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Random Ordinary Differential Equations

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1. INTRODUCTION AND NOTATION

A random ordinary differential equation is a differential equation

$$x'(t, \omega) = f(t, x(t, \omega), \omega), \qquad x(a, \omega) = x_0(\omega). \tag{1}$$

Here, x is a stochastic process, $\omega \in \Omega$ (a probability space) and x' is some sort of derivative (sample path, mean square, etc.) of x. In this paper we prove some existence and uniqueness theorems for (1) which have two distinct advantages over earlier theorems—they give in the linear case practical criteria ensuring the existence of solutions with a specified number of moments, together with bounds on their rates of convergence.

There are some existence and uniqueness theorems in the literature, especially for linear equations (see, for example, [1], [2], and [3]). However, if it is required that the solutions have (say) finite mean and variance or, equivalently, that the derivative x' in (1) be a mean square derivative, the known results are very restrictive. For example, in the problem $x'(t, \omega) = A(\omega)\, x(t, \omega)$, $x(0, \omega) = x_0(\omega)$, where A is just a random variable, these theorems apply only if A is essentially bounded. Thus, even Gaussian coefficients are not allowed!

In fact, often solutions do not exist. As we shall see later, for example, if $A(\omega) \geqslant 0$, $x' = A(\omega)\, x$, $x(0) = 1$ has a "mean square" solution on $[0, b]$ if, and only if, the Laplace transform of A is analytic for $|s| \leqslant 2b$. If $x(0) = x_0(\omega)$, then a mean square solution exists for all x_0 with finite moments of all orders if, and only if, A is essentially bounded.

Because of the large numbers of derivatives one can use in stating (1) precisely, a number of different existence problems can be stated. The next section states and compares three of these. The third section contains the basic existence and uniqueness theorems of the paper, which are essentially generalizations of the Picard existence theorem to cover certain discontinuous operators. The paper concludes with an appendix summarizing the integration theory used.

We conclude this section with some definitions and notation.

R^n is Euclidean n space with norm $|x| = \sup |x_i|$. If A is an $n \times n$ matrix, then $|A| = \sup |Ax|/|x|$ $(=\sup_i \sum_k |a_{ik}|$. See [4], p. 41).

(Ω, P) is a probability space with probability P. $L(p, n)$ is the Banach space formed from all from $x : \Omega \to R^n$ with finite p-th moments ($p \geqslant 1$) and norm $\|x\|_p = (E |x(\omega)|^p)^{1/p}$. ($E$ = expectation operator).

$I = [a, b]$ is an interval.

$x : I \times \Omega \to R^n$ is a stochastic process. It can also be thought of as a map $x : I \to L(p, n)$ if $\|x(t, \cdot)\|_p < \infty$. In this case we say x is *L^p differentiable* (*W^p pseudodifferentiable*) if the difference quotient $(x(t + h, \omega) - x(t, \omega))/h$ converges in the norm (weak) topology on $L(p, n)$. If almost all the sample paths of x are differentiable, we say x is *SP differentiable.* (See A-4 in the appendix for precise definitions.)

x is called *SP integrable* if for almost all its sample paths $\int_a^b |x(t, \omega)| \, dt$ exists and is finite. If, considered as a map $I \to L(p, n)$, x is Bochner (Pettis) integrable, we say x is *$L^p(w^p)$ integrable.* (See A-1 in the appendix.)

2. Problem Formulations

Corresponding to each of the three types of derivatives just defined, we get a different interpretation of the random differential Eq. (1). The precise formulations given make each equivalent to a related integral equation.

SP Problem. $f : I \times R^n \times \Omega \to R^n$ and $x_0 : \Omega \to R^n$ are given. $x : I \times \Omega \to R^n$ is said to solve the *SP* problem on I

$$(SP) \qquad x'(t, \omega) = f(t, x(t, \omega), \omega), \qquad x(a, \omega) = x_0(\omega)$$

iff for a.e. $\omega \in \Omega$ the following conditions are satisfied: $x(t, \omega)$ is absolutely continuous (in t) on I, $x(a, \omega) = x_0(\omega)$, and $\partial x(t, \omega)/\partial t = f(t, x(t, \omega), \omega)$ for almost all $t \in I$.

