Linear Algebra and its Applications 432 (2010) 3132–3140

Contents lists available at ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

On the eigenvalues of distance powers of circuits

J.W. Sander [a], T. Sander [b,*]

[a] *Institut für Mathematik und Angewandte Informatik, Universität Hildesheim, D-31141 Hildesheim, Germany*
[b] *Institut für Mathematik, Technische Universität Clausthal, D-38678 Clausthal-Zellerfeld, Germany*

A R T I C L E I N F O

Article history:
Received 20 October 2009
Accepted 8 January 2010
Available online 11 February 2010

Submitted by S. Kirkland

AMS classification:
Primary: 05C50
Secondary: 15A18

Keywords:
Distance power
Circuit graph
Integral eigenvalues
Eigenspace basis

A B S T R A C T

Taking the dth distance power of a graph, one adds edges between all pairs of vertices of that graph whose distance is at most d. It is shown that only the numbers $-3, -2, -1, 0, 1, 2d$ can be integer eigenvalues of a circuit distance power. Moreover, their respective multiplicities are determined and explicit constructions for corresponding eigenspace bases containing only vectors with entries -1, 0, 1 are given.

1. Introduction

Given a graph G and a positive integer d, the dth *distance power* $G^{(d)}$ of G is obtained from G by adding edges between all pairs of vertices whose distance is at most d. This implies that $G^{(1)}$ is isomorphic to G. We are interested in distance powers of the circuit graph on n vertices (denoted by C_n). They belong to the important class of *circulant graphs*.

Circulant graphs are characterized as follows. Assume that the vertices of a given graph are $0, 1, \ldots, n-1$ and consider the set N of neighbors of vertex 0. The graph is circulant if and only if under every possible cyclic rotation of the vertex numbers the set of neighbors of the new vertex 0 remains N. We shall call N the *jump set* of the graph. Circulant graphs have many fascinating properties, cf. [1], and interesting applications. For example, they play a role in the study of redundant communication networks [2].

* Corresponding author.
E-mail address: torsten.sander@tu-clausthal.de (T. Sander).

0024-3795/$ - see front matter
doi:10.1016/j.laa.2010.01.012

Moreover, circulant graphs model quantum systems. Such a system is periodic if and only if its graph is *integral*, i.e. if it has only integer eigenvalues [3]. The eigenvalues of a graph are the eigenvalues of its adjacency matrix. This matrix is defined by numbering the vertices of the graph with $0, \ldots, n-1$ and letting the entry at position (i, j) be one if the vertices numbered i and j are adjacent and zero otherwise.[1] The zero-one pattern of the adjacency matrix depends on the chosen vertex numbering of the vertices, but it follows from basic linear algebra that its eigenvalues do not. The set of eigenvalues of a graph, called its *spectrum*, reflects several structural properties of the graph (see, e.g. the books [4,5]).

There exists an elegant condition due to So [6] that asserts a given circulant graph with vertices $0, 1, \ldots, n-1$ is integral if and only if its jump set N consists of complete sets of numbers having the same gcd with n. Let us check this condition for distance powers $C_n^{(d)}$ of circuits. Clearly, the jump set of a distance power $C_n^{(d)}$ is $\{1, 2, \ldots, d, n-d, \ldots, n-2, n-1\}$. Note that for odd n the condition $k \in N$ is equivalent to $C_n^{(d)}$ being the complete graph, for even n take $k-1 \in N$. Now assume $n = 2k+1$ and let $g = \gcd(k, n)$. From the properties of the gcd it easily follows that necessarily $g|1$ and therefore $k \in N$, since $d \geqslant 1$. Similarly, it follows for $n = 2k$ and $g = \gcd(k-1, n)$ that $g|2$. So, for $d \geqslant 2$, we have $k-1 \in N$. The bottom line is that a non-complete integral circuit distance power must necessarily have $d = 1$, which means it is a circuit. The eigenvalues of C_n are the numbers $2\cos(2\pi r/n)$ for $r = 0, \ldots, n-1$. (cf. [7]). It is readily checked that the only non-complete integral circuit distance power are C_4 and C_6.

