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Couple stress theory for solids

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ABSTRACT

By relying on the definition of admissible boundary conditions, the principle of virtual work and some kinematical considerations, we establish the skew-symmetric character of the couple-stress tensor in size-dependent continuum representations of matter. This fundamental result, which is independent of the material behavior, resolves all difficulties in developing a consistent couple stress theory. We then develop the corresponding size-dependent theory of small deformations in elastic bodies, including the energy and constitutive relations, displacement formulations, the uniqueness theorem for the corresponding boundary value problem and the reciprocal theorem for linear elasticity theory. Next, we consider the more restrictive case of isotropic materials and present general solutions for two-dimensional problems based on stress functions and for problems of anti-plane deformation. Finally, we examine several boundary value problems within this consistent size-dependent theory of elasticity.

1. Introduction

Classical continuum mechanics is an approximation based on the assumption that matter is continuously distributed throughout the body. This theory provides a reasonable basis for analyzing the behavior of materials at the macro-scale, where the microstructure size-dependency can be neglected. Experiments show, however, that the mechanical behavior of materials in small scales is different from their behavior at macro-scales. Any attempt to drop the continuity assumption in a modified theory is bound to make the analysis extremely difficult and computationally intensive. Therefore, we need to develop a consistent size-dependent continuum mechanics, which accounts for the microstructure of materials. This theory must span many scales and, of course, reduce to classical continuum mechanics for macro-scale size problems.

New measures of deformation, which are length related, such as the curvature tensor, are needed in a more complete continuum theory. As a consequence, such a theory will also require the introduction of couple-stresses. The existence of couple-stress in materials was originally postulated by Voigt (1887). However, Cosserat and Cosserat (1909) were the first to develop a mathematical model to analyze materials with couple-stresses. In the original Cosserat theory, the kinematical quantities were the displacement and a material microrotation, hypothesized to be independent of the continuum mechanical rotation. This latter quantity, which may be called the macrorotation, is the usual rotation vector defined as one half of the curl of the displacement field.

A couple stress theory, using macrorotation as the true kinematical rotation, was developed much later by Toupin (1962), Mindlin and Tiersten (1962), Koiter (1964), and others for elastic bodies. In these developments, the gradient of the rotation vector is used as a curvature tensor. Unfortunately, there are some difficulties with these formulations. Perhaps the most disturbing troubles are the indeterminacy of the spherical part of the couple-stress tensor and the appearance of the body couple in the constitutive relation for the force-stress tensor (Mindlin and Tiersten, 1962). This inconsistent theory is called the indeterminate couple stress theory in the literature (Eringen, 1968). As a result of the inconsistency, a number of alternative theories have been developed.

One branch revives the idea of microrotation, inherited from Cosserat and Cosserat (1909) and is called micropolar theories (e.g., Mindlin, 1964; Eringen, 1968; Nowacki, 1986; Chen and Wang, 2001). However, microrotation, which brings extraneous degrees of freedom, is not a proper continuum mechanical concept. How can the effect of the discontinuous microstructure of matter be represented mathematically by an artificial continuous microrotation? Thus, a consistent size-dependent continuum mechanics theory should involve only true continuum kinematical quantities without recourse to any additional artificial degrees of freedom.

The other main branch, labeled second gradient theories, avoids the idea of microrotation by introducing gradients of strain, rotation or various combinations thereof (e.g., Mindlin and Eshel, 1968; Yang et al., 2002; Lazar et al., 2005). Although these theories use true continuum representations of deformation, the resulting formulations are not consistent with proper boundary condition specifications and energy conjugacy within the principle of virtual work.

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Here we develop the consistent couple stress theory by considering true continuum kinematical displacement and rotation. We demonstrate that the couple-stress tensor is skew-symmetric and the skew-symmetric part of the gradient of the rotation tensor is the consistent curvature tensor. These two tensors satisfy pair conjugacy in the virtual work principle. Although the gradient of the strain tensor might appear in the skew-symmetric force-stress constitutive relations, it is not a fundamental measure of deformation. For example, in elastic bodies, strain gradients do not appear in the stored energy density function.

Interestingly, this theory can be considered as the modification of the developments of Mindlin and Tiersten (1962) and Koiter (1964). We will see that some results from the previous indeterminate couple stress theory for two dimensional cases can still be used.

We organize the current paper in the following manner. In Section 2, we present force-stresses, couple-stresses and the equilibrium equations per the usual definitions in the existing couple stress literature. Based on purely kinematical considerations as provided in Section 3, we first suggest the mean curvature tensor as the measure of deformation compatible with the couple-stress tensor for the infinitesimal theory. Then, by using the virtual work principle in Section 4, we demonstrate that in couple stress materials, body couples must be transformed to an equivalent body force and surface traction system. More importantly, based on resolving properly the boundary conditions, we show that the couple-stress tensor is skew-symmetric and, thus, completely determinate. This also confirms the mean curvature tensor as the fundamental deformation measure, energetically conjugate to the couple-stress tensor. Afterwards, in Section 5, the general theory of small deformation elasticity is developed. The constitutive and equilibrium equations for a linear elastic material also are derived under the assumption of infinitesimal deformations in Section 6, along with the uniqueness theorem for well-posed boundary value problems and the reciprocal theorem. Section 7 provides the general solution based on stress functions for two-dimensional infinitesimal linear elasticity, while the corresponding anti-plane deformation problem is examined in Section 8. Section 9 presents solutions for several elementary elasticity problems and one more complicated case. Finally, Section 10 contains a summary and some general conclusions.

2. Stresses and equilibrium

Consider a material continuum occupying a volume V bounded by a surface S as the current configuration. For a size dependent continuum theory, it is assumed that the transfer of the interaction in the current configuration occurs between two particles of the body through a surface element dS with unit normal vector n_i by means of a force vector $t_i^{(n)}dS$ and a moment vector $m_i^{(n)}dS$, where $t_i^{(n)}$ and $m_i^{(n)}$ are force- and moment-traction vectors. Surface forces and couples are then represented by generally non-symmetric force-stress σ_{ji} and couple-stress μ_{ji} tensors, where

$$t_i^{(n)} = \sigma_{ji} n_j \tag{1}$$

$$m_i^{(n)} = \mu_{ji} n_j \tag{2}$$

The force- and couple- stress tensors can be generally decomposed into symmetric and skew-symmetric parts

$$\sigma_{ji} = \sigma_{(ji)} + \sigma_{[ji]} \tag{3}$$

$$\mu_{ji} = \mu_{(ji)} + \mu_{[ji]} \tag{4}$$

Notice that here we have introduced parentheses surrounding a pair of indices to denote the symmetric part of a second order tensor, whereas square brackets are associated with the skew-symmetric part.

Now consider an arbitrary part of the material continuum occupying a volume V_a enclosed by boundary surface S_a. Under quasistatic conditions, the linear and angular balance equations for this part of the body are

$$\int_{S_a} t_i^{(n)} dS + \int_{V_a} F_i dV = 0 \tag{5}$$

$$\int_{S_a} \left[\varepsilon_{ijk} x_j t_k^{(n)} + m_i^{(n)} \right] dS + \int_{V_a} \left[\varepsilon_{ijk} x_j F_k + C_i \right] dV = 0 \tag{6}$$

where F_i and C_i are the body force and the body couple per unit volume of the body, respectively. Here ε_{ijk} is the permutation tensor or Levi–Civita symbol.

By using the relations (1) and (2), along with the divergence theorem, and noticing the arbitrariness of volume V_a, we finally obtain the differential form of the equilibrium equations, for the usual couple stress theory, as

$$\sigma_{ji,j} + F_i = 0 \tag{7}$$

$$\mu_{ji,j} + \varepsilon_{ijk}\sigma_{jk} + C_i = 0 \tag{8}$$

where the comma denotes differentiation with respect to the spatial coordinates.

In classical continuum mechanics $\mu_{ji} = 0$ and $C_i = 0$. Therefore, angular equilibrium (8) shows that the force-stress tensor is symmetric

$$\sigma_{ji} = \sigma_{ij} = \sigma_{(ji)}, \quad \sigma_{[ji]} = 0 \tag{9a, b}$$

This means that the tensor σ_{ji} has six independent components and we have three linear equilibrium equations in (7). In the classical theory, the extra three equations are obtained by developing constitutive relations.

In a couple stress theory, the tensors σ_{ji} and μ_{ji} have 18 components altogether, but we have only six equilibrium equations. Therefore, it seems we need 12 extra equations from constitutive relations. This has been the main trouble in developing a consistent couple stress theory in the past. Similarly to the classical case, we should expect that the angular equilibrium equation (8) will furnish some insight into the subtle structure of stresses in a continuum. This will reduce the number of independent components of stresses from 18. We explore this by studying the boundary conditions, virtual work principle and some kinematical considerations and discover the skew-symmetric character of the couple-stress tensor in continuum representations of matter. We consider the kinematical aspects in the following section.

3. Kinematics

Here we consider the kinematics of a continuum under the assumptions of infinitesimal deformation. In Cartesian coordinates, we define u_i to represent the displacement field of the continuum material. Consider the neighboring points P and Q with position vectors x_i and $x_i + dx_i$ in the reference configuration. The relative displacement of point Q with respect to P is

$$du_i = u_{i,j} dx_j \tag{10}$$

where $u_{i,j}$ is the displacement gradient tensor at point P. As we know, although this tensor is important in analysis of deformation, it is not itself a suitable measure of deformation. This tensor can be decomposed into symmetric and skew-symmetric parts

$$u_{i,j} = e_{ij} + \omega_{ij} \tag{11}$$

where

$$e_{ij} = u_{(i,j)} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{12}$$

$$\omega_{ij} = u_{[i,j]} = \frac{1}{2}\left(u_{i,j} - u_{j,i}\right) \tag{13}$$

Of course, in (12) and (13), the tensors e_{ij} and ω_{ij} are the small deformation strain and rotation tensor, respectively. The rotation vector ω_i dual to the rotation tensor ω_{ij} is defined by

$$\omega_i = \frac{1}{2}\varepsilon_{ijk}\omega_{kj} = \frac{1}{2}\varepsilon_{ijk}u_{k,j} \tag{14a}$$

which in vectorial form is written

$$\boldsymbol{\omega} = \frac{1}{2}\nabla \times \mathbf{u} \tag{14b}$$

Alternatively, this rotation vector is related to the rotation tensor through

$$\omega_{ji} = \varepsilon_{ijk}\omega_k \tag{15}$$

which shows

$$\omega_1 = -\omega_{23}, \quad \omega_2 = \omega_{13}, \quad \omega_3 = -\omega_{12} \tag{16a-c}$$

Therefore, the relative displacement is decomposed into

$$du_i = du_i^{(1)} + du_i^{(2)} \tag{17}$$

where

$$du_i^{(1)} = e_{ij}dx_j \tag{18}$$

$$du_i^{(2)} = \omega_{ij}dx_j \tag{19}$$

Then, ω_{ij} is seen to generate a rigid-like rotation of element dx_i about point P, where

$$du_i^{(2)}dx_i = \omega_{ij}dx_i dx_j = 0 \tag{20}$$

Since ω_{ij} does not contribute to the elongation or contraction of element dx_i, it cannot appear in a tensor measuring material stretches. Therefore, as we know, the symmetric strain tensor e_{ij} is the suitable measure of deformation in classical infinitesimal theories, such as Cauchy elasticity.

In couple stress theory, we expect to have an additional tensor measuring the curvature of the arbitrary fiber element dx_i. To find this tensor, we consider the field of rotation vector ω_i. The relative rotation of two neighboring points P and Q is given by

$$d\omega_i = \omega_{i,j}dx_j \tag{21}$$

where the tensor $\omega_{i,j}$ is the gradient of the rotation vector at point P. It is seen that the components $\omega_{1,1}$, $\omega_{2,2}$ and $\omega_{3,3}$ represent the torsion of the fibers along corresponding coordinate directions x_1, x_2 and x_3, respectively, at point P. The off-diagonal components represent the curvature of these fibers in planes parallel to coordinate planes. For example, $\omega_{1,2}$ is the curvature of a fiber element in the x_2 direction in a plane parallel to the x_2x_3 plane, while $\omega_{2,1}$ is the curvature of a fiber element in the x_1 direction in a plane parallel to the x_1x_3 plane.

