way for arbitrary rank.

THEOREM 1.1 (SHANNON [17]). *For a uniform realizable oriented matroid $\mathcal{M}$ with $n > 4$ elements:* (1) *each element is incident with at least four mutations and* (2) *the number of mutations is at least n.*

PROOF. $\mathcal{M}$ can be represented by n points in general position in P^3. We delete point i. The subsets of three points define a plane arrangement $\mathcal{A}_i$ with $\binom{n-1}{3}$ planes. The cell containing the ith point has at least four facets generated by pairwise different 3-tuples of points. By inserting point i and moving it across the four planes, we confirm (1). The total number of incidences of a point i with a mutation is at least $4n$. Each mutation was counted four times, which implies (2). The alternating oriented matroid shows that this bound is sharp. □

In this article we investigate the non-realizable case and in particular reorientation classes of oriented matroids with n elements with the property that either condition (1), or condition (2), of Theorem 1.1 is violated. Las Vergnas conjectured that $1 \leq L(n)$, $\forall n \geq 4$, in [9]. We confirm this conjecture in Theorem 2.1 for $n < 13$. It is known that $L(n) = n \; \forall n \leq 7$, $L(8) = 7$ (see [6]), and $L(12) \leq 10$ (see [11]).

In Theorem 4.2 we show that $L(9) = 8$ and in Theorem 4.3 $2 \leq L(10) \leq 9$ and $2 \leq L(11) \leq 9$.

Finally, in Theorem 5.4, we show that $L(7k+c) \leq 5k+c$ for all integers $k \geq 0$ and $c \geq 4$, and we provide a 17 element example with a mutation free element.

2. CONJECTURE OF LAS VERGNAS FOR A SMALL NUMBER OF ELEMENTS

The following result has been mentioned in [2], but a proof has not been published so far.

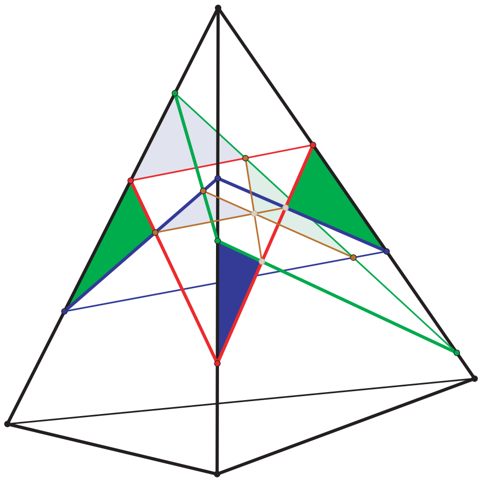

FIGURE 1. A mutation free simplex.

THEOREM 2.1 (BOKOWSKI). *Las Vergnas' conjecture, i.e.,* $1 \leq L(n)$*, is true for* $n < 13$.

PROOF. We assume that there is an oriented matroid $\mathcal{M}$ without mutations with minimal number n of elements. We consider a corresponding Folkman–Lawrence representation, i.e., an arrangement of n pseudoplanes in projective 3-space. After deleting the ith element, we obtain at least one simplicial cell C_i in the arrangement, by minimality of $\mathcal{M}$. The ith element cuts C_i into two triangular prisms. Take such a prism with triangles t_1, t_2 supported by the elements 1, 2, and with 4-gons formed by the elements 4, 5, 6 and denote it by $P_{4,5,6}$. The point of intersection $Y_{4,5,6}$ of the pseudoplanes 4, 5, 6 and the triangles t_1, t_2 define simplicial regions R_1 and R_2 in the cell decomposition of the projective 3-space. Each triangle t_i, $i \in \{1, 2\}$ must be separated from point $Y_{4,5,6}$ within R_i by at least three elements $4 + 3 \cdot i$, $5 + 3 \cdot i$, $6 + 3 \cdot i$. If there are exactly three such elements, the corresponding combinatorial structure of the cell decomposition of R_i, containing no simplicial cell, is unique up to a mirror image, compare Figure 1.