So, since integrality is out of reach for non-complete circuit distance powers, we answer the question which integers are possible eigenvalues at all. Some partial results exist in the literature. For instance, it is well known and easy to show that the circuit graph C_n itself has eigenvalue 0 if and only if $4|n$ (see [7]). Much more involved arithmetic expressions are required to describe singularity of circuit distance powers $C_n^{(d)}$ (see [8] and Theorem 6 in Section 2). In the special case of circuit squares $C_n^{(2)}$ related results were found by Davis et al. in [9]. We show that only the numbers $-3, -2, -1, 0, 1, 2d$ can be integer eigenvalues of a dth circuit distance power and determine the associated eigenvalue multiplicities. This is the first goal of the present work, covered in Section 2.

The second goal is to study the eigenspaces associated with the integral eigenvalues. We will show in Section 3 that it is always possible to choose *simply structured* bases, in the sense that the basis vectors contain only entries from the set $\{-1, 0, 1\}$. Such bases have been shown to exist for a number of graph classes. Usually, attention is restricted to the graph kernel, i.e. the eigenspace for the eigenvalue 0. The existing literature features results on trees and forests [10–12], line graphs of trees [13,14], unicyclic graphs [15,16], bipartite graphs [17], or cographs [18]. There exists analogous research concerning the incidence matrix of a graph, where the problem of finding simple kernel bases can be considered as solved (cf. [19–21]). What makes circuit distance powers interesting is that we can construct simply structured bases for *all* eigenspaces of integer eigenvalues. Such a property is obvious for the complete graphs K_n and, hence, for all usual product graphs (cf. [4]) that can be derived from them, for example Sudoku graphs (cf. [22]). However, such products are integral and it seems like the non-complete circuit distance powers are the first known class of non-integral graphs (excepting C_4, C_6) with this property.

Finally, in Section 4 we consider multiplicities of arbitrary general eigenvalues of circuit distance powers. We show that all eigenvalues of $C_n^{(d)}$ that lie in the interval $(d/3, 2d)$ have multiplicity two. Moreover, we observe only 0 and -2 may be single eigenvalues. We close with an outlook on path distance powers where the situation is quite the opposite.

2. Integer eigenvalues and their multiplicities

There exists an explicit formula for the eigenvalues of a circuit distance power. The key is the observation that they belong to the class of circulant graphs. In this section, we will tacitly assume that all considered circuit powers $C_n^{(d)}$ are non-complete, i.e. $1 \leqslant d < \frac{n-1}{2}$.

A matrix in which the ith column vector (counting from $i = 0$) can be derived from the first column vector by means of a downward rotation by i entries is called a *circulant matrix* [23]. Clearly, with respect to some suitable vertex numbering, every circulant graph has a circulant adjacency matrix.

[1] When dealing with circulant graphs it is convenient to use zero-based matrix indices.

In the following, let us abbreviate $\omega_n = e^{\frac{2\pi i}{n}}$.

Lemma 1 [7]. *Let $(a_1, a_2, \ldots, a_n)^T$ be the first column of a real circulant matrix A. Then the eigenvalues of A are exactly*

$$\lambda_r = \sum_{j=1}^{n} a_j \omega_n^{(j-1)r}, \quad r = 0, \ldots, n-1. \tag{1}$$

where $\omega_n = e^{\frac{2\pi i}{n}}$.

Theorem 2. *The eigenvalues of $C_n^{(d)}$ are exactly*

$$\lambda_0 = 2d, \quad \lambda_r = \frac{\sin\left((2d+1)\frac{r}{n}\pi\right)}{\sin\frac{r}{n}\pi} - 1 \tag{2}$$

for $r = 1, \ldots, n-1$.