The suitable measure of curvature must be a tensor measuring pure curvature of an arbitrary element dx_i. Therefore, in this tensor, the components $\omega_{1,1}$, $\omega_{2,2}$ and $\omega_{3,3}$ cannot appear. However, simply deleting these components from the tensor $\omega_{i,j}$ does not produce a tensor. Consequently, we expect that the required tensor is the skew-symmetric part of $\omega_{i,j}$. By decomposing the tensor $\omega_{i,j}$ into symmetric and skew-symmetric parts, we obtain

$$\omega_{i,j} = \chi_{ij} + \kappa_{ij} \tag{22}$$

where

$$\chi_{ij} = \omega_{(i,j)} = \frac{1}{2}\left(\omega_{i,j} + \omega_{j,i}\right) \tag{23}$$

$$\kappa_{ij} = \omega_{[i,j]} = \frac{1}{2}\left(\omega_{i,j} - \omega_{j,i}\right) \tag{24}$$

The symmetric tensor χ_{ij} results from applying the strain operator to the rotation vector, while the tensor κ_{ij} is the rotation of the rotation vector at point P. From (23)

$$\chi_{11} = \omega_{1,1}, \quad \chi_{22} = \omega_{2,2}, \quad \chi_{33} = \omega_{3,3} \tag{25a-c}$$

and

$$\chi_{12} = \chi_{21} = \frac{1}{2}(\omega_{1,2} + \omega_{2,1}) \tag{26a}$$

$$\chi_{23} = \chi_{32} = \frac{1}{2}(\omega_{2,3} + \omega_{3,2}) \tag{26b}$$

$$\chi_{13} = \chi_{31} = \frac{1}{2}(\omega_{1,3} + \omega_{3,1}) \tag{26c}$$

The diagonal elements χ_{11}, χ_{22} and χ_{33} defined in (25) represent pure torsion of fibers along the x_1, x_2 and x_3 directions, respectively, as mentioned above. On the other hand, from careful examination of (26), we find that χ_{12}, χ_{23} and χ_{13} measure the deviation from sphericity (Hamilton, 1866) of deforming planes parallel to x_1x_2, x_2x_3 and x_1x_3, respectively. Furthermore, we may recognize that this symmetric χ_{ij} tensor must have real principal values, representing the pure twists along the principal directions. Thus, we refer to χ_{ij} as the torsion tensor and we expect that this tensor will not contribute as a fundamental measure of deformation in a continuum material. Instead, we anticipate that the fundamental curvature tensor is the skew-symmetric rotation of rotation tensor κ_{ij}. This will be confirmed in the next section through consideration of couple-stresses and virtual work.

We also may arrive at this outcome by noticing that only the part of $d\omega_i$ that is normal to element dx_i produces pure curvature. Therefore, by decomposing $d\omega_i$ into

$$d\omega_i = d\omega_i^{(1)} + d\omega_i^{(2)} \tag{27}$$

where

$$d\omega_i^{(1)} = \chi_{ij}dx_j \tag{28}$$

$$d\omega_i^{(2)} = \kappa_{ij}dx_j \tag{29}$$

we notice

$$d\omega_i^{(2)}dx_i = \kappa_{ij}dx_i dx_j = 0 \tag{30}$$

This shows that $d\omega_i^{(2)}$ is the component of $d\omega_i$ normal to dx_i. Therefore, the tensor κ_{ij} seems to be the suitable curvature tensor, which is represented by

$$\left[\kappa_{ij}\right] = \begin{bmatrix} 0 & \kappa_{12} & \kappa_{13} \\ -\kappa_{12} & 0 & \kappa_{23} \\ -\kappa_{13} & -\kappa_{23} & 0 \end{bmatrix} \tag{31}$$

where the non-zero components of this tensor are

$$\kappa_{12} = -\kappa_{21} = \frac{1}{2}(\omega_{1,2} - \omega_{2,1}) \tag{32a}$$

$$\kappa_{23} = -\kappa_{32} = \frac{1}{2}(\omega_{2,3} - \omega_{3,2}) \tag{32b}$$

$$\kappa_{13} = -\kappa_{31} = \frac{1}{2}(\omega_{1,3} - \omega_{3,1}) \tag{32c}$$

Now we may recognize that κ_{12}, κ_{23}, and κ_{13} are the mean curvatures of planes parallel to the x_1x_2, x_2x_3, x_3x_1 planes, respectively, at point P after deformation. Therefore, the skew-symmetric tensor κ_{ij} will be referred to as the mean curvature tensor or simply the curvature tensor. The curvature vector κ_i dual to this tensor is defined by

$$\kappa_i = \frac{1}{2}\varepsilon_{ijk}\omega_{k,j} = \frac{1}{2}\varepsilon_{ijk}\kappa_{kj} \tag{33}$$

Thus, this axial vector is related to the mean curvature tensor through

$$\kappa_{ji} = \varepsilon_{ijk}\kappa_k \tag{34}$$

which shows

$$\kappa_1 = -\kappa_{23}, \quad \kappa_2 = \kappa_{13}, \quad \kappa_3 = -\kappa_{12} \tag{35a-c}$$

It is seen that the mean curvature vector can be expressed as

$$\boldsymbol{\kappa} = \frac{1}{2}\nabla \times \boldsymbol{\omega} \tag{36}$$

This shows that $\boldsymbol{\kappa}$ is the rotation of the rotation vector, which can also be expressed as

$$\boldsymbol{\kappa} = \frac{1}{4}\nabla \times (\nabla \times \mathbf{u}) = \frac{1}{4}\nabla(\nabla \cdot \mathbf{u}) - \frac{1}{4}\nabla^2\mathbf{u} \tag{37a}$$

$$\kappa_i = \frac{1}{4}u_{k,ki} - \frac{1}{4}u_{i,kk} = \frac{1}{4}u_{k,ki} - \frac{1}{4}\nabla^2 u_i \tag{37b}$$

From (37b) and (13), we can obtain the interesting relation

$$\kappa_i = \frac{1}{2}\omega_{ji,j} \tag{37c}$$

What we have presented here is applicable to small deformation theory, which requires the components of the strain tensor and mean curvature vector to be infinitesimal. These conditions can be written as

$$|e_{ij}| \ll 1 \tag{38}$$

$$|\kappa_i| \ll \frac{1}{l_S} \tag{39}$$

where l_S is the smallest characteristic length in the body.

While analogous measures of strain and curvature can be obtained for finite deformation theory, this would take us beyond the scope of the present work, which is directed toward the infinitesimal linear couple stress theory.

4. Principle of virtual work and its consequences for continua

Once again consider the material continuum occupying a volume V bounded by a surface S. The standard form of the equilibrium equations for this medium was given in (7) and (8). Let us multiply (7) by a virtual displacement δu_i and integrate over the volume and also multiply (8) by the corresponding virtual rotation $\delta\ \omega_i$, where

$$\delta\omega_i = \frac{1}{2}\varepsilon_{ijk}\delta u_{k,j} \tag{40}$$

and integrate this over the volume as well. Therefore, we have

$$\int_V (\sigma_{ji,j} + F_i)\delta u_i dV = 0 \tag{41}$$

$$\int_V \left(\mu_{ji,j} + \varepsilon_{ijk}\sigma_{jk} + C_i\right)\delta\omega_i dV = 0 \tag{42}$$

By noticing the relation

$$\sigma_{ji,j}\delta u_i = \left(\sigma_{ji}\delta u_i\right)_{,j} - \sigma_{ji}\delta u_{i,j} \tag{43}$$

and using the divergence theorem, the relation (41) becomes

$$\int_V \sigma_{ji}\delta u_{i,j} dV = \int_S t_i^{(n)}\delta u_i dS + \int_V F_i\delta u_i dV. \tag{44}$$

Similarly, by using the relation

$$\mu_{ji,j}\delta\omega_i + \varepsilon_{ijk}\sigma_{jk}\delta\omega_i = \left(\mu_{ji}\delta\omega_i\right)_{,j} - \mu_{ji}\delta\omega_{i,j} - \sigma_{jk}\delta\omega_{jk} \tag{45}$$

Equation (42) becomes

$$\int_V \mu_{ji}\delta\omega_{i,j} dV - \int_V \sigma_{ji}\delta\omega_{ij} dV = \int_S m_i^{(n)}\delta\omega_i dS + \int_V C_i\delta\omega_i dV \tag{46}$$

Then, by adding (44) and (46), we obtain

$$\int_V \mu_{ji}\delta\omega_{i,j} dV + \int_V \sigma_{ji}(\delta u_{i,j} - \delta\omega_{ij}) dV = \int_S t_i^{(n)}\delta u_i dS + \int_V F_i\delta u_i dV + \int_S m_i^{(n)}\delta\omega_i dS + \int_V C_i\delta\omega_i dV \tag{47}$$

However, by noticing the relation

$$\delta e_{ij} = \delta u_{i,j} - \delta\omega_{ij} \tag{48}$$

for compatible virtual displacement, we obtain the virtual work theorem as

$$\int_V \sigma_{ji}\delta e_{ij} dV + \int_V \mu_{ji}\delta\omega_{i,j} dV = \int_S t_i^{(n)}\delta u_i dS + \int_S m_i^{(n)}\delta\omega_i dS + \int_V F_i\delta u_i dV + \int_V C_i\delta\omega_i dV \tag{49}$$

Since δe_{ij} is symmetric, we also have

$$\sigma_{ji}\delta e_{ij} = \sigma_{(ji)}\delta e_{ij} \tag{50}$$

Thus, the principle of virtual work can be written as

$$\int_V \sigma_{(ji)}\delta e_{ij} dV + \int_V \mu_{ji}\delta\omega_{i,j} dV = \int_S t_i^{(n)}\delta u_i dS + \int_S m_i^{(n)}\delta\omega_i dS + \int_V F_i\delta u_i dV + \int_V C_i\delta\omega_i dV \tag{51}$$

The right hand side of (51) shows that the boundary conditions on the surface of the body can be either vectors u_i and ω_i as essential (geometrical) boundary conditions, or $t_i^{(n)}$ and $m_i^{(n)}$ as natural (mechanical) boundary conditions. The left hand side of (51) shows that $\sigma_{(ji)}$ and e_{ij} are energy conjugate tensors, and the skew symmetric part of force-stress tensor $\sigma_{[ji]}$ has no contribution to internal virtual work. At this point, it is also seen that μ_{ji} and $\omega_{i,j}$ are energy conjugate tensors. Therefore, the compatible curvature tensor must be developed from $\omega_{i,j}$. The virtual work principle (51) shows that there is no room for strain gradients as fundamental measures of deformation in a consistent couple stress theory. Interestingly, (51) can reveal more insight about the structure of this consistent couple stress theory.

Now, along those lines, we investigate the fundamental character of the body couple C_i and couple-stress μ_{ji} in a continuum. It is seen that the term

$$\int_V C_i\delta\omega_i dV \tag{52}$$

in (51) is the only term in the volume that involves $\delta\omega_i$. However, $\delta\omega_i$ is not independent of δu_i in the volume, because we have the relation

$$\delta\omega_i = \frac{1}{2}\varepsilon_{ijk}\delta u_{k,j} \tag{53}$$

Therefore, by using (53) in the integrand of (52), we find

$$C_i\delta\omega_i = \frac{1}{2}C_i\varepsilon_{ijk}\delta u_{k,j} = \frac{1}{2}\left(\varepsilon_{ijk}C_i\delta u_k\right)_{,j} - \frac{1}{2}\varepsilon_{ijk}C_{i,j}\delta u_k \tag{54}$$

and, after applying the divergence theorem, the body couple virtual work in (52) becomes

$$\int_V C_i\delta\omega_i dV = \int_V \frac{1}{2}\varepsilon_{ijk}C_{k,j}\delta u_i dV + \int_S \frac{1}{2}\varepsilon_{ijk}C_j n_k\delta u_i dS \tag{55}$$

which means that the body couple C_i transforms into an equivalent body force $\frac{1}{2}\varepsilon_{ijk}C_{k,j}$ in the volume and a force traction vector $\frac{1}{2}\varepsilon_{ijk}C_j n_k$ on the bounding surface. This shows that in a continuum theory of materials, the body couple is not distinguishable from the body force. Therefore, in couple stress theory, we must only consider body forces. This is analogous to the impossibility of distinguishing a distributed moment load in Euler–Bernoulli beam theory, in which the moment load must be replaced by the equivalent distributed force load and end concentrated loads. Therefore, for a proper couple stress theory, the equilibrium equations become

$$\sigma_{ji,j} + F_i = 0 \tag{56}$$

$$\mu_{ji,j} + \varepsilon_{ijk}\sigma_{jk} = 0 \tag{57}$$

where

$$\mathbf{F} + \frac{1}{2}\nabla \times \mathbf{C} \rightarrow \mathbf{F} \quad \text{in } V \tag{58a}$$

$$\mathbf{t}^{(n)} + \frac{1}{2}\mathbf{C} \times \mathbf{n} \rightarrow \mathbf{t}^{(n)} \quad \text{on } S \tag{58b}$$

and the virtual work theorem reduces to

$$\int_V \sigma_{(ji)}\delta e_{ij}dV + \int_V \mu_{ji}\delta\omega_{i,j}dV = \int_S t_i^{(n)}\delta u_i dS + \int_S m_i^{(n)}\delta\omega_i dS + \int_V F_i\delta u_i dV \tag{59}$$

Next, we investigate the fundamental character of the couple-stress tensor based on boundary conditions.