In what follows, if we write $f : I \times L(p, n) \to L(p, n)$, we mean that for each $t \in I$ $f(t, \cdot)$ is defined on some subset of $L(p, n)$, in general varying with t.

W^p Problem. $\hat{f} : I \times L(p, n) \to L(p, n)$ and $\hat{x}_0 \in L(p, n)$ are given. $\hat{x} : I \to L(p, n)$ is said to solve the w^p problem on I

$$(w^p) \quad \hat{x}'(t) = \hat{f}(t, \hat{x}(t)), \qquad \hat{x}(a) = \hat{x}_0$$

IFF $\hat{x}$ is $L(p, n)$ absolutely continuous (see A.2 in the appendix), $\hat{x}(a) = \hat{x}_0$, $\hat{x}(t)$ is in the domain of $\hat{f}(t, \cdot)$ for a.e. $t \in I$, and $\hat{f}(t, \hat{x}(t))$ is the $L(p, n)$ pseudo-derivative of $\hat{x}$ on I.

L^p PROBLEM. $\hat{f} : I \times L(p, n) \to L(p, n)$ and $\hat{x}_0 \in L(p, n)$ are given. $\hat{x} : I \to L(p, n)$ is said to solve the L^p problem on I

$$(L^p) \qquad \hat{x}'(t) = \hat{f}(t, \hat{x}(t)), \qquad \hat{x}(a) = \hat{x}_0$$

IFF $\hat{x}$ is $L(p, n)$ strongly absolutely continuous, $\hat{x}(t)$ is in the domain of $\hat{f}(t, \cdot)$ for a.e. $t \in I$, and $\hat{f}(t, \hat{x}(t))$ is the $L(p, n)$ derivative of $\hat{x}$ on I.

THEOREM 1. (a) *$x : I \times \Omega \to R^n$ solves the SP problem* IFF *for a.e. $\omega \in \Omega$, $x(t, \omega) = x_0(\omega) + (SP) \int_a^t f(s, x(s, \omega), \omega)\, ds$ for all $t \in I$.*

(b) *$\hat{x} : I \to L(p, n)$ solves the w^p problem* IFF

$$\hat{x}(t) = \hat{x}_0 + (w^p) \int_a^t \hat{f}(s, \hat{x}(s))\, ds \qquad \text{for all} \quad t \in I.$$

(c) *$\hat{x} : I \to L(p, n)$ solves the L^p problem* IFF

$$\hat{x}(t) = \hat{x}_0 + (L^p) \int_a^t \hat{f}(s, \hat{x}(s))\, ds \qquad \text{for all} \quad t \in I.$$

Proof. (a) is very well-known, and (b) and (c) are characterizations of the different integrals. See A.5. Q.E.D.

DEFINITION 2. If $x : \Omega \to R^n$ has a finite p-th absolute moment, we denote by $\hat{x}$ its equivalence class in $L(p, n)$, and we say that $\hat{x}$ and x are equivalent.

If $f : I \times R^n \times \Omega \to R^n$, then by $\hat{f} : I \times L(p, n) \to L(p, n)$ we mean the function defined by $\hat{f}(t, \hat{x}) = f(t, x(\omega), \omega)\hat{}$. We say f and $\hat{f}$ are equivalent.

We now give a complete solution to the problem of interrelationships between the various problems, based on the interrelationships between the different integrals developed in the appendix.