Proof. Use Lemma 1 and the following well-known trigonometric identity for the functions $D_q(x)$ of the *Dirichlet kernel* [24]:

$$D_q(x) = \sum_{j=-q}^{q} e^{iqx} = \frac{\sin\left(\left(q+\frac{1}{2}\right)x\right)}{\sin\frac{x}{2}}. \quad \square \tag{3}$$

Let us now investigate which integer eigenvalues a circuit distance power can have. Writing the second part of (2) as

$$(\lambda_r + 1)\sin\frac{r}{n}\pi - \sin\left((2d+1)\frac{r}{n}\pi\right) = 0 \tag{4}$$

we see that it is a trigonometric Diophantine equation of the form

$$A\sin 2\pi a + B\sin 2\pi b = C \tag{5}$$

with rational numbers A, B, C, a, b.

Conway and Jones have outlined how to find the solutions for such equations. Theorem 7 of their paper [25] considers the similar case

$$A\cos 2\pi a + B\cos 2\pi b + C\cos 2\pi c + D\cos 2\pi d = E. \tag{6}$$

Adapting their results, we get all nontrivial solutions of Eq. (5) as follows:

Theorem 3. *Consider at most two distinct rational multiples of π lying in the interval $(0, \pi/2)$ for which some rational linear combination of their sines, but of no proper subset, is rational. The only possible linear combinations, up to multiplication with a rational nonzero factor, are:*

$$\sin\frac{\pi}{6} = \frac{1}{2}, \qquad \sin\frac{3\pi}{10} - \sin\frac{\pi}{10} = \frac{1}{2}. \tag{7}$$

Proposition 4. *The set of integer eigenvalues of a circuit distance power $C_n^{(d)}$ is a subset of $\{-3, -2, -1, 0, 1, 2d\}$.*

Proof. Consider an integer solution λ_r of Eq. (4) with $0 < r < n$. For $|\lambda_r + 1| \geqslant 3$, Theorem 3 implies that the equation has no solutions with distinct rational degree sine arguments in the interval $(0, \pi/2)$. But even permitting arbitrary rational degree sine arguments does not help, so that the equation cannot be solved. $\square$

It is well known that the degree of regularity of a connected regular graph is an eigenvalue of multiplicity one (cf. [7]). Therefore $2d$ is always a single eigenvalue of $C_n^{(d)}$. This also follows from Theorem 2.

Next, we determine when and with which multiplicity the integers $-3, -2, -1, 0, 1$ occur as eigenvalues of circuit distance powers.

Theorem 5. *Let* $g = \gcd(2d+1, n)$. *Then the multiplicity of* -1 *as an eigenvalue of* $C_n^{(d)}$ *equals* $g - 1$.

Proof. For $\lambda_r = -1$, Eq. (4) simply becomes

$$\sin\left((2d+1)\frac{r}{n}\pi\right) = 0,$$

so that, equivalently, we need to find all positive integers $r < n$ such that $(2d+1)r = ln$ for some integer l. With the coprime integers $d' := (2d+1)/g$ and $n' := n/g$ the last identity becomes $d'r = n'l$. Hence $l = d'l'$ for a suitable integer l', and therefore $1 \leqslant r = n'l' < n$. This means $1 \leqslant r < g$. $\square$

Let $\operatorname{ord}_p(n)$ denote the order of the prime divisor p with respect to n, i.e.

$$\operatorname{ord}_p(n) = \max\{j \in \mathbb{N}_0 : p^j | n\}.$$

Theorem 6 [8]. *For given* $n, d \in \mathbb{N}$ *let* $g := \gcd(n, d)$ *and* $h := \gcd(n, d+1)$. *Then the multiplicity of* 0 *as an eigenvalue of* $C_n^{(d)}$ *is*

$$\begin{cases} g-1 & \text{if } \operatorname{ord}_2(d+1) \geqslant \operatorname{ord}_2(n), \\ g+h-1 & \text{if } \operatorname{ord}_2(d+1) < \operatorname{ord}_2(n) \text{ and } 2 \nmid d, \\ g+h-2 & \text{if } 2|n \text{ and } 2|d. \end{cases}$$

Proof. For $\lambda_r = 0$, Eq. (4) takes the form

$$\sin\frac{r}{n}\pi = \sin\left((2d+1)\frac{r}{n}\pi\right),$$

so that we need to determine all integers $0 < r < n$ and $l \in \mathbb{N}_0$ such that $dr = ln$ or $2(d+1)r = (2l+1)n$. A detailed proof can be found in [8]. $\square$

Let us point out that, since circuit squares $C_n^{(2)}$ are 4-circulant graphs of type $4C_n(1,2)$, Theorem 5 in [9] proves our Theorem 6 in the special case $d = 2$.