Hence, a minimal example must contain at least $5 + 2 \cdot 3 = 11$ elements. A minimal example containing only 11 or 12 elements must contain a pair of adjacent triangular prisms ATP (glued along a triangle) of Figure 1. We could have started our investigation with such a cell ATP instead of C_i, and hence a minimal example must contain at least $6 + 2 \cdot 3 = 12$ elements. We now assume that $n = 12$, and we denote the starting cell by $ATP_{4,5,6}$. In the simplicial regions R_1 and R_2 we find altogether six adjacent triangular prisms, each of which we call a *child* of $ATP_{4,5,6}$. Using these adjacent triangular prisms as starting cells yield a further six adjacent triangular prisms in each case, etc. Consider all adjacent triangular prisms $ATP_{i,j,k}$ (glued along a triangle) in the cell decomposition of the projective 3-space. We define a directed graph (it might have several connected components) having all adjacent triangular prisms $ATP_{i,j,k}$ as its vertices and a directed edge $(ATP_{i,j,k}, ATP_{o,p,q})$ whenever $ATP_{o,p,q}$ is a child of $ATP_{i,j,k}$ The number of edges in this graph going out from a point is always six. However, the number of edges going to a particular pair of adjacent triangular prisms cannot exceed three, e.g., $ATP_{4,5,6}$ can only be a child of $ATP_{1,2,4}$, $ATP_{1,2,5}$, or $ATP_{1,2,6}$, compare again Figure 1. This contradiction implies that a mutation free example must have at least 13 elements. □

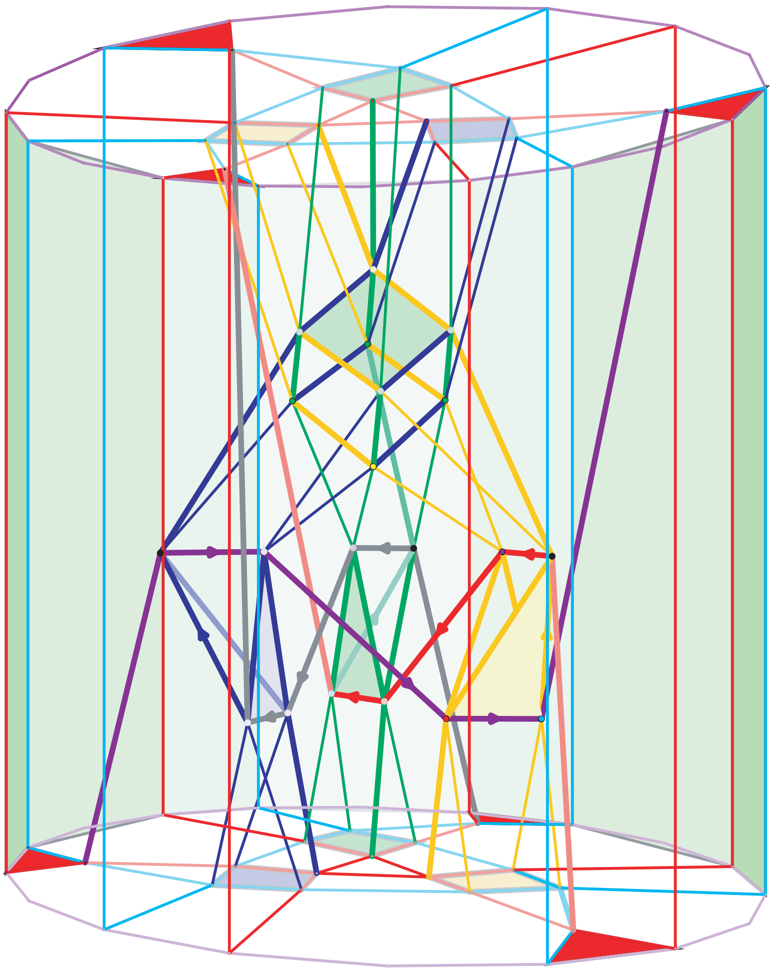

FIGURE 2. Folkman–Lawrence representation, seven mutations, eight elements.

3. THE SEVEN-MUTATION EXAMPLE WITH EIGHT ELEMENTS

We look for oriented matroids with n elements in which $\operatorname{mut}(\mathcal{M}) < n$. The smallest reorientation class with this property is the seven-mutation example $X(8)$ which is unique within the set of reorientation classes with eight elements, compare [6]. It is a minor of all known examples for which the number of mutations is smaller than the number of elements. In particular, it appears as a minor not only in the infinite class of Richter-Gebert, but also in our infinite classes of examples. We have depicted the Folkman–Lawrence representation of this example in affine 3-space in Figure 2.