THEOREM 3. *We assume throughout that f and $\hat{f}$, and x_0 and $\hat{x}_0$ appearing in the different problems are related as follows: f and $\hat{f}$ are equivalent, x_0 and $\hat{x}_0$ are equivalent. If $x : I \times \Omega \to R^n$ is product integrable and has absolutely continuous sample paths, then $f(t, x(t, \omega), \omega)$ is product measurable.*

(a) *If $\hat{x} : I \to L(p, n)$ solves the L^p problem, there exists an equivalent $x : I \times \Omega \to R^n$, product measurable, solving the SP problem. Conversely, if x solves the SP problem, the equivalent $\hat{x}$ solves the L^p problem if, and only if, $\int_a^b \|f(t, x(t, \omega), \omega)\|_p\, dt < \infty$*

(b) *If $\hat{x} : I \to L(p, n)$ solves the W^p problem and there exists an equivalent product integrable $x : I \times \Omega \to R^n$ then x solves the SP problem. Conversely, if x solves the SP problem and $p > 1$, then the equivalent $\hat{x}$ solves the W^p problem* IFF *$x^*(\hat{f}(t, \hat{x}(t)))$ is integrable for each $x^* \in (L(p, n))^*$.*

(c) *If $\hat{x}$ solves the L^p problem it always also solves the W^p problem. If $\hat{x}$ solves the W^p problem it solves the L^p problem* IFF *$\int_a^b \|\hat{f}(t, \hat{x}(t))\|_p \, dt < \infty$ and $\hat{f}(t, \hat{x}(t))$ is almost separably valued* ([7], p. 72).

Proof. All these results follow from the relationships between integrals given in A.6. We prove (a), for example.

Assume $\hat{x}$ solves the L^p problem. Then there exists a product integrable $y : I \times \Omega \to R^n$ equivalent to $\hat{f}(t, \hat{x}(t))$ by Theorem A.6 (a). Define $x(t, \omega) = x_0(\omega) + (SP) \int_a^t y(s, \omega) \, ds$. This x is easily seen to be the equivalent SP solution. The converse follows from A.6 (c). Q.E.D.

COROLLARY 4. *Under the same assumption about $f, \hat{f}, x_0$, and $\hat{x}_0$, if the SP problem has at most one solution, then the L^p problem also has at most one solution.*

3. EXISTENCE THEOREMS FOR THE L^p PROBLEM

Most existence theorems for random differential equations which have appeared in the literature are variants on the following result, which is an easy generalization of Picard's classical proof of the convergence of the successive iterates.

THEOREM 5. *If $f : I \times L(p, n) \to L(p, n)$ satisfies*

$$\|f(t, x) - f(t, y)\|_p \leqslant k(t) \, \| x - y \|_p ,$$

where $\int_a^b k(t) \, dt < \infty$, then there exists a unique solution to the L^p problem for any initial condition.

Unfortunately, this theorem has very limited applicability even in the linear case, as the following example shows:

EXAMPLE. Consider the L^p problem $x' = A(\omega) \, x$ where $A(\omega)$ is a random variable. This equation has a unique SP solution $x(t, \omega) = x_0(\omega) \, e^{tA(\omega)}$ for initial condition $x(0) = x_0$. Since each L^p solution is also an SP solution, $\hat{x}(t)$ is the only possible L^p solution. However, $\hat{x}(t) \in L^p$ for all $x_0 \in L^p$ IFF $A(\omega) \leqslant K$ almost surely. To see this, let $T(\omega) = e^{tA(\omega)}$ for some fixed $t \in I$. The map $x_0 \to Tx_0$ in L^p is easily seen to be closed. But if one sets $x_n = T^n / \| T^n \|_p$, then $\| Tx_n \| / \| x_n \| = \| T^{n+1} \| / \| T^n \|$ is bounded IFF T is bounded. By the closed graph theorem, therefore, $\hat{x}(t) \notin L^p$ for any t for some initial conditions $x_0 \in L^p$, if A is not bounded on the right.

The rest of this section is devoted to trying to rescue the essential element of the Picard theorem—the convergence of the successive iterates—even though the operator not only does not satisfy a norm Lipschitz condition but is not even $L(p, n)$ continuous. Two basic methods are used. The first (Lemma 6 and its consequences) is based on a sample path Lipschitz condition. The second (Theorem 9) gives somewhat sharper results for linear systems but does not give equally simple existence criteria. The proof of Theorem 9 is contructive and may be used to compute the moments of the solution and to find rates of convergence to the moments (see Corollary 11).