As we mentioned, the prescribed boundary conditions on the surface of the body can be either vectors u_i and ω_i, or $t_i^{(n)}$and $m_i^{(n)}$, which makes a total number of six boundary values for either case. However, this is in contrast to the number of geometric boundary conditions that can be imposed (Koiter, 1964). In particular, if components of u_i are specified on the boundary surface, then the normal component of the rotation ω_i corresponding to twisting

$$\omega_i^{(n)} = \omega^{(nn)}n_i = \omega_k n_k n_i \tag{60}$$

where

$$\omega^{(nn)} = \omega_k n_k \tag{61}$$

cannot be prescribed independently. However, the tangential component of rotation ω_i corresponding to bending, that is

$$\omega_i^{(ns)} = \omega_i - \omega^{(nn)}n_i = \omega_i - \omega_k n_k n_i \tag{62}$$

may be specified in addition, and the number of geometric or essential boundary conditions that can be specified is therefore five.

Next, we let $m^{(nn)}$ and $m_i^{(ns)}$ represent the normal and tangential components of the surface moment traction vector $m_i^{(n)}$, respectively, where

$$m^{(nn)} = m_k^{(n)}n_k = \mu_{ji}n_i n_j \tag{63}$$

causes twisting, while

$$m_i^{(ns)} = m_i^{(n)} - m^{(nn)}n_i \tag{64}$$

is responsible for bending.

From kinematics, since $\omega^{(nn)}$ is not an independent generalized degree of freedom, its apparent corresponding generalized force must be zero. Thus, for the normal component of the surface moment traction vector $m_i^{(n)}$, we must enforce the condition

$$m^{(nn)} = m_k^{(n)}n_k = \mu_{ji}n_i n_j = 0 \quad \text{on } S \tag{65}$$

Furthermore, the boundary moment surface virtual work in (55) becomes

$$\int_S m_i^{(n)}\delta\omega_i dS = \int_S m_i^{(ns)}\delta\omega_i dS = \int_S m_i^{(ns)}\delta\omega_i^{(ns)} dS \tag{66}$$

This shows that a material in couple stress theory does not support independent distributions of normal surface moment (or twisting) traction $m^{(nn)}$, and the number of mechanical boundary conditions also is five. In practice, it might seem that a given $m^{(nn)}$ has to be replaced by an equivalent shear stress and force system. Koiter (1964) gives the detail analogous to the Kirchhoff bending theory of plates. However, we should realize that there is a difference between couple stress theory and the Kirchhoff bending theory of plates. Plate theory is an approximation for elasticity, which is a continuum mechanics theory. However, couple stress theory is a continuum mechanics theory itself without any approximation.

From the above discussion, we should realize that on the surface of the body, a normal moment $m^{(nn)}$ cannot be applied. By continuing this line of reasoning, we may reveal the subtle character of the couple-stress tensor. First, we notice that the virtual work theorem can be written for every arbitrary volume V_a with surface S_a within the body V. Thus

$$\int_{V_a} \sigma_{ji}\delta e_{ij}dV + \int_{V_a} \mu_{ji}\delta\omega_{i,j}dV = \int_{S_a} t_i^{(n)}\delta u_i dS + \int_{S_a} m_i^{(n)}\delta\omega_i dS + \int_{V_a} F_i\delta u_i dV \tag{67}$$

For any point on this arbitrary surface with unit normal n_i, we must have

$$m^{(nn)} = \mu_{ji}n_i n_j = 0 \quad \text{in } V \tag{68}$$

Since $n_i n_j$ is symmetric and arbitrary in (68), μ_{ji} must be skew-symmetric. Thus

$$\mu_{ji} = -\mu_{ij} \quad \text{in } V \tag{69}$$

This is the fundamental property of the couple-stress tensor in continuum mechanics, which has not been recognized previously. Here we can see the crucial role of the virtual work theorem in this result.

In terms of components, the couple-stress tensor now can be written as

$$[\mu_{ij}] = \begin{bmatrix} 0 & \mu_{12} & \mu_{13} \\ -\mu_{12} & 0 & \mu_{23} \\ -\mu_{13} & -\mu_{23} & 0 \end{bmatrix} \tag{70}$$

and one can realize that the couple-stress actually can be considered as an axial vector. This couple-stress vector μ_i dual to the tensor μ_{ij} can be defined by

$$\mu_i = \frac{1}{2}\varepsilon_{ijk}\mu_{kj} \tag{71}$$

where we also have

$$\varepsilon_{ijk}\mu_k = \mu_{ji} \tag{72}$$

These relations simply show

$$\mu_1 = -\mu_{23}, \quad \mu_2 = \mu_{13}, \quad \mu_3 = -\mu_{12} \tag{73a-c}$$

It is seen that the surface moment traction vector can be expressed as

$$m_i^{(n)} = \mu_{ji}n_j = \varepsilon_{ijk}n_j\mu_k \tag{74}$$

which can be written in vectorial form

$$\mathbf{m}^{(n)} = \mathbf{n} \times \boldsymbol{\mu} \tag{75}$$

This obviously shows that the moment traction vector $\mathbf{m}^{(n)}$ is tangent to the surface.

After discovering the skew-symmetric character of couple-stress tensor μ_{ji}, we investigate the structure of the force-stress tensor σ_{ji}. Using (72), the angular equilibrium equation (57) can be expressed as

$$\varepsilon_{ijk}\left(\mu_{k,j} + \sigma_{jk}\right) = 0 \tag{76}$$

which indicates that $\mu_{k,j}+\sigma_{jk}$ is symmetric. Therefore, its skew-symmetric part vanishes and

$$\sigma_{[ji]} = -\mu_{[i,j]} = -\frac{1}{2}\left(\mu_{i,j} - \mu_{j,i}\right) \tag{77}$$

which produces the skew-symmetric part of the force-stress tensor in terms of the couple-stress vector. Therefore, the sole duty of the angular equilibrium equation (57) is to produce the skew-symmetric part of the force-stress tensor. This relation can be elaborated if we consider the axial vector s_i dual to the skew-symmetric part of the force-stress tensor $\sigma_{[ij]}$, where

$$s_i = \frac{1}{2}\varepsilon_{ijk}\sigma_{[kj]} \tag{78}$$

which also satisfies

$$\varepsilon_{ijk}s_k = \sigma_{[ji]} \tag{79}$$

or simply

$$s_1 = -\sigma_{[23]}, \quad s_2 = \sigma_{[13]}, \quad s_3 = -\sigma_{[12]} \tag{80a-c}$$

By using (77) in (78), we obtain

$$s_i = -\frac{1}{2}\varepsilon_{ijk}\mu_{[j,k]} = \frac{1}{2}\varepsilon_{ijk}\mu_{k,j} \tag{81a}$$

which can be written in vectorial form

$$\mathbf{s} = \frac{1}{2}\nabla \times \boldsymbol{\mu} \tag{81b}$$

This simply shows that half of the curl of the couple-stress vector $\boldsymbol{\mu}$ produces the skew-symmetric part of the force-stress tensor through $\mathbf{s}$. Interestingly, it is seen that

$$\nabla \cdot \mathbf{s} = 0 \tag{82}$$

Returning to the virtual work theorem, we notice since μ_{ji} is skew-symmetric

$$\mu_{ji}\delta\omega_{i,j} = \mu_{ji}\delta\kappa_{ij} \tag{83}$$

which shows that the skew-symmetric mean curvature tensor κ_{ij} is energetically conjugate to the skew-symmetric couple-stress tensor μ_{ji}. This confirms our speculation of κ_{ij} as a suitable curvature tensor in Section 2. Furthermore, the virtual work theorem (59) becomes

$$\int_V \sigma_{(ji)}\delta e_{ij}dV + \int_V \mu_{ji}\delta\kappa_{ij}dV = \int_S t_i^{(n)}\delta u_i dS + \int_S m_i^{(n)}\delta\omega_i dS + \int_V F_i\delta u_i dV \tag{84}$$

Interestingly, by using the dual vectors of these tensors, we have

$$\begin{aligned}\mu_{ji}\delta\kappa_{ij} &= \varepsilon_{ijp}\mu_p\varepsilon_{jiq}\delta\kappa_q = -\varepsilon_{ijp}\varepsilon_{ijq}\mu_p\delta\kappa_q = -2\delta_{pq}\mu_p\delta\kappa_q \\ &= -2\mu_i\delta\kappa_i\end{aligned} \tag{85}$$

which shows the conjugate relation between twice the mean curvature vector $-2\kappa_i$ and the couple-stress vector μ_i. Thus, the principle of virtual work can be written

$$\int_V (\sigma_{(ji)}\delta e_{ij} - 2\mu_i\delta\kappa_i)dV = \int_S t_i^{(n)}\delta u_i dS + \int_S m_i^{(n)}\delta\omega_i dS + \int_V F_i\delta u_i dV \tag{86}$$

Now it is time to explain a very important aspect of specifying boundary conditions. The natural and essential boundary conditions $t_i^{(n)}$ and u_i, respectively, are specified as in classical Cauchy theory. On the other hand, the two new boundary conditions for couple stress theory need more elaboration. In practice, the actual boundary S is usually free of moment traction, which means this natural boundary condition is zero ($m_i^{(n)} = 0$) everywhere on S. However, couple-stresses μ_{ji} can be created inside the volume V and non-zero $m_i^{(n)}$ may exist on any arbitrary internal surface S_a. The essential boundary condition, which is the tangential component of ω_i, usually cannot be specified on the actual boundary S, but once again non-zero couple-stresses μ_{ji} can be generated inside the domain V.

What we have presented so far is a continuum mechanics theory of couple stress materials, independent of the material properties. We have shown that the actual number of independent components of stresses are 9 components of $\sigma_{(ji)}$ and μ_i, and the linear equilibrium equation (56) reduces to

$$\left[\sigma_{(ji)} - \mu_{[i,j]}\right]_{,j} + F_i = 0 \tag{87}$$

Thus, we have nine stress components and three linear equilibrium equations. The six extra required equations are obtained from constitutive relations. In the following section, we specialize the theory for elastic materials.

5. Infinitesimal size-dependent elasticity

Now, we develop the size-dependent theory of small deformation for elastic materials. In an elastic material, there is a stored elastic energy density function W, where for arbitrary virtual deformations about the equilibrium position, we have

$$\delta W = \sigma_{ji}\delta e_{ij} + \mu_{ji}\delta\kappa_{ij} = \sigma_{(ji)}\delta e_{ij} - 2\mu_i\delta\kappa_i \tag{88}$$

Therefore, W is a positive definite function of the symmetric strain tensor e_{ij} and the mean curvature vector κ_i. Thus

$$W = W(\mathbf{e}, \boldsymbol{\kappa}) = W\left(e_{ij}, \kappa_i\right) \tag{89}$$

However, for a variational analysis, the relation (88) should be written as

$$\delta W = \sigma_{(ji)}\delta u_{i,j} - 2\mu_i\delta\kappa_i \tag{90}$$

where the all components of $\delta u_{i,j}$ and $\delta\kappa_i$ can be taken independent of each other. From the relations (89) and (90), we obtain

$$\sigma_{(ji)} = \frac{\partial W}{\partial u_{i,j}} \tag{91}$$

$$2\mu_i = -\frac{\partial W}{\partial \kappa_i} \tag{92}$$

However, it is seen that

$$\frac{\partial W}{\partial u_{i,j}} = \frac{\partial W}{\partial e_{kl}}\frac{\partial e_{kl}}{\partial u_{i,j}} \tag{93}$$

By noticing

$$e_{kl} = \frac{1}{2}\left(u_{k,l} + u_{l,k}\right) \tag{94}$$

we obtain

$$\frac{\partial e_{kl}}{\partial u_{i,j}} = \frac{1}{2}\left(\delta_{ki}\delta_{lj} + \delta_{li}\delta_{kj}\right) \tag{95}$$

Therefore

$$\frac{\partial W}{\partial u_{i,j}} = \frac{1}{2}\frac{\partial W}{\partial e_{kl}}\left(\delta_{ki}\delta_{lj} + \delta_{li}\delta_{kj}\right) \tag{96}$$

which shows

$$\frac{\partial W}{\partial u_{i,j}} = \frac{1}{2}\left(\frac{\partial W}{\partial e_{ij}} + \frac{\partial W}{\partial e_{ji}}\right) \tag{97}$$