Note that the terms $h + g - 1$ and $h + g - 2$ in Theorem 6 are fairly interesting. With the greatest common divisors $g := \gcd(n, d)$ and $h := \gcd(n, d+1)$ we see that the terms are essentially sums of two multiplicative objects – a somewhat irritating fact for number theorists.

In the same manner as Theorem 6 we can prove the conditions for eigenvalue -2.

Theorem 7. *For given* $n, d \in \mathbb{N}$ *let* $g := \gcd(n, d)$ *and* $h := \gcd(n, d+1)$. *Then the multiplicity of* -2 *as an eigenvalue of* $C_n^{(d)}$ *with* $d > 1$ *is*

$$\begin{cases} h-1 & \text{if } \operatorname{ord}_2(d) \geqslant \operatorname{ord}_2(n), \\ g+h-1 & \text{if } \operatorname{ord}_2(d) < \operatorname{ord}_2(n) \text{ and } 2|d, \\ g+h-2 & \text{if } 2|n \text{ and } 2 \nmid d. \end{cases}$$

Theorem 8. *A circuit distance power* $C_n^{(d)}$ *has eigenvalue* 1 *if and only if* $6|n$ *and* $d \equiv 1 \bmod 6$. *In this case, the multiplicity of the eigenvalue equals two.*

Proof. For $(\lambda_r + 1) = 2$ we see from Theorem 3 that Eq. (4) can only have a solution if one of the arguments is a multiple of $\pi/2$. To be precise, the first sine term must equal $1/2$ and the second sine term must equal 1. This leads to the two solutions $r/n = \pi/6$ or $r/n = 5\pi/6$ for the first term (recall that $0 < r < n$) and to $(2d+1)r/n = \pi + 2k\pi$ with $k \in \mathbb{Z}$ for the second term. Hence, $6|n$ and $d \equiv 1 \bmod 6$. $\square$

Analogously, we obtain the following theorem:

Theorem 9. *A circuit distance power $C_n^{(d)}$ has eigenvalue -3 if and only if $6|n$ and $d \equiv 4 \bmod 6$. In this case, the multiplicity of the eigenvalue equals two.*

3. Eigenspaces for integer eigenvalues

According to Davis [23], the column vectors of the matrix

$$F^* = n^{-\frac{1}{2}} \left(\omega_n^{ij}\right)_{i,j=0,\ldots,n-1} \in \mathbb{C}^{n\times n},$$

which is the conjugate transpose of the *Fourier matrix* $F \in \mathbb{C}^{n\times n}$, constitute a complete and universal set of complex eigenvectors for *every* circulant matrix M of order n. Moreover, the rth column of F^*, denoted by $\mathrm{col}(r)$, yields a complex eigenvector for eigenvalue λ_r of Theorem 2.

In the following, we use this fact to establish *real* eigenspace bases. Even more, we assert that for all integer eigenvalues of a circuit distance power $C_n^{(d)}$ there exist associated simply structured eigenspace bases.

Theorem 10. *Every integer eigenvalue of $C_n^{(d)}$ admits a simply structured eigenspace basis.*

Proof. *Case $\lambda_r = 2d$*: It is well-known [7] that the all-ones vector forms a corresponding eigenspace basis.

Case $\lambda_r = -3$ or $\lambda_r = 1$: It it easily verified that in both cases the vectors

$$(1, 1, 0, -1, -1, 0, \ldots)^T, (1, 0, -1, -1, 0, 1, \ldots)^T$$

form a corresponding simply structured basis.