LEMMA 6. *Let $f : I \times R^n \to R^n$ and $x_0 : \Omega \to R^n$ satisfy the following conditions*:

(a) *If $x : I \times \Omega \to R^n$ is product measurable and absolutely continuous in its first variable almost surely, then $f(t, x(t, \omega), \omega)$ is product measurable.*

(b) $x_0 \in L(p, n)$.

(c) $|f(t, x_0(\omega), \omega)|$ *is L^p integrable on I.*

(d) *There is a product measurable $k : R \times \Omega \to R$, such that for all $x, y \in R^n$*

$$|f(t, x, \omega) - f(t, y, \omega)| \leqslant k(t, \omega) \, | x - y |$$

almost surely, and such that $\int_a^b k(t, \omega) \, dt < \infty$ almost surely.

Then if, in addition, the linear homogeneous L^p problem

$$\begin{aligned} \xi'(t, \omega) &= k(t, \omega) \, \xi(t, \omega) \\ \xi(a, \omega) &= \int_a^b | f(s, x_0(\omega), \omega)| \, ds \end{aligned} \tag{2}$$

has an L^p solution on $[a, b]$, then the L^p problem

$$\begin{aligned} x'(t, \omega) &= f(t, x(t, \omega), \omega) \\ x(a, \omega) &= x_0(\omega) \end{aligned} \tag{3}$$

has a unique solution $x(t, \omega)$ on $[a, b]$.

Proof. Let $\xi(t, \omega)$ and $x(t, \omega)$ be the SP solutions of (2) and (3), respectively. Then ξ (or more precisely, $\hat{\xi}$—see Definition 2) is the unique L^p solution of (2).

First, we briefly study (2). Let $L(\omega) = \int_a^b |f(s, x_0(\omega), \omega)|\, ds$, and define iteratively $\xi_0(t, \omega) = 0$, $\xi_{n+1}(t, \omega) = L(\omega) + \int_a^t k(s, \omega)\, \xi_n(s, \omega)\, ds$. This sequence has the following properties:

(i) $\xi_n(t, \omega)$ is a nondecreasing function of t.

(ii) $\xi_n(t, \omega) = 1 + \sum_{i=1}^n 1/i!\, (\int_a^t k(s, \omega)\, ds)^i L(\omega)$

(iii) $\xi_n(t, \omega) \nearrow \xi(t, \omega)$ [the solution of (2)] uniformly in t a.s., uniformly in L^p, and in the L^p integral sense (i.e., $\int_a^b \| \xi(t, \omega) - \xi_n(t, \omega)\|_p\, dt \to 0$.)

(iv) $\xi_n' \geqslant 0$.

(v) $\xi_n'(t, \omega) \nearrow \xi'(t, \omega)$ a.s., in L^p norm, and in the L^p integral sense.

Now define $x_0(t, \omega) = x_0(\omega)$,

$$x_{n+1}(t, \omega) = x_0 + \int_a^t f(s, x_n(s, \omega), \omega)\, ds.$$

By the Picard theorem applied to sample paths, $x_{n+1} \to x$ uniformly in t for a.e. ω. Note that

$$| x_{n+1}(t, \omega) - x_n(t, \omega)| \leqslant \int_a^t k(s, \omega)\, | x_n(s, \omega) - x_{n-1}(s, \omega)|\, ds.$$

Since

$$| x_1(t, \omega) - x_0 | \leqslant L(\omega) = | \xi_1(t, \omega) - \xi_0(t, \omega)|$$

and

$$| \xi_{n+1}(t, \omega) - \xi_n(t, \omega)| = \int^t k(x, \omega)(\xi_n(s, \omega) - \xi_{n-1}(s, \omega))\, ds,$$

it is easy to induce that

$$| x_{n+1}(t, \omega) - x_n(t, \omega)| \leqslant \xi_{n+1}(t, \omega) - \xi_n(t, \omega).$$

Also,

$$\begin{aligned} | x_{n+1}'(t, \omega) - x_n'(t, \omega)| &\leqslant k(t, \omega)(\xi_{n-1}(t, \omega) - \xi_{n-1}(t, \omega)) \\ &\leqslant \xi_{n+1}'(t, \omega) - \xi_n'(t, \omega). \end{aligned}$$