Then

$$\sigma_{(ji)} = \frac{1}{2}\left(\frac{\partial W}{\partial e_{ij}} + \frac{\partial W}{\partial e_{ji}}\right) \tag{98}$$

$$\mu_i = -\frac{1}{2}\frac{\partial W}{\partial \kappa_i} \tag{99}$$

If we further agree to construct W, such that

$$\frac{\partial W}{\partial e_{ij}} = \frac{\partial W}{\partial e_{ji}} \tag{100}$$

we can write in place of (98)

$$\sigma_{(ji)} = \frac{\partial W}{\partial e_{ij}} \tag{101}$$

It is also seen that

$$\mu_{[i,j]} = \frac{1}{2}\left(\mu_{i,j} - \mu_{j,i}\right) = -\frac{1}{4}\left[\left(\frac{\partial W}{\partial \kappa_i}\right)_{,j} - \left(\frac{\partial W}{\partial \kappa_j}\right)_{,i}\right] \tag{102}$$

Therefore, for the skew-symmetric part of the force-stress tensor, we have

$$\sigma_{[ji]} = -\mu_{[i,j]} = \frac{1}{4}\left[\left(\frac{\partial W}{\partial \kappa_i}\right)_{,j} - \left(\frac{\partial W}{\partial \kappa_j}\right)_{,i}\right] \tag{103}$$

Finally, we obtain the constitutive relations as

$$\sigma_{ji} = \frac{1}{2}\left(\frac{\partial W}{\partial e_{ij}} + \frac{\partial W}{\partial e_{ji}}\right) + \frac{1}{4}\left[\left(\frac{\partial W}{\partial \kappa_i}\right)_{,j} - \left(\frac{\partial W}{\partial \kappa_j}\right)_{,i}\right] \tag{104}$$

$$\mu_i = -\frac{1}{2}\frac{\partial W}{\partial \kappa_i} \tag{105}$$

The total potential energy functional for an elastic body is defined as

$$\Pi\{\mathbf{u}\} = \int_V W dV - \int_V F_i u_i dV - \int_S t_i^{(n)} u_i dS - \int_S m_i^{(n)} \omega_i dS \tag{106}$$

It can be easily shown that this functional attains a minimum when the displacement field corresponds to the elastic solution that satisfies the equilibrium equations. The kinematics of deformation and variation of (106) reveal an important character of the stored energy density function W. We know there are two sets of equilibrium equations (56) and (57) corresponding to linear and angular equilibrium of an infinitesimal element of material. Therefore, the geometrical boundary conditions are the displacement u_i and rotation ω_i as we discussed previously. As we showed in Section 4, continuum mechanics supports the geometrical boundary conditions u_i and $\omega_i^{(ns)}$, and their corresponding energy conjugate mechanical boundary conditions $t_i^{(n)}$ and $m_i^{(ns)}$. Consequently, there is no other possible type of boundary condition in size-dependent continuum mechanics. Therefore, in the variation of the total potential energy Π in (106), the stored energy density function W at most can be in the form (89). This means at most the stored energy density function W is a function of the second derivative of deformation in the form of the mean curvature vector κ_i, not strain gradient. In other words, the continuum mechanics stored energy function W cannot depend on third and higher order derivatives of deformation.

6. Infinitesimal size-dependent linear elasticity

6.1. Strain energy and constitutive relations

For a linear elastic material, based on our development, the quadratic positive definite stored energy density must be in the general form

$$W(\mathbf{e}, \boldsymbol{\kappa}) = \frac{1}{2}A_{ijkl}e_{ij}e_{kl} + \frac{1}{2}B_{ij}\kappa_i\kappa_j + C_{ijk}e_{ij}\kappa_k \tag{107}$$

The tensors A_{ijkl}, B_{ij} and C_{ijk} contain the elastic constitutive coefficients and are such that W is positive definite. As a result, tensors A_{ijkl} and B_{ij} are positive definite. The tensor A_{ijkl} is actually equivalent to its corresponding tensor in Cauchy elasticity. The symmetry relations

$$A_{ijkl} = A_{klij} = A_{jikl} \tag{108}$$

$$B_{ij} = B_{ji} \tag{109}$$

$$C_{ijk} = C_{jik} \tag{110}$$

show that for the most general case the number of distinct components for A_{ijkl} , B_{ij} and C_{ijk} are 21, 6 and 18, respectively. Therefore, the most general linear elastic anisotropic material is described by 45 independent constitutive coefficients.

It is seen that the couple-stress vector and symmetric part of force-stress tensor can be found as

$$\mu_i = -\frac{1}{2}B_{ij}\kappa_j - \frac{1}{2}C_{kji}e_{kj} \tag{111}$$

$$\sigma_{(ji)} = A_{ijkl}e_{kl} + C_{ijk}\kappa_k \tag{112}$$

Additionally, we find that

$$\mu_{i,j} = -\frac{1}{2}B_{im}\kappa_{m,j} - \frac{1}{2}C_{kmi}e_{km,j} \tag{113}$$

and the skew-symmetric part of this tensor is

$$\begin{aligned}\mu_{[i,j]} &= -\sigma_{[ji]} \\ &= -\frac{1}{4}B_{im}\kappa_{m,j} + \frac{1}{4}B_{jm}\kappa_{m,i} - \frac{1}{4}C_{kmi}e_{km,j} + \frac{1}{4}C_{kmj}e_{km,i}\end{aligned} \tag{114}$$

Therefore, for the force-stresses, we find

$$\sigma_{ji} = A_{ijkl}e_{kl} + C_{ijk}\kappa_k + \frac{1}{4}B_{im}\kappa_{m,j} - \frac{1}{4}B_{jm}\kappa_{m,i} + \frac{1}{4}C_{kmi}e_{km,j} - \frac{1}{4}C_{kmj}e_{km,i} \tag{115}$$

It is seen that the strain gradient does appear in the skew-symmetric part of the force-stress tensor. However, the strain gradient cannot be considered as a fundamental measure of deformation, because it does not appear directly in the stored energy density (107).

For an isotropic material, the symmetry relations require

$$A_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \mu\delta_{il}\delta_{jk} \tag{116}$$

$$B_{ij} = 16\eta\delta_{ij} \tag{117}$$

$$C_{ijk} = 0 \tag{118}$$

The moduli λ and μ have the same meaning as the Lamé constants for an isotropic material in Cauchy elasticity. It is seen that only one extra material constant η accounts for couple stress effects in an isotropic material and the stored energy becomes

$$W(\mathbf{e}, \boldsymbol{\kappa}) = \frac{1}{2}\lambda(e_{kk})^2 + \mu e_{ij}e_{ij} + 8\eta\kappa_i\kappa_i \tag{119}$$

with the following restrictions on elastic constants for positive definite stored energy

$$3\lambda + 2\mu > 0, \quad \mu > 0, \quad \eta > 0 \tag{120}$$

Then, the constitutive relations can be written

$$\mu_i = -8\eta\kappa_i \tag{121}$$

$$\sigma_{(ji)} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} \tag{122}$$

Interestingly, it is seen that for an isotropic material

$$\nabla \cdot \boldsymbol{\mu} = \mu_{i,i} = 0 \tag{123}$$

By using the relation

$$\kappa_i = \frac{1}{4}u_{k,ki} - \frac{1}{4}u_{i,kk} \tag{124}$$

we obtain

$$\mu_i = 2\eta\left(\nabla^2 u_i - u_{k,ki}\right) \tag{125a}$$

or in vectorial form

$$\boldsymbol{\mu} = 2\eta\left[\nabla^2\mathbf{u} - \nabla(\nabla\cdot\mathbf{u})\right] \tag{125b}$$

Additionally,

$$\mu_{i,j} = 2\eta\left(\nabla^2 u_{i,j} - u_{k,kij}\right) \tag{126}$$

Therefore,

$$\mu_{[i,j]} = \eta\nabla^2\left(u_{i,j} - u_{j,i}\right) \tag{127}$$

or

$$\mu_{[i,j]} = 2\eta\nabla^2\omega_{ij} \tag{128}$$

and we obtain

$$\sigma_{[ji]} = -\mu_{[i,j]} = 2\eta\nabla^2\omega_{ji} \tag{129}$$

or by exchanging indices

$$\sigma_{[ij]} = 2\eta\nabla^2\omega_{ij} \tag{130}$$

Recall that the axial vector s_i is dual to $\sigma_{[ij]}$, as shown in (78). Then, from (81a) and (121), s_i can be written in terms of the curvature vector as

$$s_i = -4\eta\varepsilon_{ijk}\kappa_{k,j} \tag{131}$$

Therefore, the constitutive relation for vector **s** is

$$\mathbf{s} = -4\eta\nabla\times\boldsymbol{\kappa} \tag{132a}$$

which can be written alternatively as

$$\mathbf{s} = -2\eta\nabla\times\nabla\times\boldsymbol{\omega} = 2\eta\nabla^2\boldsymbol{\omega} \tag{132b}$$

or

$$\mathbf{s} = -\eta\nabla\times\nabla\times\nabla\times\mathbf{u} \tag{132c}$$

This remarkable result shows that in an isotropic material the vector **s**, corresponding to the skew-symmetric part of the stress tensor, is proportional to the curl of the curl of the curl of the displacement vector **u**.

By using the relations (122) and (129), the total force-stress tensor can be written as

$$\sigma_{ji} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} + 2\eta\nabla^2\omega_{ji} \tag{133}$$

We also notice that

$$\mu_{ji} = -8\eta\kappa_{ji} = 4\eta\left(\omega_{i,j} - \omega_{j,i}\right) \tag{134}$$

which is more useful than μ_i in practice.

It is seen that these relations are similar to those in the indeterminate couple stress theory (Mindlin and Tiersten, 1962; Koiter, 1964), when $\eta' = -\eta$. Here we have derived the couple stress theory for materials in which all former troubles with indeterminacy disappear. There is no spherical indeterminacy and the second couple stress coefficient η' depends on η, such that the couple-stress tensor becomes skew-symmetric.

Interestingly, the ratio

$$\frac{\eta}{\mu} = l^2 \tag{135}$$

specifies a characteristic material length l, which is absent in Cauchy elasticity, but is fundamental to small deformation couple stress elasticity. We realize that this is the characteristic length in an elastic material and that $l_S \to l$ in (39). Thus, the requirements for small deformation elasticity are

$$|e_{ij}| \ll 1 \tag{136a}$$

$$|\kappa_i| \ll \frac{1}{l} \tag{136b}$$

6.2. Displacement formulations

When the force-stress tensor (115) is written in terms of displacements, as follows

$$\begin{aligned}\sigma_{ji} &= A_{ijkl}e_{kl} + C_{ijk}\kappa_k + \frac{1}{4}C_{kmi}e_{km,j} - \frac{1}{4}C_{kmj}e_{km,i} + \frac{1}{4}B_{im}\kappa_{m,j} - \frac{1}{4}B_{jm}\kappa_{m,i} \\ &= A_{ijkl}u_{k,l} + \frac{1}{4}C_{ijk}\left(u_{m,mk} - \nabla^2 u_k\right) + \frac{1}{4}C_{kmi}u_{k,mj} - \frac{1}{4}C_{kmj}u_{k,mi} \\ &\quad + \frac{1}{16}B_{ik}\left(u_{m,mkj} - \nabla^2 u_{k,j}\right) - \frac{1}{16}B_{jk}\left(u_{m,mki} - \nabla^2 u_{k,i}\right)\end{aligned} \tag{137}$$

and is carried into the linear equilibrium equation (56), we obtain

$$\begin{aligned}&A_{ijkl}u_{k,lj} + \frac{1}{4}C_{ijk}\left(u_{m,mjk} - \nabla^2 u_{k,j}\right) + \frac{1}{4}C_{kmi}\nabla^2 u_{k,m} - \frac{1}{4}C_{kmj}u_{k,mij} \\ &\quad + \frac{1}{16}B_{ik}\left(\nabla^2 u_{m,mk} - \nabla^2\nabla^2 u_k\right) - \frac{1}{16}B_{jk}\left(u_{m,mkij} - \nabla^2 u_{k,ij}\right) + F_i = 0\end{aligned} \tag{138}$$

For an isotropic material, the force-stress tensor becomes

$$\begin{aligned}\sigma_{ji} &= \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} - 2\eta\nabla^2\omega_{ij} \\ &= \lambda u_{k,k}\delta_{ij} + \mu(u_{i,j} + u_{j,i}) - \eta\nabla^2(u_{i,j} - u_{j,i})\end{aligned} \tag{139}$$

and for the linear equilibrium equation, we have

$$\left(\lambda + \mu + \eta\nabla^2\right)u_{k,ki} + \left(\mu - \eta\nabla^2\right)\nabla^2 u_i + F_i = 0 \tag{140a}$$

which can be written in the vectorial form

$$\left(\lambda + \mu + \eta\nabla^2\right)\nabla(\nabla\cdot\mathbf{u}) + \left(\mu - \eta\nabla^2\right)\nabla^2\mathbf{u} + \mathbf{F} = \mathbf{0} \tag{140b}$$

This relation can also be written as

$$(\lambda + 2\mu)\nabla(\nabla\cdot\mathbf{u}) - \left(\mu - \eta\nabla^2\right)\nabla\times\nabla\times\mathbf{u} + \mathbf{F} = \mathbf{0} \tag{141}$$

which was derived previously by Mindlin and Tiersten (1962) within the context of the indeterminate couple stress theory. However, recall that the Mindlin–Tiersten formulation involved two couple stress parameters η and η'. In hindsight, the fact that η' does not appear in (141) should have been an indication that this coefficient is not independent of η. We now know that $\eta' = -\eta$.