The two elements appearing as the bottom and the lid are parallel (inseparable) elements. Their boundaries have to be identified (they meet at infinity). We see the inner mutation incident with the lid and an additional three mutations below. The remaining three mutations are incident with the bottom and the lid at the dark triangles. The arrows mark a cyclic component. The understanding of this structure was decisive for our findings, namely Theorem 5.4.

4. Small Numbers of Elements and Mutations

The overview for our mutation problem for $n = 8$ was done in [6] with the result that the reorientation class with seven mutations is the unique smallest example within this class of eight elements.

Theorem 4.1 (Bokowski and Richter-Gebert [6]). $L(8) = 7$.

A fast algorithm for the inductive generation of oriented matroids has been described in [5]. We have used this method and a modified version of the corresponding C++ program to generate all reorientation classes of oriented matroids with a small number of elements for which there is an element that is incident with less than four mutations (condition (1) in Theorem 1.1). This set of reorientation classes contains in particular those examples having less mutations than elements (condition (2) in Theorem 1.1).

Within the class with nine elements we found precisely five reorientation classes with less than nine mutations, namely with eight mutations. All these five reorientation classes are extensions of $X(8)$.

Theorem 4.2. $L(9) = 8$.

There are altogether 9 276 595 reorientation classes with nine elements. Precisely 650 of these reorientation classes have an element which is incident with less than four mutations (condition (1) in Theorem 1.1). They are all non-Euclidean examples.

A similar overview for 10 elements would require several CPU years, so we only looked at all extensions with up to 10 elements of the interesting example $X(8)$. Within this subclass there are 179 reorientation classes with $\mathrm{mut}(\mathcal{M}) < 10$. All of them have nine mutations and a minor with nine elements and eight mutations.

Within the class of 11 elements we only considered those being extensions of one of the above-mentioned 179 reorientation classes with less than 10 mutations. Here we found precisely two reorientation classes with 11 elements and only nine mutations. Figures 3 and 4 show the rank 3 contractions of these interesting examples $X(11, 9)_a$ and $X(11, 9)_b$ generated by the omawin software. Omawin can be obtained from [4]. Both reorientation classes have a symmetry generated by the permutation $(1, 2, 3)(4, 5, 6)(7, 8, 9)$.

We summarize our findings for $10 \le n \le 12$.

Theorem 4.3. $2 \le L(10) \le 9, \quad 2 \le L(11) \le 9, \quad 1 \le L(12) \le 10.$

The lower bound for $L(12)$ is due to Theorem 2.1. For $n = 10$ and $n = 11$ it can be improved by using refined arguments similar to those in the proof of Theorem 2.1. Assume that $L(10) = 1$. Since $L(9) = 8$, we can assume that there exists an element cutting a simplicial cell into two triangular prisms. In the restriction of the Folkman–Lawrence arrangement to the five elements involved, we have two pairs of simplicial regions (R_1, R_2) and (R'_1, R'_2). The argument of the proof of Theorem 2.1 shows that $2 \le L(10)$. Now we assume that $L(11) = 1$. Since $2 \le L(10)$, we can assume that there exists an element cutting a simplicial cell into two triangular prisms. In the restriction of the Folkman–Lawrence arrangement to the five elements involved, we have two pairs of simplicial regions (R_1, R_2) and (R'_1, R'_2). Either we have an additional simplicial region or the cell decomposition has the structure of Figure 1 with two pairs of adjacent triangular prisms. In this case we can look at the Folkman–Lawrence arrangement restricted to the six elements defining two pairs of adjacent triangular prisms. Using the above argument again on the two pairs of simplicial regions at the ends of these ATPs shows that $2 \le L(11)$. The upper bound for 12 elements can be obtained by an appropriate lexicographic extension of, e.g., example $X(11, 9)_a$, or we use an example of Richter-Gebert [11].