Case $\lambda_r = 0$: Let $g = \gcd(n, d)$ and $h = \gcd(n, d+1)$. It follows from the proof of Theorem 6 that the vectors $u_1, \ldots, u_{g-1}$ with $u_k = \sqrt{n} \cdot \mathrm{col}(kn/g)$ form a basis of a subspace of $\ker C_n^{(d)}$. We will show that the vectors $u'_1, \ldots, u'_{g-1}$ with

$$u'_k = \sum_{m=0}^{n/g-1} e_{k+mg} - e_{g+mg}$$

constitute an alternative (real) basis of this subspace.

Let M be the matrix with columns $u_1, \ldots, u_{g-1}$. Fix some $1 \leqslant \iota \leqslant g-1$ and let M' be the matrix with columns $u_1, \ldots, u_{g-1}, u'_\iota$. Clearly, $\mathrm{rk}\, M' \geqslant \mathrm{rk}\, M = g-1$. Actually, we have $\mathrm{rk}\, M' = g-1$ since the sum of all row vectors of M' vanishes. To see this, consider the summation of the values in a single column. We have $u_k = (\omega_n^{0kn/g}, \omega_n^{1kn/g}, \omega_n^{2kn/g}, \ldots, \omega_n^{(g-1)kn/g})^T$ so that its component sum is

$$\sum_{m=0}^{g-1} \omega_n^{mkn/g} = \sum_{m=0}^{g-1} \omega_g^{km} \tag{8}$$

and therefore a Gaussian period. Because of $1 \leqslant k \leqslant g-1$ we have $g \nmid k$ so that, according to the theory of Gaussian periods (see [26,27]), the component sum in Eq. (8) is zero. Moreover, the component sum of u'_ι is zero, too. Hence it follows that u'_ι is a linear combination of the vectors u_k. Since the vectors $u'_1, \ldots, u'_{g-1}$ are obviously linearly independent we see that they are a basis for the space spanned by $u_1, \ldots, u_{g-1}$.

In the case that $\mathrm{ord}_2(d+1) < \mathrm{ord}_2(n)$, equivalently $2h|n$, the vectors $v_1, \ldots, v_h$ with $v_k = \sqrt{n} \cdot \mathrm{col}(kn/h - n/(2h))$ form a basis of another subspace of $\ker C_n^{(d)}$.

A similar argument as for the vectors u'_k shows that the vectors $v'_1, \ldots, v'_h$ with

$$v'_k = \sum_{m=0}^{n/h-1} (-1)^m e_{k+mh}$$

constitute a basis of the subspace of $\ker C_n^{(d)}$ spanned by the vectors $v_1, \ldots, v_h$.

Let us consider the cases listed in Theorem 6:

- If $\mathrm{ord}_2(d+1) \geqslant \mathrm{ord}_2(n)$, then $\{u'_1, \ldots, u'_{g-1}\}$ is a basis of $\ker C_n^{(d)}$.
- If $\mathrm{ord}_2(d+1) < \mathrm{ord}_2(n)$ and $2 \nmid d$, then $\{u'_1, \ldots, u'_{g-1}, v'_1, \ldots, v'_h\}$ is a basis of $\ker C_n^{(d)}$.
- If $2|n$ and $2|d$, then $\{u'_1, \ldots, u'_{g-1}, v'_1, \ldots, v'_h\}$ can be reduced to a basis of $\ker C_n^{(d)}$.

All of the above bases are simply structured.

Case $\lambda_r = -2$: Use Theorem 7. This case is analogous to case $\lambda = 0$, only with swapped roles of g and h. We have complex subspace basis vectors $u_1, \ldots, u_g$ with $u_k = \sqrt{n} \cdot \mathrm{col}(kn/g - n/(2g))$ and can find real basis vectors $u'_1, \ldots, u'_g$ with

$$u'_k = \sum_{m=0}^{n/g-1} (-1)^m e_{k+mg}$$

for the same subspace. Likewise, we have complex vectors $v_1, \ldots, v_{h-1}$ with $v_k = \sqrt{n} \cdot \mathrm{col}(kn/h)$ and real vectors $v'_1, \ldots, v'_{h-1}$ with

$$v'_k = \sum_{m=0}^{n/h-1} e_{k+mh} - e_{h+mh}.$$

Case $\lambda_r = -1$: With the help of Theorem 5, we can reason as in the first part of case $\lambda = 0$, but with $g = \gcd(n, 2d+1)$. The same complex vectors $u_1, \ldots, u_{g-1}$ and real vectors $u'_1, \ldots, u'_{g-1}$ are obtained. $\square$