The general solution for the displacement in isotropic elasticity also has been derived by Mindlin and Tiersten (1962) in terms of a vector function **G** and scalar function G_0 as

$$\mathbf{u} = \mathbf{G} - l^2\nabla\nabla\cdot\mathbf{G} - \frac{1}{4(1-\nu)}\nabla\left[\mathbf{r}\cdot\left(1 - l^2\nabla^2\right)\mathbf{G} + G_0\right] \tag{142}$$

where $\nu = \frac{\lambda}{2(\lambda+\mu)}$ is the Poisson's ratio. These functions satisfy the relations

$$\mu(1 - l^2\nabla^2)\nabla^2\mathbf{G} = -\mathbf{F} \tag{143a}$$

$$\mu\nabla^2 G_0 = \mathbf{r}\cdot\mathbf{F} \tag{143b}$$

These functions reduce to the Papkovich functions in the classical theory, when $l = 0$. In general, it is easily seen that

$$\nabla\cdot\mathbf{u} = \frac{1-2\nu}{2(1-\nu)}\left(1 - l^2\nabla^2\right)\nabla\cdot\mathbf{G} \tag{144a}$$

$$2\boldsymbol{\omega} = \nabla\times\mathbf{u} = \nabla\times\mathbf{G} \tag{144b}$$

6.3. Uniqueness theorem for boundary value problems

Now we investigate the uniqueness of the linear size-dependent elasticity boundary value problem. The proof follows from the concept of stored energy, similar to the approach for Cauchy elasticity. By replacing the virtual deformation with the actual deformation in the virtual work theorem (86), we obtain

$$\int_V (\sigma_{(ji)} e_{ij} - 2\mu_i \kappa_i) dV = \int_S t_i^{(n)} u_i dS + \int_S m_i^{(n)} \omega_i dS + \int_V F_i u_i dV \tag{145}$$

Using the constitutive relations (111) and (112), we have

$$\sigma_{(ji)} e_{ij} - 2\mu_i \kappa_i = A_{ijkl} e_{ij} e_{kl} + B_{ij} \kappa_i \kappa_j + 2C_{ijk} e_{ij} \kappa_k = 2W(\mathbf{e}, \boldsymbol{\kappa}) \tag{146}$$

Therefore, (145) can be written as

$$2\int_V W dV = \int_S t_i^{(n)} u_i dS + \int_S m_i^{(n)} \omega_i dS + \int_V F_i u_i dV \tag{147}$$

This relation gives twice the total stored energy in terms of the work of external body forces and surface tractions.

Now, we consider the general boundary value problem. The prescribed boundary conditions on the surface of the body can be any well-posed combination of vectors u_i and ω_i, $t_i^{(n)}$ and $m_i^{(n)}$ as discussed on Section 4. Assume that there exist two different solutions $\left\{u_i^{(1)}, e_{ij}^{(1)}, \kappa_i^{(1)}, \sigma_{ji}^{(1)}, \mu_i^{(1)}\right\}$ and $\left\{u_i^{(2)}, e_{ij}^{(2)}, \kappa_i^{(2)}, \sigma_{ji}^{(2)}, \mu_i^{(2)}\right\}$ to the same problem with identical body forces and boundary conditions. Thus, we have the equilibrium equations

$$\sigma_{ji,j}^{(\alpha)} + F_i = 0 \tag{148}$$

$$\sigma_{[ji]}^{(\alpha)} = -\mu_{[i,j]}^{(\alpha)} \tag{149}$$

where

$$\mu_i^{(\alpha)} = -\frac{1}{2} B_{ij} \kappa_j^{(\alpha)} - \frac{1}{2} C_{kji} e_{kj}^{(\alpha)} \tag{150a}$$

$$\sigma_{(ji)}^{(\alpha)} = A_{ijkl} e_{kl}^{(\alpha)} + C_{ijk} \kappa_k^{(\alpha)} \tag{150b}$$

and the superscript $^{(\alpha)}$ references the solutions $^{(1)}$ and $^{(2)}$.

Let us now define the *difference solution*

$$u_i' = u_i^{(2)} - u_i^{(1)} \tag{151a}$$

$$e_{ij}' = e_{ij}^{(2)} - e_{ij}^{(1)} \tag{151b}$$

$$\kappa_i' = \kappa_i^{(2)} - \kappa_i^{(1)} \tag{151c}$$

$$\sigma_{ji}' = \sigma_{ji}^{(2)} - \sigma_{ji}^{(1)} \tag{151d}$$

$$\mu_i' = \mu_i^{(2)} - \mu_i^{(1)} \tag{151e}$$

Since the solutions $\left\{u_i^{(1)}, e_{ij}^{(1)}, \kappa_i^{(1)}, \sigma_{ji}^{(1)}, \mu_i^{(1)}\right\}$ and $\left\{u_i^{(2)}, e_{ij}^{(2)}, \kappa_i^{(2)}, \sigma_{ji}^{(2)}, \mu_i^{(2)}\right\}$ correspond to the same body forces and boundary conditions, the *difference solution* must satisfy the equilibrium equations

$$\sigma_{ji,j}' = 0 \tag{152}$$

$$\sigma_{[ji]}' = -\mu_{[i,j]}' \tag{153}$$

with zero corresponding boundary conditions. Consequently, twice the total strain energy (147) for the *difference solution* is

$$\int_V 2W' dV = \int_V \left(A_{ijkl} e_{ij}' e_{kl}' + B_{ij} \kappa_i' \kappa_i' + 2C_{ijk} e_{ij}' \kappa_k'\right) dV = 0 \tag{154}$$

Since the stored energy density of the *difference solution* W' is non-negative, this relation requires

$$2W' = A_{ijkl} e_{ij}' e_{kl}' + B_{ij} \kappa_i' \kappa_j' + 2C_{ijk} e_{ij}' \kappa_k' = 0 \quad \text{in } V \tag{155}$$

However, the tensors A_{ijkl} and B_{ij} are positive definite and the tensor C_{ijk} is such that the energy W' is non-negative. Therefore the strain, curvature and associated stresses for the *difference solution* must vanish

$$e_{ij}' = 0, \quad \kappa_i' = 0, \quad \sigma_{ij}' = 0, \quad \mu_i' = 0 \tag{156a-d}$$

These require that the *difference displacement* u_i' can be at most a rigid body motion. However, if displacement is specified on part of the boundary such that rigid body motion is prevented, then the *difference displacement* vanishes everywhere and we have

$$u_i^{(1)} = u_i^{(2)} \tag{157a}$$

$$e_{ij}^{(1)} = e_{ij}^{(2)} \tag{157b}$$

$$\kappa_i^{(1)} = \kappa_i^{(2)} \tag{157c}$$

$$\sigma_{ji}^{(1)} = \sigma_{ji}^{(2)} \tag{157d}$$

$$\mu_i^{(1)} = \mu_i^{(2)} \tag{157e}$$

Therefore, the solution to the boundary value problem is unique. On the other hand, if only force and moment tractions are specified over the entire boundary, then the displacement is not unique and is determined only up to an arbitrary rigid body motion.

6.4. Reciprocal theorem

We derive now the general reciprocal theorem for the equilibrium states of a linear elastic material under different applied loads. Consider two sets of equilibrium states of compatible elastic solutions $\left\{u_i^{(1)}, \omega_i^{(1)}, t_i^{(n)(1)}, m_i^{(n)(1)}, F_i^{(1)}\right\}$ and $\left\{u_i^{(2)}, \omega_i^{(2)}, t_i^{(n)(2)}, m_i^{(n)(2)}, F_i^{(2)}\right\}$. Let us apply the virtual work theorem (86) in the forms

$$\int_V \left(\sigma_{(ji)}^{(1)} e_{ij}^{(2)} - 2\mu_i^{(1)} \kappa_i^{(2)}\right) dV = \int_S t_i^{(n)(1)} u_i^{(2)} dS + \int_S m_i^{(n)(1)} \omega_i^{(2)} dS + \int_V F_i^{(1)} u_i^{(2)} dV \tag{158}$$

$$\int_V \left(\sigma_{(ji)}^{(2)} e_{ij}^{(1)} - 2\mu_i^{(2)} \kappa_i^{(1)}\right) dV = \int_S t_i^{(n)(2)} u_i^{(1)} dS + \int_S m_i^{(n)(2)} \omega_i^{(1)} dS + \int_V F_i^{(2)} u_i^{(1)} dV \tag{159}$$

By using the general constitutive relations

$$\sigma_{(ji)}^{(1)} = A_{ijkl} e_{kl}^{(1)} + C_{ijk} \kappa_k^{(1)} \tag{160}$$

$$\mu_i^{(1)} = -\frac{1}{2} B_{ij} \kappa_j^{(1)} - \frac{1}{2} C_{kji} e_{kj}^{(1)} \tag{161}$$

$$\sigma_{(ji)}^{(2)} = A_{ijkl} e_{kl}^{(2)} + C_{ijk} \kappa_k^{(2)} \tag{162}$$

$$\mu_i^{(2)} = -\frac{1}{2} B_{ij} \kappa_j^{(2)} - \frac{1}{2} C_{kji} e_{kj}^{(2)} \tag{163}$$

it is seen that

$$\sigma_{ji}^{(1)} e_{ij}^{(2)} - 2\mu_i^{(1)} \kappa_i^{(2)} = A_{ijkl} e_{kl}^{(1)} e_{ij}^{(2)} + C_{ijk} \kappa_k^{(1)} e_{ij}^{(2)} + C_{kji} e_{kj}^{(1)} \kappa_i^{(2)} + B_{ij} \kappa_j^{(1)} \kappa_i^{(2)} \tag{164}$$

$$\sigma_{ji}^{(2)} e_{ij}^{(1)} - 2\mu_i^{(2)} \kappa_i^{(1)} = A_{ijkl} e_{kl}^{(2)} e_{ij}^{(1)} + C_{ijk} \kappa_k^{(2)} e_{ij}^{(1)} + C_{kji} e_k^{(2)} + B_{ij} \kappa_j^{(2)} \kappa_i^{(1)} \tag{165}$$

By using the symmetry relations (108)–(110) in (164) and (165), we obtain

$$\sigma_{ji}^{(1)} e_{ij}^{(2)} - 2\mu_i^{(1)} \kappa_i^{(2)} = \sigma_{ji}^{(2)} e_{ij}^{(1)} - 2\mu_i^{(2)} \kappa_i^{(1)} \tag{166}$$

which shows

$$\int_V \left(\sigma_{ji}^{(1)} e_{ij}^{(2)} - 2\mu_i^{(1)} \kappa_i^{(2)}\right) dV = \int_V \left(\sigma_{ji}^{(2)} e_{ij}^{(1)} - 2\mu_i^{(2)} \kappa_i^{(1)}\right) dV \tag{167}$$

Therefore, the general reciprocal theorem for these two elastic solutions is

$$\int_S t_i^{(n)(1)} u_i^{(2)} dS + \int_S m_i^{(n)(1)} \omega_i^{(2)} dS + \int_V F_i^{(1)} u_i^{(2)} dV = \int_S t_i^{(n)(2)} u_i^{(1)} dS + \int_S m_i^{(n)(2)} \omega_i^{(1)} dS + \int_V F_i^{(2)} u_i^{(1)} dV \quad (168)$$

7. Two-dimensional infinitesimal linear isotropic elasticity theory

In this section, we consider the two-dimensional infinitesimal linear isotropic couple stress theory of elasticity. It is seen that the results have similarity to the results of indeterminate couple stress theory (Mindlin, 1963). We start this development by assuming that the displacement components are two-dimensional, where in cartesian coordinates

$$u_1 = u(x, y), \quad u_2 = v(x, y), \quad u_3 = 0 \quad (169\text{a-c})$$

This is exactly the conditions for plane strain theory in Cauchy elasticity. The non-zero components of strains are

$$e_{xx} = \frac{\partial u}{\partial x}, \quad e_{yy} = \frac{\partial v}{\partial x}, \quad e_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \quad (170\text{a-c})$$

and the only non-zero rotation component is

$$\omega = \omega_z = \omega_{yx} = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \quad (171)$$