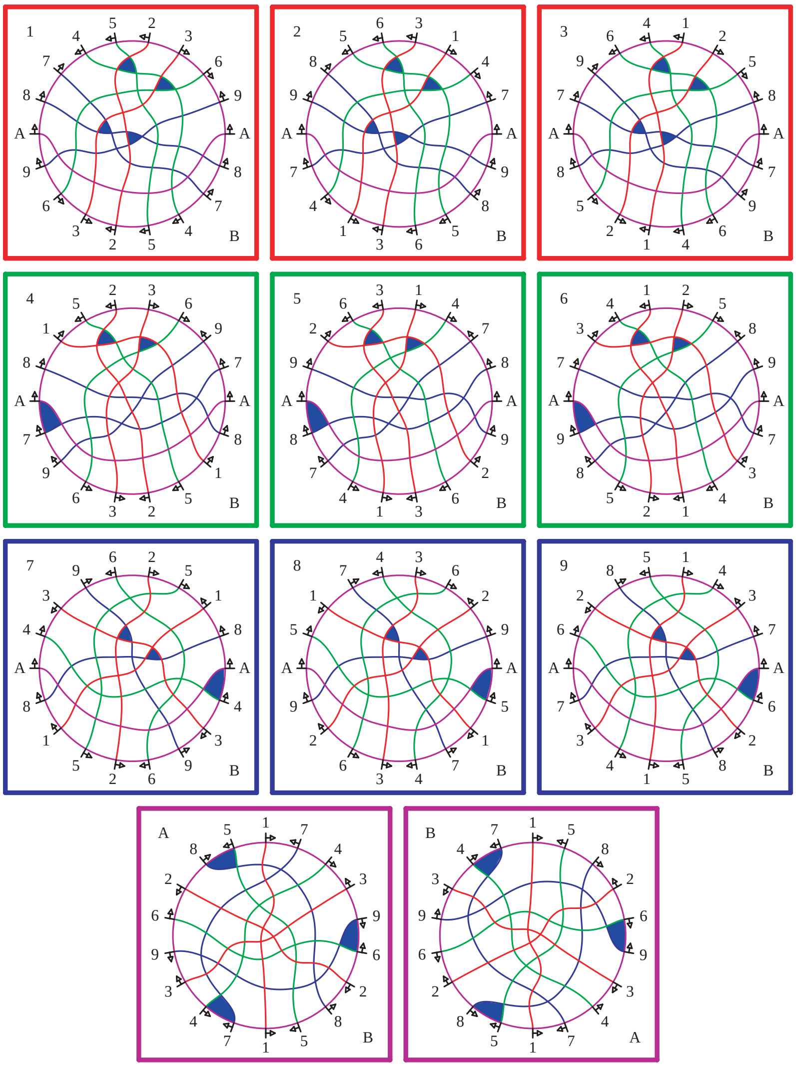

FIGURE 3. Rank 3 contractions, 11 elements and nine mutations.

4.1. The reorientation class $X(11, 9)_a$. In Figure 3 we have depicted all rank 3 contractions of the reorientation class $X(11, 9)_a$. The nine simplicial cells are listed as their facet elements:

$$(1, 2, 4, 5)\ (1, 2, 8, 9)\ (1, 3, 4, 6)\ (1, 3, 7, 8)\ (2, 3, 5, 6)$$
$$(2, 3, 7, 9)\ (4, 7, 10, 11)\ (5, 8, 10, 11)\ (6, 9, 10, 11).$$

Deleting the following 3-tuples of elements

$$(1, 2, 3)\ (7, 8, 10)\ (7, 8, 11)\ (7, 8, 9)\ (7, 9, 10)\ (7, 9, 11)\ (8, 9, 10)\ (8, 9, 11)$$

leads to the reorientation class $X(8)$.

We have one inseparable pair: $(10, 11)$.