Example 11. Theorem 10 asserts that the vectors

$$\begin{aligned}
&(1, 0, -1, 0, 1, 0, -1, 0, \ldots, 1, 0, -1, 0)^T,\\
&(0, 1, 0, -1, 0, 1, 0, -1, \ldots, 0, 1, 0, -1)^T,\\
&(1, 0, -1, 1, 0, -1, 1, 0, -1, \ldots, 1, 0, -1)^T,\\
&(0, 1, -1, 0, 1, -1, 0, 1, -1, \ldots, 0, 1, -1)^T,
\end{aligned}$$

form a simply structured eigenspace basis of $C_{36}^{(14)}$ for eigenvalue -2.

Remark 12. Note that some of the simply structured bases constructed in Theorem 10 are actually orthogonal.

For $\lambda = 2d$ and $\lambda = -1$ this is always the case. For $\lambda = -3$ and $\lambda = 1$ one can never obtain a simply structured basis. For $\lambda = 0$, we see by Theorem 6 that the constructed basis is simply structured if $\mathrm{ord}_2(d+1) < \mathrm{ord}_2(n)$ and $\gcd(n, d) \leqslant 2$. Theorem 7 implies an analogous statement for $\lambda = -2$.

4. Eigenvalue multiplicities in general

Let us revisit Eqs. (1) and (3) by considering the functions $f_d : [0, 2\pi] \to \mathbb{R}$ with

$$f_d(\varphi) := \frac{\sin((2d+1)\varphi/2)}{\sin(\varphi/2)}$$

for $\varphi \in (0, 2\pi)$ and the continuous extension $f_d(0) = f_d(2\pi) = 2d + 1$.

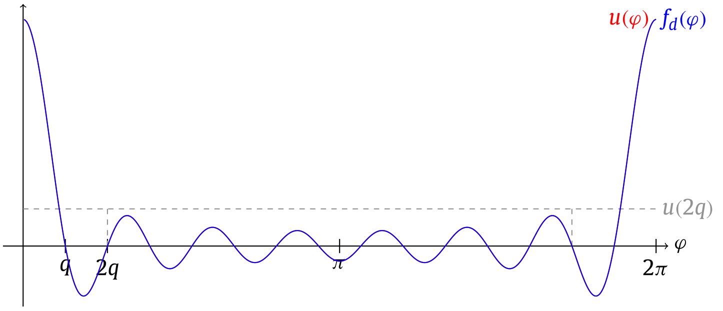

Fig. 1. Obtaining a lower bound for eigenvalues with multiplicity two.

Let us point out some obvious properties of f_d. We have $f_d(\pi) = \sin((2d+1)\pi/2) \in \{1, -1\}$. Moreover, $f_d(\varphi)$ is axis symmetric with respect to $\varphi = \pi$. The zeros of f_d are exactly the integer multiples kq of $q := 2\pi/(2d+1)$ with $k = 1, \ldots, 2d$.

Since we can find the eigenvalues of $C_n^{(d)}$ as $\lambda_0 = 2d$ and $\lambda_r = f_d(2\pi r/n) - 1 \neq 2d$ for $r = 1, \ldots, n-1$, the following fact is obvious:

Observation 13. Any eigenvalue of $C_n^{(d)}$ of odd multiplicity must necessarily be $2d$, 0 or -2.

Theorem 14. *Every eigenvalue of $C_n^{(d)}$ that is greater than $\frac{d}{\pi} - 1$ and less than $2d$ has multiplicity two.*

Proof. A simple upper bound for f_d is

$$u : \varphi \mapsto \frac{1}{\sin(\varphi/2)}.$$

This bound is strictly decreasing on $(0, \pi)$. Observing symmetry, it follows that $u(2q) \geqslant f_d(\varphi)$ for $\varphi \in (2q, 2\pi - 2q)$.