Therefore, the components of the mean curvature vector are

$$\kappa_x = -\kappa_{yz} = \frac{1}{2}\frac{\partial \omega}{\partial y}, \quad \kappa_y = \kappa_{xz} = -\frac{1}{2}\frac{\partial \omega}{\partial x} \quad (172\text{a, b})$$

It is seen that the compatibility equations for this case are

$$\frac{\partial^2 e_{xx}}{\partial y^2} + \frac{\partial^2 e_{yy}}{\partial x^2} = 2\frac{\partial^2 e_{xy}}{\partial x \partial y} \quad (173\text{a})$$

$$\frac{\partial \kappa_{yz}}{\partial x} = \frac{\partial \kappa_{xz}}{\partial y} \quad (173\text{b})$$

$$\frac{\partial \omega}{\partial x} = \frac{\partial e_{xy}}{\partial x} - \frac{\partial e_{xx}}{\partial y} \quad (173\text{c})$$

$$\frac{\partial \omega}{\partial y} = \frac{\partial e_{yy}}{\partial x} - \frac{\partial e_{xy}}{\partial y} \quad (173\text{d})$$

Then, the corresponding couple-stress components can be written

$$\mu_x = -\mu_{yz} = -4\eta\frac{\partial \omega}{\partial y}, \quad \mu_y = \mu_{xz} = 4\eta\frac{\partial \omega}{\partial x} \quad (174\text{a, b})$$

while the skew-symmetric and symmetric force-stress components are

$$\sigma_{[xy]} = -2\eta\nabla^2\omega \quad (175\text{a})$$

$$\sigma_{[yx]} = 2\eta\nabla^2\omega \quad (175\text{b})$$

$$\sigma_{(xx)} = \frac{2\mu}{1-2\nu}\left[(1-\nu)e_{xx} + \nu e_{yy}\right] \quad (176\text{a})$$

$$\sigma_{(yy)} = \frac{2\mu}{1-2\nu}\left[\nu e_{xx} + (1-\nu)e_{yy}\right] \quad (176\text{b})$$

$$\sigma_{(xy)} = 2\mu e_{xy} \quad (176\text{c})$$

Finally, total force-stress components are given by

$$\sigma_{xx} = \frac{2\mu}{1-2\nu}\left[(1-\nu)e_{xx} + \nu e_{yy}\right] \quad (177\text{a})$$

$$\sigma_{yy} = \frac{2\mu}{1-2\nu}\left[\nu e_{xx} + (1-\nu)e_{yy}\right] \quad (177\text{b})$$

$$\sigma_{xy} = 2\mu e_{xy} - 2\eta\nabla^2\omega \quad (177\text{c})$$

$$\sigma_{yx} = 2\mu e_{xy} + 2\eta\nabla^2\omega \quad (177\text{d})$$

where

$$\sigma_{xy} + \sigma_{yx} = 4\mu e_{xy} \quad (177\text{e})$$

Similarly to plane strain Cauchy elasticity, we have

$$\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy}) \quad (177\text{f})$$

When there is no body force, these stresses satisfy the equilibrium equations

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} = 0 \quad (178\text{a})$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0 \quad (178\text{b})$$

$$\frac{\partial \mu_{xz}}{\partial x} + \frac{\partial \mu_{yz}}{\partial y} + \sigma_{xy} - \sigma_{yx} = 0 \quad (178\text{c})$$

To solve for stresses, we need to derive compatibility equations in terms of stresses as follows. It is seen that

$$e_{xx} = \frac{1}{2\mu}\left[(1-\nu)\sigma_{xx} - \nu\sigma_{yy}\right] \quad (179\text{a})$$

$$e_{yy} = \frac{1}{2\mu}\left[(1-\nu)\sigma_{yy} - \nu\sigma_{xx}\right] \quad (179\text{b})$$

$$2e_{xy} = \frac{1}{2\mu}(\sigma_{xy} + \sigma_{yx}) \quad (179\text{c})$$

$$\nabla^2\omega = \frac{1}{4\eta}(\sigma_{yx} - \sigma_{xy}) \quad (179\text{d})$$

By inserting these in (173), we obtain the compatibility equations in terms of the force and couple-stress tensors. Thus,

$$\frac{\partial^2 \sigma_{xx}}{\partial y^2} + \frac{\partial^2 \sigma_{yy}}{\partial x^2} - \nu\nabla^2(\sigma_{xx} + \sigma_{yy}) = \frac{\partial^2}{\partial x \partial y}(\sigma_{yx} + \sigma_{xy}) \quad (180\text{a})$$

$$\frac{\partial \mu_{xz}}{\partial y} = \frac{\partial \mu_{yz}}{\partial x} \quad (180\text{b})$$

$$\mu_{xz} = l^2\frac{\partial}{\partial x}(\sigma_{yx} + \sigma_{xy}) - 2l^2\frac{\partial}{\partial y}\left[\sigma_{xx} - \nu(\sigma_{xx} + \sigma_{yy})\right] \quad (180\text{c})$$

$$\mu_{yz} = 2l^2\frac{\partial}{\partial x}\left[\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{yy})\right] - l^2\frac{\partial}{\partial y}(\sigma_{yx} + \sigma_{xy}) \quad (180\text{d})$$

By combining these with the equilibrium equations, we obtain the following full set of equations in terms of stresses

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} = 0 \quad (181\text{a})$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0 \quad (181\text{b})$$

$$\frac{\partial \mu_{xz}}{\partial x} + \frac{\partial \mu_{yz}}{\partial y} + \sigma_{xy} - \sigma_{yx} = 0 \quad (181\text{c})$$

$$\nabla^2(\sigma_{xx} + \sigma_{yy}) = 0 \quad (181\text{d})$$

$$\frac{\partial \mu_{xz}}{\partial y} = \frac{\partial \mu_{yz}}{\partial x} \quad (181\text{e})$$

$$\mu_{xz} = l^2\frac{\partial}{\partial x}(\sigma_{yx} + \sigma_{xy}) - 2l^2\frac{\partial}{\partial y}\left[\sigma_{xx} - \nu(\sigma_{xx} + \sigma_{yy})\right] \quad (181\text{f})$$

$$\mu_{yz} = 2l^2\frac{\partial}{\partial x}\left[\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{yy})\right] - l^2\frac{\partial}{\partial y}(\sigma_{yx} + \sigma_{xy}) \quad (181\text{g})$$

The equilibrium equations (181a-c) can be identically satisfied by choosing the representations

$$\sigma_{xx} = \frac{\partial^2 \Phi}{\partial y^2} - \frac{\partial^2 \Psi}{\partial x \partial y} \tag{182a}$$

$$\sigma_{yy} = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial x \partial y} \tag{182b}$$

$$\sigma_{xy} = -\frac{\partial^2 \Phi}{\partial x \partial y} - \frac{\partial^2 \Psi}{\partial y^2} \tag{182c}$$

$$\sigma_{yx} = -\frac{\partial^2 \Phi}{\partial x \partial y} + \frac{\partial^2 \Psi}{\partial x^2} \tag{182d}$$

$$\mu_{xz} = \frac{\partial \Psi}{\partial x} \tag{183a}$$

$$\mu_{yz} = \frac{\partial \Psi}{\partial y} \tag{183b}$$

where $\Phi = \Phi(x,y)$ and $\Psi = \Psi(x,y)$ are stress functions. While the compatibility equation (181e) is self-satisfied, the compatibility equations (181d), (181f) and (181g) reduce to

$$\nabla^2 \nabla^2 \Phi = 0 \tag{184}$$

$$\frac{\partial}{\partial x}\left(\Psi - l^2 \nabla^2 \Psi\right) = -2(1-\nu) l^2 \frac{\partial}{\partial y}\left(\nabla^2 \Phi\right) \tag{185}$$

$$\frac{\partial}{\partial y}\left(\Psi - l^2 \nabla^2 \Psi\right) = 2(1-\nu) l^2 \frac{\partial}{\partial x}\left(\nabla^2 \Phi\right) \tag{186}$$

Combining (185) and (186) by eliminating Φ, we obtain

$$\nabla^2 \Psi - l^2 \nabla^4 \Psi = 0 \tag{187}$$

All these relations are exactly the equations derived by Mindlin (1963). This shows that the solutions for two-dimensional cases based on Mindlin's development, such as stress concentration relations for a plate with a circular hole, still can be used. However, we should notice that in Mindlin (1963), there are two size-dependent constants η and η', along with an indeterminacy in the spherical part of couple-stress tensor. Based upon Mindlin, the couple-stresses μ_{xx}, μ_{yy} and μ_{zz} are indeterminate, while μ_{zx} and μ_{zy} are given by

$$\mu_{zx} = \frac{\eta'}{\eta} \mu_{xz} = 4\eta' \frac{\partial \omega}{\partial x} \tag{188a}$$

$$\mu_{zy} = \frac{\eta'}{\eta} \mu_{yz} = 4\eta' \frac{\partial \omega}{\partial y} \tag{188b}$$

In the present couple stress theory, we have only one single size-dependent constant η and the couple-stress tensor is skew-symmetric without indeterminacy. Interestingly, the relations (188) become identical to those in the present theory, when we take $\eta' = -\eta$. Thus, we may solve the boundary value problem in an identical manner to Mindlin (1963), but then evaluate the determinate couple-stresses through a postprocessing operation.

More specifically, by comparing the relations (174) and (183), we can see

$$\mu_{xz} = 4\eta \frac{\partial \omega}{\partial x} = \frac{\partial \Psi}{\partial x} \tag{189a}$$

$$\mu_{yz} = 4\eta \frac{\partial \omega}{\partial y} = \frac{\partial \Psi}{\partial y} \tag{189b}$$

Therefore, we can take

$$\Psi = 4\eta\omega + c \tag{190}$$

where c is an arbitrary constant, which can be chosen as zero. If Ψ is zero (or constant), there are no couple-stress tensor components, and the relations for the force-stress tensor reduce to the relations in classical elasticity, where Φ is the Airy stress function.

For force and moment traction components, we have

$$t_x^{(n)} = \sigma_{xx} n_x + \sigma_{yx} n_y \tag{191a}$$

$$t_y^{(n)} = \sigma_{xy} n_x + \sigma_{yy} n_y \tag{191b}$$

$$m = m_z^{(n)} = \mu_{xz} n_x + \mu_{yz} n_y \tag{191c}$$

which can be written in terms of stress functions as

$$t_x^{(n)} = \left(\frac{\partial \Phi}{\partial y^2} - \frac{\partial^2 \Psi}{\partial x \partial y}\right) n_x + \left(-\frac{\partial^2 \Phi}{\partial x \partial y} + \frac{\partial^2 \Psi}{\partial x^2}\right) n_y \tag{192a}$$

$$t_y^{(n)} = \left(-\frac{\partial^2 \Phi}{\partial x \partial y} - \frac{\partial^2 \Psi}{\partial y^2}\right) n_x + \left(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial x \partial y}\right) n_y \tag{192b}$$

$$m = \frac{\partial \Psi}{\partial x} n_x + \frac{\partial \Psi}{\partial y} n_y \tag{192c}$$

If the location on the boundary contour in the x–y plane is specified by the coordinate s in a positive sense, we have

$$n_x = \frac{dy}{ds} \tag{193a}$$

$$n_y = -\frac{dx}{ds} \tag{193b}$$

Therefore

$$t_x^{(n)} = \frac{d}{ds}\left(\frac{\partial \Phi}{\partial y} - \frac{\partial \Psi}{\partial x}\right) \tag{194a}$$

$$t_y^{(n)} = -\frac{d}{ds}\left(\frac{\partial \Phi}{\partial x} + \frac{\partial \Psi}{\partial y}\right) \tag{194b}$$

$$m = \frac{\partial \Psi}{\partial n} = 4\eta \frac{\partial \omega}{\partial n} \tag{194c}$$

In polar coordinates, the equilibrium equations become

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta r}}{\partial \theta} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0 \tag{195a}$$