Therefore, by (iii) and (v) $x_n \to x$ and $x_n' \to x'$ uniformly in t, a.s., uniformly in L^p, and in the $L(p, n)$ integral sense. We now show that x is the desired solution. If $f(t, x(t, \omega), \omega)$ were $L(p, n)$ integrable,

$$\begin{aligned} &\left\| x(t, \omega) - x_0 - (L^p) \int_a^t f(s, x(s, \omega), \omega)\, ds \right\|_p \\ &\leqslant \| x(t, \omega) - x_{n+1}(t, \omega)\|_p \\ &\quad + \left\| x_{n+1}(t, \omega) - x_0 - (L^p) \int_a^t f(s, x_n(s, \omega), \omega)\, ds \right\|_p \\ &\quad + (L^p) \int_a^t \|(f(s, x_n(s, \omega), \omega) - f(s, x(s, \omega), \omega)\|_p\, ds. \end{aligned}$$

We know the first two terms on the right go to zero. We now show simultaneously the $L(p, n)$ integrability of $f(t, x(t, \omega), \omega)$ and the convergence to zero of the third term. Since $f(t, x(t, \omega), \omega)$ is product measurable [Assumption (a)], by A.6 we need only show

$$\int_a^b \|f(t, x(t, \omega), \omega)\|_p \, dt < \infty.$$

$$\begin{aligned} \|f(t, x(t, \omega), \omega) &- f(t, x_k(t, \omega), \omega)\|_p \\ &\leqslant \left\| \sum_{j=k}^{\infty} |f(t, x_{j+1}(t, \omega)) - f(t, x_j(t, \omega), \omega)| \right\|_p \\ &\leqslant \left\| \sum_{j=k}^{\infty} k(t, \omega)(\xi_{j+1}(t, \omega) - \xi_j(t, \omega)) \right\|_p \\ &\leqslant \| \xi'(t, \omega) - \xi_k'(t, \omega)\|_p . \end{aligned}$$

Also,

$$\begin{aligned} \|f(t, x_k(t, \omega), \omega)\|_p \leqslant{} & \|f(t, x_0(\omega), \omega)\|_p \\ & + \left\| \sum_{j=1}^{k} |f(t, x_j(t, \omega), \omega) - f(t, x_{j-1}(t, \omega)\omega)| \right\|_p \\ \leqslant{} & \|f(t, x_0(\omega), \omega)\|_p + \| \xi_k'(t, \omega) - \xi_1'(t, \omega)\|_p \\ \leqslant{} & \|f(t, x_0(\omega), \omega)\|_p + \| \xi'(t, \omega)\|_p . \end{aligned}$$

Thus, $\|f(t, x_k(t, \omega), \omega)\|_p$ is bounded by an integrable function. Applying the generalized dominated convergence theorem completes the existence proof.

Uniqueness follows from Corollary 4, since the *SP* problem satisfies a Lipschitz condition and, hence, has only one solution. Q.E.D.

We now apply the lemma to linear systems to get two explicit existence criteria. While the criteria do not always give the best result, each is sharp in the sense that counterexamples exist if any condition is relaxed.

COROLLARY 7. *Consider the linear L^p problem*

$$x'(t) = A(t)\, x(t) + P(t), \qquad x(a) = x_0 , \tag{4}$$

where A is an $n \times n$ matrix, and P and x_0 n vectors, of integrable processes. Let $k(t, \omega) = | A(t, \omega)|$.

Suppose $A(t)$, $P(s)$, and x_0 are independent for all $a \leqslant s, t \leqslant b$.

Define $L(s, t) = E(e^{sk(t,\omega)})$. Then if $\int_a^b L(\bar{s}, t)\, dt < \infty$, *the L^p problem* (2) *has a* (*unique*) *solution on* $[a, b]$ *if $p < \bar{s}/b - a$.*

Proof. Using the independence assumptions, it is easily seen that Lemma 6 can be applied if we can show that

$$\int_a^b \left\| k(t, \omega) \int_a^b k(s, \omega)\, ds \exp\left(\int_a^t k(s, \omega)\, ds\right)\right\|_p dt < \infty \tag{5}$$

if $p < \bar{s}/(b - a)$, for then the SP solution of (4) will have an L^p integrable derivative.