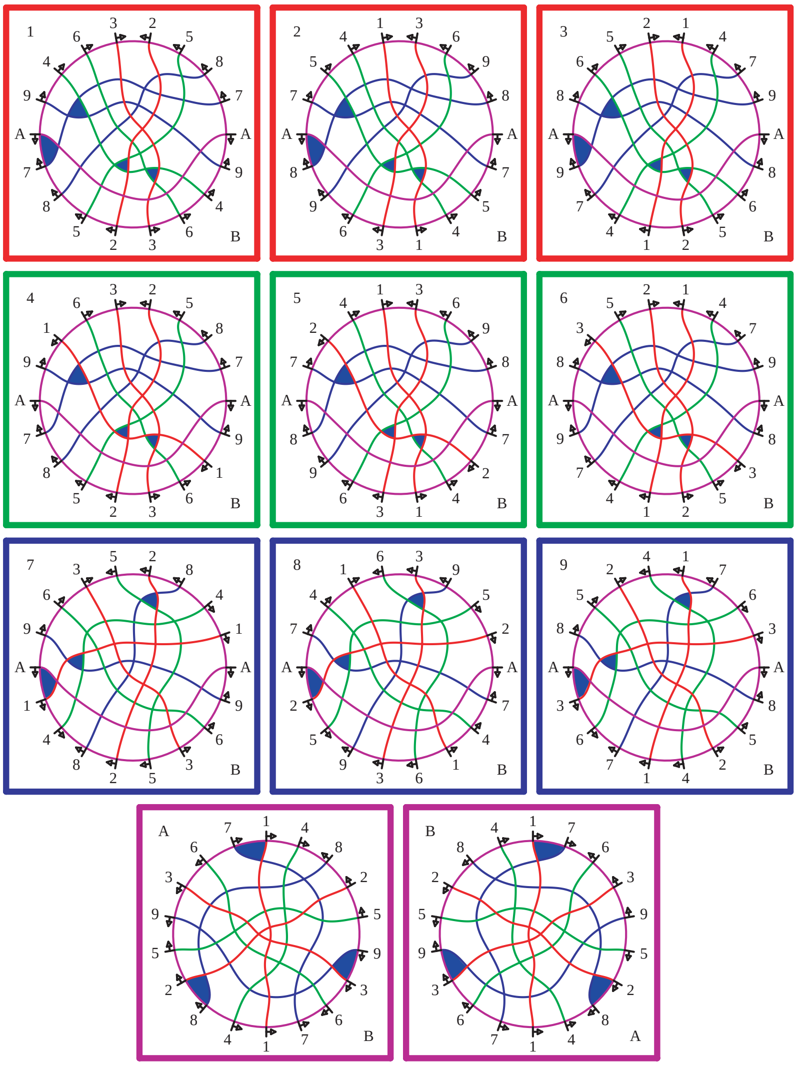

FIGURE 4. Rank 3 contractions, 11 elements and nine mutations.

4.2. The reorientation class $X(11, 9)_b$. In Figure 4 we have depicted all rank 3 contractions of the reorientation class $X(11, 9)_b$. The nine simplicial cells are listed as their facet elements.

$$(1, 2, 4, 5)\ (1, 3, 4, 6)\ (1, 4, 7, 9)\ (1, 7, 10, 11)\ (2, 3, 5, 6)$$
$$(2, 5, 7, 8)\ (2, 8, 10, 11)\ (3, 6, 8, 9)\ (3, 9, 10, 11).$$

Deleting the following 3-tuples of elements

$$(1, 2, 3)\ (1, 2, 6)\ (1, 3, 5)\ (1, 5, 6)\ (2, 3, 4)\ (2, 4, 6)\ (3, 4, 5)\ (4, 5, 6)$$
$$(7, 8, 10)\ (7, 8, 11)\ (7, 8, 9)\ (7, 9, 10)\ (7, 9, 11)\ (8, 9, 10)\ (8, 9, 11)$$

leads to the reorientation class $X(8)$.

We have four inseparable pairs: $(1, 4)\quad (2, 5)\quad (3, 6)\quad (10, 11)$.

5. Infinite Sequences

So far we have studied the mutation problem for a small number of elements. We now consider the general case and, in particular, what can be said about the asymptotic behavior when the number of elements tends to infinity. So far the best result in this direction was due to Richter-Gebert, $\lim_{n\to\infty} \frac{L(n)}{n} \leq \frac{3}{4}$. We do not copy the proof from [11] which would need several pages. We assume the reader is familiar with these results. Our result in this section, $\lim_{n\to\infty} \frac{L(n)}{n} \leq \frac{5}{7}$, uses these building blocks together with additional arguments. Theorem 5.1 follows from Theorem 5.2 by induction.