We have $f_d(0) = 2d + 1$. So it is clear that f_d is nonnegative in the interval $(0, q)$ and non-positive in the interval $(q, 2q)$.

Claim: f_d is strictly decreasing in the interval $(0, q)$.

We consider the derivative

$$f_d' : \varphi \mapsto \frac{(2d+1)\cos((2d+1)\varphi/2)\sin(\varphi/2) - \sin((2d+1)\varphi/2)\cos(\varphi/2)}{2\sin^2(\varphi/2)}$$

and show that $f_d'(\varphi) \leqslant 0$ for $0 \leqslant \varphi \leqslant q$. It is easy to see that the first term of the numerator is positive for $0 < \varphi < q/2$ and negative for $q/2 < \varphi < q$ whereas the second term of the numerator is positive for $0 < \varphi < q$. Clearly, $f_d'(\varphi) < 0$ for $q/2 < \varphi < q$. For $0 < \varphi < q/2$ consider the ratio of the two numerator terms, which is

$$\frac{(2d+1)\cos((2d+1)\varphi/2)\sin(\varphi/2)}{\sin((2d+1)\varphi/2)\cos(\varphi/2)} = \frac{(2d+1)\tan(\varphi/2)}{\tan((2d+1)\varphi/2)} \leqslant 1$$

as can be concluded, for instance, from the tangent function's Taylor expansion. This proves the claim.

It follows readily that any eigenvalue of $C_n^{(d)}$ that is greater than $u(2q) - 1$ must have multiplicity two, see Fig. 1. An immediate asymptotic relaxation is obtained by observing that $u(2q) > d/\pi$ for all $d \in \mathbb{N}$ and $\lim_{d\to\infty} \frac{u(2q)}{d/\pi} = 1$. $\square$

If we denote by φ_0 the unique $\varphi \in (2q, 3q)$ such that $f_d(\varphi)$ is maximal, then it is clear that one could improve the bound $d/\pi - 1$ in Theorem 14 to a bound $d/u(\varphi_0) - 1$. To do so, however, requires tedious calculations and we could not get anything considerably smaller than $4d/15 - 1$.

For fixed $1 \leqslant d \leqslant 4$, the respective smallest graphs with eigenvalues that fulfil the condition of Theorem 14 are $C_5^{(1)}$, $C_7^{(2)}$, $C_{10}^{(3)}$, and $C_{12}^{(4)}$.

We conclude with some remarks on path distance powers $P_n^{(d)}$.

Clearly, one may remove a suitable number of consecutive vertices from a circuit distance power to obtain a path graph. It follows from this observation that there is a certain relation between the spectra of path and circuit distance powers since the eigenvalues of a graph and an induced subgraph interlace (cf. [28]).

However, path distance powers are not circulant and possess many spectral properties that are quite unlike those of circuit distance powers. With respect to the questions considered so far in this and the previous sections, we pose a number of conjectures which we derived from computer experiments:

Conjecture 15. *For every integer k there exists a pair* (n, d) *such that k is an eigenvalue of* $P_n^{(d)}$.

Conjecture 16. *The complete graph* K_2 *is the only path distance power with eigenvalue* 1.

Conjecture 17. *Every eigenvalue* $\lambda \notin \{-2, -1, 0\}$ *of* $P_n^{(d)}$ *is simple.*

This is a clear contrast to Observation 13 and Theorem 14. Towards proving Conjecture 17, it has be shown that for $\frac{n}{2} < d < n - 1$ every multiple eigenvalue $\lambda \neq -1$ of $P_n^{(d)}$ has multiplicity two [29]. But this result does not apply for values of d outside this range, cf. the graph $P_{15}^{(6)}$ with triple eigenvalue zero.