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\sigma_{r\theta} - \sigma_{\theta r}}{r} = 0 \tag{195b}$$

$$\frac{\partial \mu_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \mu_{\theta z}}{\partial \theta} + \frac{\mu_{rz}}{r} + \sigma_{r\theta} - \sigma_{\theta r} = 0 \tag{195c}$$

while the strain-deformation relations are

$$e_{rr} = \frac{\partial u_r}{\partial r}, \quad e_{\theta\theta} = \frac{1}{r}\left(u_r + \frac{\partial u_\theta}{\partial \theta}\right), \quad e_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) \tag{196a-c}$$

and the only non-zero rotation component is

$$\omega = \omega_z = \omega_{r\theta} = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r} - \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right) \tag{197}$$

Therefore, the components of the mean curvature vector are

$$\kappa_r = \kappa_{z\theta} = \frac{1}{2}\frac{1}{r}\frac{\partial \omega}{\partial \theta}, \quad \kappa_\theta = -\kappa_{zr} = -\frac{1}{2}\frac{\partial \omega}{\partial r} \tag{198a, b}$$

The constitutive relations are

$$\sigma_{rr} = \frac{2\mu}{1-2\nu}\left[(1-\nu) e_{rr} + \nu e_{\theta\theta}\right] \tag{199a}$$

$$\sigma_{\theta\theta} = \frac{2\mu}{1-2\nu}\left[\nu e_{rr} + (1-\nu) e_{\theta\theta}\right] \tag{199b}$$

$$\sigma_{r\theta} = 2\mu e_{r\theta} - 2\eta \nabla^2 \omega \tag{199c}$$

$$\sigma_{\theta r} = 2\mu e_{r\theta} + 2\eta \nabla^2 \omega \tag{199d}$$

where

$$\sigma_{r\theta} + \sigma_{\theta r} = 4\mu e_{r\theta} \tag{199e}$$

It is also seen that

$$\sigma_{zz} = \nu(\sigma_{rr} + \sigma_{\theta\theta}) \tag{199f}$$

In the polar coordinate case, stresses can be expressed in terms of stress functions as

$$\sigma_{rr} = \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} - \frac{1}{r}\frac{\partial^2 \Psi}{\partial r \partial \theta} + \frac{1}{r^2}\frac{\partial \Psi}{\partial \theta} \tag{200a}$$

$$\sigma_{\theta\theta} = \frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial^2 \Psi}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial \Psi}{\partial \theta} \tag{200b}$$

$$\sigma_{r\theta} = -\frac{1}{r}\frac{\partial^2 \Phi}{\partial r \partial \theta} + \frac{1}{r^2}\frac{\partial \Phi}{\partial \theta} - \frac{1}{r}\frac{\partial \Psi}{\partial r} - \frac{1}{r^2}\frac{\partial^2 \Psi}{\partial \theta^2} \tag{200c}$$

$$\sigma_{\theta r} = -\frac{1}{r}\frac{\partial^2 \Phi}{\partial r \partial \theta} + \frac{1}{r^2}\frac{\partial \Phi}{\partial \theta} + \frac{\partial^2 \Psi}{\partial r^2} \tag{200d}$$

$$\mu_\theta = \mu_{rz} = \frac{\partial \Psi}{\partial r} \tag{201a}$$

$$-\mu_r = \mu_{\theta z} = \frac{1}{r}\frac{\partial \Psi}{\partial \theta} \tag{201b}$$

where

$$\nabla^2\nabla^2 \Phi = 0 \tag{202}$$

$$\nabla^2 \Psi - l^2 \nabla^4 \Psi = 0 \tag{203}$$

and

$$\frac{\partial}{\partial r}\left(\Psi - l^2\nabla^2\Psi\right) = -2(1-\nu)l^2\frac{1}{r}\frac{\partial}{\partial \theta}\left(\nabla^2\Phi\right) \tag{204}$$

$$\frac{1}{r}\frac{\partial}{\partial \theta}\left(\Psi - l^2\nabla^2\Psi\right) = 2(1-\nu)l^2\frac{\partial}{\partial r}\left(\nabla^2\Phi\right) \tag{205}$$

8. Anti-plane deformation infinitesimal linear isotropic elasticity theory

We assume the displacement components are

$$u_1 = 0, \quad u_2 = 0, \quad u_3 = w(x,y) \tag{206a-c}$$

These are exactly the conditions for anti-plane deformation in Cauchy elasticity. The non-zero components of strains are

$$e_{zx} = \frac{1}{2}\frac{\partial w}{\partial x}, \quad e_{zy} = \frac{1}{2}\frac{\partial w}{\partial y} \tag{207a, b}$$

and the non-zero rotation components are

$$\omega_x = \frac{1}{2}\frac{\partial w}{\partial y}, \quad \omega_y = -\frac{1}{2}\frac{\partial w}{\partial x} \tag{208a, b}$$

Therefore, the only non-zero component of the mean curvature vector is

$$\kappa_z = \kappa_{yx} = \frac{1}{2}\left(\frac{\partial \omega_y}{\partial x} - \frac{\partial \omega_x}{\partial y}\right) = -\frac{1}{4}\nabla^2 w \tag{209}$$

Then, the corresponding couple-stress component is given by

$$\mu_z = \mu_{yx} = 2\eta\nabla^2 w \tag{210}$$

while the skew-symmetric and symmetric force-stress components are

$$\sigma_{(zx)} = \mu\frac{\partial w}{\partial x}, \quad \sigma_{(zy)} = \mu\frac{\partial w}{\partial y} \tag{211a, b}$$

$$\sigma_{[xz]} = -\frac{1}{2}\frac{\partial \mu_z}{\partial x} = -\eta\frac{\partial}{\partial x}\nabla^2 w, \quad \sigma_{[yz]} = -\frac{1}{2}\frac{\partial \mu_z}{\partial y} = -\eta\frac{\partial}{\partial y}\nabla^2 w \tag{211c, d}$$

Therefore

$$\sigma_{xz} = \mu\frac{\partial w}{\partial x} - \eta\frac{\partial}{\partial x}\nabla^2 w, \quad \sigma_{zx} = \mu\frac{\partial w}{\partial x} + \eta\frac{\partial}{\partial x}\nabla^2 w \tag{212a, b}$$

$$\sigma_{yz} = \mu\frac{\partial w}{\partial y} - \eta\frac{\partial}{\partial y}\nabla^2 w, \quad \sigma_{zy} = \mu\frac{\partial w}{\partial y} + \eta\frac{\partial}{\partial y}\nabla^2 w \tag{212c, d}$$

When there is no body force, these stresses satisfy the equilibrium equation

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} = 0 \tag{213}$$

which in terms of displacement gives the single fourth order equation

$$\nabla^2 w - l^2\nabla^2\nabla^2 w = 0 \tag{214}$$

For force and moment traction components on the boundary contour in the x-y plane, we have

$$t_z^{(n)} = \sigma_{xz}n_x + \sigma_{yz}n_y = \mu\frac{\partial}{\partial n}(w - l^2\nabla^2 w) \tag{215a}$$

$$m_x^{(n)} = \mu_z n_y = 2\eta\nabla^2 w n_y \tag{215b}$$

$$m_y^{(n)} = -\mu_z n_x = -2\eta\nabla^2 w n_y \tag{215c}$$

It is important to note that if the moment traction vector $\mathbf{m}^{(n)}$ is zero on the boundary, the solution reduces to the classical Cauchy elasticity solution

$$\nabla^2 w = 0 \tag{216}$$

with

$$\mu_z = \mu_{yx} = 0 \tag{217a}$$

$$\sigma_{zx} = \sigma_{xz} = \mu\frac{\partial w}{\partial x} \tag{217b}$$

$$\sigma_{yz} = \sigma_{zy} = \mu\frac{\partial w}{\partial y} \tag{217c}$$

everywhere in the domain. However, specification of tangential rotation $\omega^{(s)} = -\frac{1}{2}\frac{\partial w}{\partial n}$ as a geometrical boundary condition creates couple-stresses in the body. In that case, the classical solution cannot be used. As we mentioned, this essential boundary condition cannot usually be specified in practice. Therefore, anti-plane deformation usually follows the classical Cauchy elasticity.

9. Sample problems of isotropic elasticity

In this section, several problems in classical Cauchy elasticity are reconsidered within the framework of the present infinitesimal linear size-dependent theory. Koiter (1964) examined the first three elementary problems in which classical deformations are assumed. Some differences appear between his results and those obtained from the present consistent theory. The fourth example is more involved and requires solution to a boundary value problem.

9.1. Twist of a cylindrical bar

Consider the x_3-axis of our coordinate system along the axis of a cylindrical bar with constant cross section. We assume the displacement components are in the form as in the classical theory and examine the corresponding stress field in the couple stress theory. The assumed displacement components are

$$u_1 = -\theta x_2 x_3, \quad u_2 = \theta x_1 x_3, \quad u_3 = 0 \tag{218a-c}$$

where θ is the constant angle of twist per unit length. The non-zero components of the strain tensor and rotation vector are

$$e_{13} = -\frac{1}{2}\theta x_2, \quad e_{23} = \frac{1}{2}\theta x_1 \tag{219a, b}$$

$$\omega_1 = \omega_{32} = -\frac{1}{2}\theta x_1, \quad \omega_2 = \omega_{13} = -\frac{1}{2}\theta x_2, \quad \omega_3 = \omega_{21} = \theta x_3 \tag{220a-c}$$

Interestingly, it is seen that the curvature vector vanishes

$$\boldsymbol{\kappa} = \frac{1}{2}\nabla \times \boldsymbol{\omega} = \mathbf{0} \tag{221}$$

Therefore, the force-stress distribution is the classical result

$$\sigma_{13} = -\mu\theta x_2, \quad \sigma_{23} = \mu\theta x_1 \tag{222a, b}$$

and the twist of a cylindrical bar does not generate couple-stresses. This is in contrast with the Koiter (1964) result, in which couple-stresses appear.

9.2. Cylindrical bending of a flat plate

Consider a flat material plate of thickness h bent into a cylindrical shell with generators parallel to the x_3-axis. Let R denote the radius of curvature of the middle plane x_1x_3 in the deformed configuration. We assume the displacement components are similar to those in Cauchy elasticity. Thus,

$$u_1 = -\frac{1}{R}x_1x_2, \quad u_2 = \frac{1}{2}\frac{1}{R}x_1^2 + \frac{1}{2}\frac{\nu}{1-\nu}\frac{1}{R}x_2^2, \quad u_3 = 0 \tag{223a-c}$$

The non-zero components of the strain tensor, rotation vector and mean curvature vector are

$$e_{11} = -\frac{1}{R}x_2, \quad e_{22} = \frac{\nu}{1-\nu}\frac{1}{R}x_2 \tag{224a, b}$$

$$\omega_3 = \omega_{21} = \frac{x_1}{R} \tag{225}$$

$$\kappa_{31} = -\kappa_2 = \frac{1}{2R} \tag{226}$$

Therefore, the non-zero force and couple-stresses are written as

$$\sigma_{11} = -\frac{2\mu}{1-\nu}\frac{x_2}{R}, \quad \sigma_{33} = -\frac{2\mu\nu}{1-\nu}\frac{x_2}{R} \tag{227a, b}$$

$$\mu_2 = \mu_{13} = -\mu_{31} = 4\frac{\eta}{R} \tag{228}$$

Notice that unlike the previous example of twisting deformation, bending does produce couple-stresses. This is due to the existence of non-zero mean curvature.

9.3. Pure bending of a bar with rectangular cross-section

We take the x_1-axis to coincide with the centerline of the rectangular beam and the other axes parallel to the sides of the cross section of the beam. Let R denote the radius of curvature of the central axis of the beam after bending in the x_1x_3-plane. We assume the displacement components are the same as in the classical Cauchy elasticity theory as follows:

$$u_1 = \frac{1}{R}x_1x_3, \quad u_2 = -\frac{\nu}{R}x_2x_3, \quad u_3 = \frac{\nu}{2R}\left(x_2^2 - x_3^2\right) - \frac{1}{2R}x_1^2 \tag{229a-c}$$

Then, the strains, rotations and mean curvatures can be written

$$e_{11} = \frac{x_3}{R}, \quad e_{22} = e_{33} = -\frac{\nu x_3}{R} \tag{230a, b}$$

$$\omega_1 = \omega_{32} = \frac{\nu x_2}{R}, \quad \omega_2 = \omega_{13} = \frac{x_1}{R} \tag{231a, b}$$

$$\kappa_1 = \kappa_{32} = \frac{1}{2}(\omega_{3,2} - \omega_{2,3}) = 0 \tag{232a}$$

$$\kappa_2 = \kappa_{13} = \frac{1}{2}(\omega_{1,3} - \omega_{3,1}) = 0 \tag{232b}$$

$$\kappa_3 = \kappa_{21} = \frac{1}{2}(\omega_{2,1} - \omega_{1,2}) = \frac{1-\nu}{2R} \tag{232c}$$

As a result, the non-zero force- and couple-stresses take the form

$$\sigma_{11} = 2\mu(1+\nu)\frac{x_3}{R} \tag{233a}$$

$$\mu_3 = \mu_{21} = -\mu_{12} = -4\eta\frac{1-\nu}{R} \tag{233b}$$

Again, for this problem, we find non-zero mean curvature and couple-stresses.