Theorem 5.1 (Richter-Gebert [11]). *$L(4k) \leq 3k + 1$ for all integers $k \geq 2$.*

Theorem 5.2 (Richter-Gebert [11]). *Given an oriented matroid $\mathcal{M}$ with $n \geq 5$ elements, a mutation $[1, 2, 3, 4]$, and an inseparable pair $(1, 2)$, there is an extension $\mathcal{M}_{(1,2,3,4)}$ of $\mathcal{M}$ with $n + 4$ elements and $\mathrm{mut}(M_{(1,2,3,4)}) = \mathrm{mut}(\mathcal{M}) + 3$.*

The idea behind Theorem 5.2 is the following. Take an affine arrangement of pseudoplanes representing $\mathcal{M}$ (with an arbitrary element $g > 4$ as the plane at infinity). Insert four new pseudoplanes $a_1, \ldots, a_4$ between the elements 1 and 2, so that the intersection of these six elements is the pseudoline $1 \cap 2$. A small deformation of these six elements can be used to obtain a uniform arrangement in which the restriction to the elements $\{1, 2, 3, 4, a_1, \ldots, a_4\}$ is isomorphic to $X(8)$, and all cocircuits formed by these elements are within the former cell $[1, 2, 3, 4]$.

Figure 2 shows the inner structure of the former mutation $[1, 2, 3, 4]$ after the extension. Bottom and lid represent the elements 3 and 4. The old mutation is replaced by four new ones.

By generalizing the above construction we obtain the following theorem.

Theorem 5.3. *Let $\mathcal{M}$, and $\mathcal{N}$ be oriented matroids with $m \geq 5$, and $n \geq 5$ elements, respectively, with the following properties:*

(i) $\mathcal{M}$ has a mutation $[1, 2, 3, 4]$ and an inseparable pair $(1, 2)$,
(ii) $\mathcal{N}$ has a mutation $[1, 2, 3, 4]$ and an inseparable pair $(3, 4)$,
(iii) in the pseudoplane 3 of $\mathcal{N}$, the line $3 \cap 4$ is incident with exactly three triangles.

Then there exists a uniform oriented matroid $\mathcal{M} \diamond \mathcal{N}$ with $m + n - 4$ elements and $|\mathrm{mut}(\mathcal{M})| + |\mathrm{mut}(\mathcal{N})| - 4$ mutations. If $\mathcal{M}$ or $\mathcal{N}$ has an inseparable pair (a, b) with $a, b \notin \{1, 2, 3, 4\}$, then there is an inseparable pair in $\mathcal{M} \diamond \mathcal{N}$.

The proof of Theorem 5.3 is similar to Theorem 5.2 (see [11, 14] for details). $\mathcal{N} \setminus \{1, 2, 3, 4\}$ plays the role of the elements $a_1 \ldots a_4$ inserted into the mutation $[1, 2, 3, 4]$ of $\mathcal{M}$. The elements 1, 2, 3, 4 of $\mathcal{M}$ and $\mathcal{N}$ are identified in $\mathcal{M} \diamond \mathcal{N}$.

Relabeling the elements of $X(11, 9)_b$ as follows leads to an oriented matroid $\mathcal{N}$ that fulfills the assumptions of Theorem 5.3.

$$3 \to 1, \qquad 9 \to 2, \qquad 10 \to 3, \qquad 11 \to 4.$$

This leads to the following theorem.

Theorem 5.4. *There are infinite sequences of oriented matroids showing that:*

(1) $L(7k + c) \leq 5k + c$ for all integers $k \geq 0$ and $c \geq 4$.
(2) $L(7k + c) \leq 5k + c - 1$ for all integers $k \geq 0$ and $c \geq 8$.

Proof. Take the alternating oriented matroid with c elements as $\mathcal{M}$ and take for $\mathcal{N}$ the oriented matroid described above. Applying Theorem 5.3 k times confirms (1). For (2) set $\mathcal{M} = X(8)$. □

6. A Mutation Free Element

We found a reorientation class $E(17)$ with 17 elements in which one element is not incident with any mutation. The smallest previously known example with this property had 20 elements and was described in [10, 11]. We present our new example as a list of rank 3 contractions in Figure 5. The example is interesting in the context of extension spaces of oriented matroids, see [16]. It leads to the smallest known (non-realizable) Lawrence polytope with triangulations which do not admit any flip.