References

[1] F. Boesch, R. Tindell, Circulants and their connectivities, J. Graph Theory 8 (1984) 487–499.
[2] J.-C. Bermond, F. Comellas, D. Hsu, Distributed loop computer networks: a survey, J. Parallel Distrib. Comput. 24 (1995) 2–10.
[3] N. Saxena, S. Severini, I. Shparlinski, Parameters of integral circulant graphs and periodic quantum dynamics, Int. J. Quantum Inf. 5 (2007) 417–430.
[4] D. Cvetković, P. Rowlinson, S. Simić, Eigenspaces of graphs, Encyclopedia of Mathematics and its Applications, vol. 66, Cambridge University Press, 1997.
[5] D.M. Cvetković, M. Doob, H. Sachs, Spectra of Graphs. Theory and Application, second ed., VEB Deutscher Verlag der Wissenschaften, Berlin, 1982.
[6] W. So, Integral circulant graphs, Discrete Math. 306 (2006) 153–158.
[7] N. Biggs, Algebraic Graph Theory, second ed., Cambridge Mathematical Library, Cambridge University Press, 1993.
[8] T. Sander, Singular distance powers of circuits, Tokyo J. Math. 4 (2007) 491–498.
[9] G.J. Davis, G.S. Domke, 4-Circulant Graphs, Ars Combin. 65 (2000) 97–110.
[10] P.E. John Jr., G. Schild, Calculating the characteristic polynomial and the eigenvectors of a tree, Match (34) (1996) 217–237.
[11] J. Sander, T. Sander, On simply structured bases of tree kernels, AKCE J. Graphs. Combin. 2 (2005) 45–56.
[12] S. Akbari, A. Alipour, E. Ghorbani, G. Khosrovshahi, $\{-1, 0, 1\}$-basis for the null space of a forest, Linear Algebra Appl. 414 (2006) 506–511.
[13] M.C. Marino, I. Sciriha, S.K. Simić, D.V. Tošić, More about singular line graphs of trees, Publ. Inst. Math. (Beograd) 79 (2006) 70–85.
[14] I. Sciriha, The two classes of singular line graphs of trees, in: 5th Workshop on Combinatorics (Messina, 1999), Rend. Sem. Mat. Messina Ser. II, vol. 5, 1999, pp. 167–180.
[15] M. Nath, B.K. Sarma, On the null-spaces of acyclic and unicyclic singular graphs, Linear Algebra Appl. 427 (2007) 42–54.
[16] T. Sander, J. Sander, On simply structured kernel bases of unicyclic graphs, AKCE J. Graphs. Combin. 4 (2007) 61–82.
[17] D.M. Cvetković, I.M. Gutman, The algebraic multiplicity of the number zero in the spectrum of a bipartite graph, Mat. Vesnik 9 (1972) 141–150.
[18] T. Sander, On certain eigenspaces of cographs, Electron. J. Combin. 15 (2008).
[19] R.H. Villarreal, Rees algebras of edge ideals, Commun. Algebra 39 (1995) 3513–3524.
[20] F. Hazama, On the kernels of the incidence matrices of graphs, Discrete Math. 254 (2002) 165–174.
[21] S. Akbari, N. Ghareghani, G.B. Khosrovshahi, H.R. Maimani, The kernels of the incidence matrices of graphs revisited, Linear Algebra Appl. 414 (2006) 617–625.
[22] T. Sander, Sudoku graphs are integral, Electr. J. Combin. 16 (2009) research Note N25, 7 pp. (electronic).
[23] P.J. Davis, Circulant matrices, John Wiley & Sons, New York-Chichester-Brisbane, 1979.
[24] A. Zygmund, Trigonometrical Series, second ed., Dover Publications, New York, 1955.

[25] J.H. Conway, A.J. Jones, Trigonometric diophantine equations (On vanishing sums of roots of unity), Acta Arithmetica 30 (1976) 229–240.

[26] H. Davenport, Multiplicative number theory. 2nd ed. Rev. by Hugh L. Montgomery., Graduate Texts in Mathematics, Springer, 1980.

[27] T. Nagell, Introduction to Number Theory, Wiley, New York, 1951.

[28] W.H. Haemers, Interlacing eigenvalues and graphs, Linear Algebra Appl. 226/228 (1995) 593–616.

[29] T. Sander, Eigenspace Structure of Certain Graph Classes, Ph.D. Thesis, TU Clausthal, 2004.