9.4. Deformation of a plane ring

As a final example, we consider a plane ring, rigidly fixed on the external circular boundary at $r = b$, under deformation due to a rigid displacement of the internal circular boundary at $r = a$ with magnitude U in the x_1 direction. For the displacement components in polar coordinates at $r = a$, we have

$$u_r = U\cos\theta \tag{234a}$$

$$u_\theta = -U\sin\theta \tag{234b}$$

The appropriate stress functions for this problem are

$$\Phi = \left[\frac{A_2}{r} + A_3r^3 + A_4r\ln r\right]\cos\theta + A_5r\theta\sin\theta \tag{235}$$

$$\Psi = 4\mu l^2\omega = \left[B_1r + \frac{B_2}{r} + B_3I_1\left(\frac{r}{l}\right) + B_4K_1\left(\frac{r}{l}\right)\right]\sin\theta \tag{236}$$

where I_n and K_n are the modified Bessel functions of first and second kind of order n, respectively. The eight constants $A_2, A_3, A_4, A_5, B_1, B_2, B_3$ and B_4 are to be determined. From (201) and (200), for couple- and force- stresses, we have

$$\mu_{rz} = \mu_\theta = \left\{B_1 - \frac{B_2}{r^2} + B_3\left[\frac{1}{l}I_0\left(\frac{r}{l}\right) - \frac{1}{r}I_1\left(\frac{r}{l}\right)\right] - B_4\left[\frac{1}{l}K_0\left(\frac{r}{l}\right) + \frac{1}{r}K_1\left(\frac{r}{l}\right)\right]\right\}\sin\theta \tag{237a}$$

$$\mu_{\theta z} = -\mu_r = \left[B_1 + \frac{B_2}{r^2} + B_3\frac{1}{r}I_1\left(\frac{r}{l}\right) + B_4\frac{1}{r}K_1\left(\frac{r}{l}\right)\right]\cos\theta \tag{237b}$$

$$\sigma_{rr} = \left\{\begin{array}{l} -\frac{2A_2}{r^3} + 2A_3r + \frac{A_4}{r} + \frac{2A_5}{r} + \frac{2B_2}{r^3} \\ -B_3\left[\frac{1}{lr}I_0\left(\frac{r}{l}\right) - \frac{2}{r^2}I_1\left(\frac{r}{l}\right)\right] + B_4\left[\frac{1}{lr}K_0\left(\frac{r}{l}\right) + \frac{2}{r^2}K_1\left(\frac{r}{l}\right)\right]\end{array}\right\}\cos\theta \tag{237c}$$

$$\sigma_{\theta\theta} = \left\{\begin{array}{l} \frac{2A_2}{r^3} + 6A_3r + \frac{A_4}{r} - \frac{2B_2}{r^3} \\ +B_3\left[\frac{1}{lr}I_0\left(\frac{r}{l}\right) - \frac{2}{r^2}I_1\left(\frac{r}{l}\right)\right] - B_4\left[\frac{1}{lr}K_0\left(\frac{r}{l}\right) + \frac{2}{r^2}K_1\left(\frac{r}{l}\right)\right]\end{array}\right\}\cos\theta \tag{237d}$$

$$\sigma_{r\theta} = \left\{\begin{array}{l} -\frac{2A_2}{r^3} + 2A_3r + \frac{A_4}{r} + \frac{2B_2}{r^3} \\ -B_3\left[\frac{1}{lr}I_0\left(\frac{r}{l}\right) - \frac{2}{r^2}I_1\left(\frac{r}{l}\right)\right] + B_4\left[\frac{1}{lr}K_0\left(\frac{r}{l}\right) + \frac{2}{r^2}K_1\left(\frac{r}{l}\right)\right]\end{array}\right\}\sin\theta \tag{237e}$$

$$\sigma_{\theta r} = \left\{\begin{array}{l} -\frac{2A_2}{r^3} + 2A_3r + \frac{A_4}{r} + \frac{2B_2}{r^3} + B_3\left[\frac{1}{l^2}I_1\left(\frac{r}{l}\right) - \frac{1}{lr}I_0\left(\frac{r}{l}\right) + \frac{2}{r^2}I_1\left(\frac{r}{l}\right)\right] \\ +B_4\left[\frac{1}{l^2}K_1\left(\frac{r}{l}\right) + \frac{1}{lr}K_0\left(\frac{r}{l}\right) + \frac{2}{r^2}K_1\left(\frac{r}{l}\right)\right]\end{array}\right\}\sin\theta \tag{237f}$$

By using (237e) and (237f) in (199e), we obtain

$$4\mu e_{r\theta} = \left\{\begin{array}{l} -\frac{4A_2}{r^3} + 4A_3r + \frac{2A_4}{r} + \frac{4B_2}{r^3} + B_3\left[\frac{1}{l^2}I_1\left(\frac{r}{l}\right) - \frac{2}{lr}I_0\left(\frac{r}{l}\right) + \frac{4}{r^2}I_1\left(\frac{r}{l}\right)\right] \\ +B_4\left[\frac{1}{l^2}K_1\left(\frac{r}{l}\right) + \frac{2}{lr}K_0\left(\frac{r}{l}\right) + \frac{4}{r^2}K_1\left(\frac{r}{l}\right)\right]\end{array}\right\}\sin\theta \tag{238}$$

By using (199a-b) to obtain strains and then (196), we obtain the displacement components

$$2\mu u_r = \begin{bmatrix} A_1 + \frac{A_2}{r^2} + (1-4\nu)A_3 r^2 + (1-2\nu)A_4 \ln r + 2(1-\nu)A_5 \ln r \\ -\frac{B_2}{r^2} - B_3 \frac{1}{r} I_1\left(\frac{r}{l}\right) - B_4 \frac{1}{r} K_1\left(\frac{r}{l}\right) \end{bmatrix} \cos\theta \tag{239a}$$

$$2\mu u_\theta = \begin{Bmatrix} -A_1 + \frac{A_2}{r^2} + (5-4\nu)A_3 r^2 \\ +(1-2\nu)A_4(1-\ln r) - 2[\nu + (1-\nu)\ln r]A_5 \\ -\frac{B_2}{r^2} + B_3\left[\frac{1}{l} I_0\left(\frac{r}{l}\right) - \frac{1}{r} I_1\left(\frac{r}{l}\right)\right] - B_4\left[\frac{1}{l} K_0\left(\frac{r}{l}\right) + \frac{1}{r} K_1\left(\frac{r}{l}\right)\right] \end{Bmatrix} \sin\theta \tag{239b}$$

where the additional terms involving A_1 account for rigid-body translation of the ring in the x_1 direction. By using the displacement components from (239) in (196c) and (197), we obtain alternative expressions for $e_{r\theta}$ and ω

$$4\mu e_{r\theta} = \begin{Bmatrix} -\frac{4A_2}{r^3} + 4A_3 r - \frac{2(1-2\nu)A_4}{r} - \frac{2(1-2\nu)A_5}{r} \\ +\frac{4B_2}{r^3} + B_3\left[\frac{1}{l^2} I_1\left(\frac{r}{l}\right) - \frac{2}{lr} I_0\left(\frac{r}{l}\right) + \frac{4}{r^2} I_1\left(\frac{r}{l}\right)\right] \\ +B_4\left[\frac{1}{l^2} K_1\left(\frac{r}{l}\right) + \frac{2}{lr} K_0\left(\frac{r}{l}\right) + \frac{4}{r^2} K_1\left(\frac{r}{l}\right)\right] \end{Bmatrix} \sin\theta \tag{240}$$

$$4\mu\omega = \left[16(1-\nu)A_3 r - 2\frac{A_5}{r} + B_3 \frac{1}{l^2} I_1\left(\frac{r}{l}\right) + B_4 \frac{1}{l^2} K_1\left(\frac{r}{l}\right)\right] \sin\theta \tag{241}$$

After comparing (240) with (238) and (241) with (236), we obtain the constraint equations among coefficients

$$A_4 = -\frac{1-2\nu}{2(1-\nu)} A_5 \tag{242a}$$

$$B_1 = 16(1-\nu) l^2 A_3 \tag{242b}$$

$$B_2 = -2l^2 A_5 \tag{242c}$$

Therefore, it is seen that

$$2\mu u_r = \begin{bmatrix} A_1 + \frac{A_2}{r^2} + (1-4\nu)A_3 r^2 + A_5\left[\frac{3-4\nu}{2(1-\nu)} \ln r + \frac{2l^2}{r^2}\right] \\ -B_3 \frac{1}{r} I_1\left(\frac{r}{l}\right) - B_4 \frac{1}{r} K_1\left(\frac{r}{l}\right) \end{bmatrix} \cos\theta \tag{243a}$$

$$2\mu u_\theta = \begin{Bmatrix} -A_1 + \frac{A_2}{r^2} + (5-4\nu)A_3 r^2 - A_5\left[\frac{(3-4\nu)\ln r + 1}{2(1-\nu)} - \frac{2l^2}{r^2}\right] \\ +B_3\left[\frac{1}{l} I_0\left(\frac{r}{l}\right) - \frac{1}{r} I_1\left(\frac{r}{l}\right)\right] - B_4\left[\frac{1}{l} K_0\left(\frac{r}{l}\right) + \frac{1}{r} K_1\left(\frac{r}{l}\right)\right] \end{Bmatrix} \sin\theta \tag{243b}$$

$$\mu_{rz} = \mu_\theta = \begin{Bmatrix} 16(1-\nu) l^2 A_3 + \frac{2l^2}{r^2} A_5 \\ +B_3\left[\frac{1}{l} I_0\left(\frac{r}{l}\right) - \frac{1}{r} I_1\left(\frac{r}{l}\right)\right] - B_4\left[\frac{1}{l} K_0\left(\frac{r}{l}\right) + \frac{1}{r} K_1\left(\frac{r}{l}\right)\right] \end{Bmatrix} \sin\theta \tag{243c}$$

Similarly, for $\mu_{\theta z}$, σ_{rr}, $\sigma_{\theta\theta}$, $\sigma_{r\theta}$ and $\sigma_{\theta r}$, we can write expressions that involve only the six coefficients A_1, A_2, A_3, A_5, B_3 and B_4.

We have the following six boundary conditions. At $r = a$

$$u_r = U\cos\theta, \quad u_\theta = -U\sin\theta, \quad m = -\mu_{rz} = 0 \tag{244a-c}$$

and at $r = b$,

$$u_r = 0, \quad u_\theta = 0, \quad m = \mu_{rz} = 0 \tag{244d-f}$$

Notice that we have taken the moment traction m to vanish on the whole boundary, which is consistent with the usual reality. Using the boundary conditions (244), we can obtain the six unknown coefficients to complete the solution.

10. Conclusion

By considering further the consequences of the kinematics of a continuum, definition of admissible boundary conditions and the principle of virtual work, we find that couple stress theory can be formulated as a practical theory without any ambiguity. In the resulting theory, independent body couples cannot be specified in the volume and surface moments can only exist in the tangent plane at each boundary point. As a consequence, the couple-stress tensor is found to be skew-symmetric and energetically conjugate to the mean curvature tensor, which also is skew-symmetrical. This is a general result, independent of material properties, which makes size-dependent continuum mechanics possible.

For infinitesimal or small deformation linear elasticity, we can write constitutive relations for all of the components of the force-stress and couple-stress tensors. The most general anisotropic elastic material is described by 45 independent constitutive coefficients. This includes six coefficients relating mean curvatures to couple-stresses and 18 coefficients relating strain and mean curvatures to couple-stresses and the symmetric part of force-stresses, respectively. At the other extreme, for isotropic materials, the two Lamé parameters and one length scale completely characterize the behavior. In addition, stored energy relations, along with uniqueness and reciprocal theorems, have been developed for linear elasticity. General formulations for two-dimensional and anti-plane problems are also elucidated for the isotropic case. The former employs a pair of stress functions, as introduced previously by Mindlin for the indeterminate theory. Finally, several elementary problems are examined within the context of small deformation elasticity, along with a more complicated boundary value problem.

The present theory provides a fundamental basis for the development of consistent scale-dependent material response from a continuum mechanics view. Additional aspects of the linear elastic theory, including fundamental solutions and computational mechanics formulations, will be addressed in forthcoming work. Beyond this, the present theory should be useful for the development of nonlinear elastic, elastoplastic, viscoplastic and damage mechanics formulations that may govern the behavior of solid continua at the smallest scales.

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