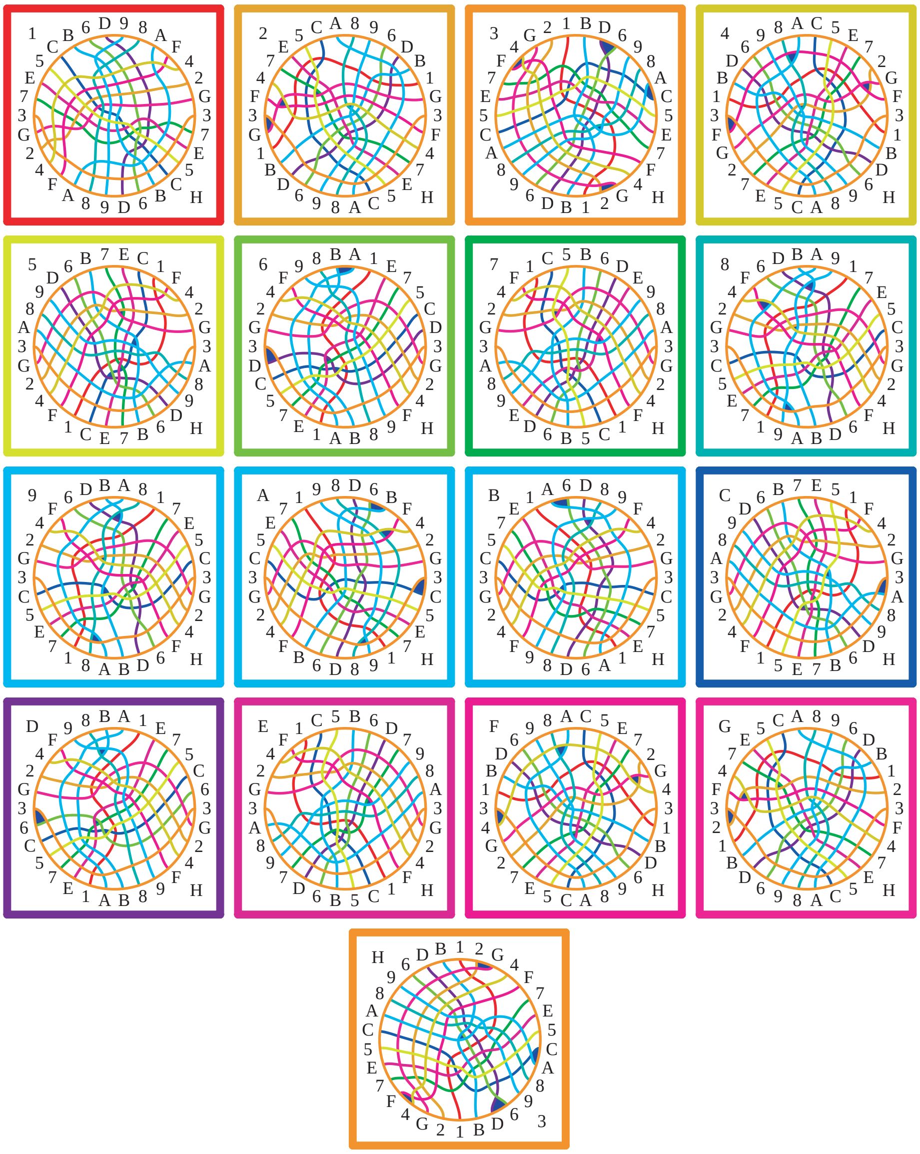

FIGURE 5. Reorientation class $E(17)$ with a mutation free element 1.

Consider the reorientation class $\mathcal{M} = X(11, 9)_b \setminus \{4, 5, 6\}$. It is isomorphic to an $X(8)$, with 'inner' mutation $[1, 2, 3, 10]$, which means 1, 2, 3, 10 are the unique four elements within $\mathcal{M}$ that are incident with four mutations. Element 10 in $X(11, 9)_b$ is incident with only three mutations; in other words, the insertion of the three new elements eliminated one of the mutations incident with 10. We obtain $X(8)$ from the unique reorientation class with five elements by the same extension.

Even though the construction does not work for arbitrary mutations, it could be applied to one of the oriented matroids which was found during the computer search. This example had 11 elements and one element incident with only two mutations. Applying the above extension twice led to the oriented matroid $E(17)$ with 17 elements, and a mutation free element.

7. Remarks

We do not know any example $\mathcal{M}$ with $\text{mut}(\mathcal{M}) < n$ that does not have $X(8)$ as a minor. We do not know any Euclidean example that contradicts condition (1) of Theorem 1.1.

The data of all oriented matroids mentioned in this paper are available from [13]. A list of all reorientation classes with $n \leq 9$ elements is available from the second author.

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