For any $\epsilon > 0$, the integral in (5) is bounded by

$$\int_a^b \| k(t, \omega)\|_{p(1+\epsilon)/2\epsilon} \left\| \int_a^b k(s, \omega)\, ds \right\|_{p(1+\epsilon)/2\epsilon} \left\| \exp \int_a^b k(s, \omega)\, ds \right\|_{p(1+\epsilon)} dt$$

$$\leqslant \int_a^b \| k(t, \omega)\|_{p(1+\epsilon)/2\epsilon}\, dt \left\| \exp \int_a^b k(s, \omega)\, ds \right\|_{p(1+\epsilon)} \int_a^b \| k(t, \omega)\|_{p(1+\epsilon)/2\epsilon}\, dt.$$

We show these three terms are all finite. Since $L(s, t)$ is a moment-generating function of $k(t, \omega)$, if $\int_a^b L(s, t)\, dt < \infty$ for *any* positive s, $\int_a^b \| k(s, \omega)\|_r\, ds < \infty$ for any $r < \infty$. Hence, the first and third terms are finite. For the second term, we apply an inequality due to R. Edsinger [5] to show that

$$\begin{aligned}
&\left\| \exp\left(\int_a^b k(s, \omega)\, ds\right)\right\|_{p(1+\epsilon)} \\
&\quad\leqslant \frac{1}{b - a} \int_a^b \| \exp(b - a)\, k(s, \omega)\|_{p(1+\epsilon)}\, ds \\
&\quad\leqslant \frac{1}{b - a} \int_a^b E(\exp((b - a)(1 + \epsilon)\, pk(s, \omega))\, ds + \exp(b - a) \\
&\quad\leqslant \frac{1}{b - a} \int_a^b L(p(1 + \epsilon)(b - a), s)\, ds + \exp(b - a)
\end{aligned}$$

and, picking $\epsilon > 0$ small enough, we have $p(1 + \epsilon) < \bar{s}/(b - a)$, as desired. Hence, all the integrals are finite and we are done. Q.E.D.

COROLLARY 8. *Consider the same L^p problem*

$$x'(t) = A(t)\, x(t) + P(t), \qquad x(a) = x_0, \tag{4}$$

where A is an $n \times n$ matrix, and P and x_0 are n vectors of integrable processes. Let $A(t, \omega) = (a_{ij}(t, \omega))$.

Assume that all the processes $a_{ij}(t)$ are mutually independent, and assume also that $A(t)$, $P(s)$, and x_0 are independent.

Define $L_{ij}(s, t) = E(e^{s a_{ij}(t, \omega)})$. Then

(i) *If for some* $\bar{s} > 0$ *and each* (i, j)

$$\int_a^b L_{ij}(\bar{s}, t)\, dt < \infty \qquad \text{and} \qquad \int_a^b L_{ij}(-\bar{s}, t)\, dt < \infty,$$

the L^p *problem* (2) *has a solution if* $p < \bar{s}/n^2(b - a)$).

(ii) *If for some* $\bar{s} > 0$ *and each* (i, j)

$$\int_a^b L_{ij}^{n^2}(\bar{s}, t)\, dt < \infty \qquad \text{and} \qquad \int_a^b L_{ij}^{n^2}(-\bar{s}, t)\, dt < \infty$$

the L^p *problem has a solution if* $p < \bar{s}/(b - a)$.

In both cases the solution is necessarily unique.

Proof. Let $k(t, \omega) = |A(t, \omega)|$ and $L(s, t) = Ee^{sk(t,\omega)}$. Then

$$k(t, \omega) \leqslant \sum_{i,j=1}^{n} |a_{ij}(t, \omega)|.$$

But since the a_{ij} are independent, if $s > 0$, then

$$L(s, t) \leqslant \prod_{i,j=1}^{n} E(e^{s|a_{ij}(t,\omega)|}).$$

Since $e^{s|a_{ij}(t,\omega)|} \leqslant e^{sa_{ij}(t,\omega)} + e^{-sa_{ij}(t,\omega)}$, $E(e^{s|a_{ij}(t,\omega)|})$ is L^r integrable on $[a, b]$ if the same is true for $e^{sa_{ij}(t,\omega)}$ and $e^{-sa_{ij}(t,\omega)}$. Elementary estimates using the Hölder inequality then show we can apply the preceding theorem. Q.E.D.

If $A(t)$ has some special form, one can often get better estimates. For example, if $A(t, \omega)$ is the matrix arising from an n-th degree linear equation, then all the n^2 in the previous corollary can be replaced by n.

If all the Laplace transforms $L_{ij}(s, t)$ in the preceding corollary are everywhere defined (as functions of s) and are locally L^∞ functions of t, then the equation has an L^p solution on the whole interval being considered for all p for which $P(s)$ is L^p-integrable. Hence, the following example is true.

EXAMPLE. Suppose x_0, $A(t)$ and $P(t)$ are as in the preceding corollary. Suppose further that the coefficients of $A(t)$ and $P(t)$ are normally distributed with means and variances which are bounded on $[a, b]$. Then

$$x' = A(t)\, x(t) + P(t), \qquad x(0) = x_0$$

has a unique L^p solution for all p on $[a, b]$.

We remark that it is possible to use the methods of proof of Lemma 6 to bound $\| x(t) - x_n(t)\|_p$ by $\| \xi(t) - \xi_n(t)\|_p$ and, hence, get error estimates for the convergence of moments of the solution. For linear systems the estimates arising from the next theorem are slightly better.

THEOREM 9. *Let $A(t)$ be an $n \times n$ matrix of L^p integrable processes. Let $P : I \to L(p, n)$ be L^p integrable, and let $x_0 \in L(p, n)$. Then the L^p problem*

$$x'(t) = A(t)\,x(t) + P(t), \qquad x(a) = x_0 \tag{4}$$

has a (necessarily unique) solution on I if

(a) $$\sum_{k=1}^{\infty} \int_a^b \int_a^{s_k} \cdots \int_a^{s_2} \| A(s_k)\, A(s_{k-1}) \cdots A(s_2)\, P(s_1) \|_p \, ds_1 \cdots ds_k < \infty$$

(b) $$\sum_{k=1}^{\infty} \int_a^b \int_a^{s_k} \cdots \int_a^{s_2} \| A(s_k) \cdots A(s_1)\, x_0 \|_p \, ds_1 \cdots ds_k < \infty.$$

Proof. Define the iterates

$$\begin{aligned} x_0(t) &= x_0 \text{ (the initial condition)} \\ x_{m+1}(t) &= x_0 + (L^p) \int_a^t A(s)\, x_m(s)\, ds + (L^p) \int_a^t P(s)\, ds. \end{aligned} \tag{6}$$

The iterates obviously exist and converge uniformly in the $L(p, n)$ norm to some process $x(t)$. In fact, x_m has the explicit representation

$$\begin{aligned} x_m(t) = x_0 &+ \sum_{k=1}^{m} \int_a^t \int_a^{s_k} \cdots \int_a^{s_2} A(s_k) \cdots A(s_1)\, x_0 \, ds_1 \cdots ds_k \\ &+ \sum_{k=1}^{m} \int_a^t \int_a^{s_k} \cdots \int_a^{s_2} A(s_k) \cdots A(s_2)\, P(s_1)\, ds_k \cdots ds_k \end{aligned} \tag{7}$$

The desired convergence then follows from the assumptions and Theorem A.7.

To complete the proof, we must show

$$x(t) = x_0 + \int_a^t A(s)\, x(s)\, ds + \int_a^t P(s)\, ds.$$

We first show that $A(s)\, x(s)$ is $L(p, n)$ integrable. $x(t)$ is clearly $L(p, n)$ norm continuous and, hence, (since it is then L^p integrable) there is an equivalent product measurable process. Hence, if $\int_a^b \| A(s)\, x(s) \|_p \, ds < \infty$, then $A(s)\, x(s)$ is $L(p, n)$ integrable. But since $A(s)\, x_m(s)$ is easily seen to be L^p integrable and since

$$\| A(s)\, x(s) \|_p \leqslant \| A(s)(x(s) - x_m(s)) \|_p + \| A(s)\, x_m(s) \|_p ,$$

we need only show $\int_a^b \| A(s)(x(s) - x_m(s))\|_p < \infty$ for some m. But by assumption,

$$\int_a^b \| A(s)(x(s) - x_m(s))\|_p \, ds$$
$$= \sum_{k=m+1}^{\infty} \int_a^b \int_a^{s_k} \cdots \int_a^{s_2} \| A(s_k) \cdots A(s_1)\, x_0 \|_p \, ds_1 \cdots ds_k$$
$$+ \sum_{k=m+1}^{\infty} \int_a^b \int_a^{s_k} \cdots \int_a^{s_2} \| A(s_k) \cdots A(s_2)\, P(s_1)\|_p \, ds_1 \cdots ds_k < K$$

uniformly for some K. Observe also that this inequality also shows $\int_a^b \| A(s)(x(s) - x_n(s))\|_p \, ds \to 0$. Hence, we may apply the bounded convergence theorem for Bochner integrals to conclude, in addition,

$$\lim_{m\to\infty} \left\| (L^p) \int_a^t A(s)\, x(s)\, ds - (L^p) \int_a^t A(s)\, x_m(s)\, ds \right\|_p = 0.$$

But

$$\left\| x(t) - x_0 - (L^p) \int_a^t A(s)\, x(s)\, ds - (L^p) \int_a^t P(s)\, ds \right\|_p$$
$$\leqslant \| x(t) - x_{m+1}(t)\| + \left\| x_{m+1}(t) - x_0 - \int_a^t A(s)\, x_m(s)\, ds - \int_a^t P(s)\, ds \right\|_p$$
$$+ \left\| (L^p) \int_a^t A(s)\, x(s)\, ds - (L^p) \int_a^t A(s)\, x_m(s)\, ds \right\|_p$$

and all the terms on the right go to zero. Hence, x is the desired solution.

Uniqueness follows from Corollary 4. Q.E.D.

COROLLARY 10. *Let A, P, and x_0 be as in the theorem. If x_0 and P are independent of A, then conditions* (a) *and* (b) *may be replaced by the single condition*

$$\text{(a}'\text{)} \quad \sum_{k=1}^{\infty} \int_a^b \int_a^{s_k} \cdots \int_a^{s_2} \| A(s_k) \cdots A(s_1)\|_p \, ds_1 \cdots ds_k < \infty.$$

We note in passing that the iterates defined in Eq. (7) can be used to compute the moments of the solution. The method is summarized in the following corollary.

COROLLARY 11. *Let $x(t) = (x^1(t),..., x^n(t))$ solve* (2) *and let the m-th iterate [defined in* (6)] *be $x_m(t) = (x_m{}^1(t),...,x_m{}^n(t))$.*

Let

$$M(t_1, \dots, t_r; i_1, \dots, i_r) = E(x^{i_1}(t_1) \cdots x^{i_r}(t_r))$$

$$M_m(t_1, \dots, t_r; i_1, \dots, i_r) = E(x_m^{i_1}(t_1) \cdots x_m^{i_r}(t_r))$$

$$R_m(t) = \sum_{k=m+1}^{\infty} \int_a^t \int_a^{s_k} \cdots \int_a^{s_2} [\| A(s_k) \cdots A(s_2) P(s_1) \|_p + \| A(s_k) \cdots A(s_2) x_0 \|_p] \, ds_1 \cdots ds_k .$$

Then if $r < P$ and $t_1, t_2, \dots, t_r$ are in I,

(a) $$| M(t_1, t_2, \dots, t_r; i_1, \dots, i_r) - M_m(t_1, \dots, t_r; i_1, \dots, i_r)| \leqslant (R_1(b))^{r-1} \sum_{i=1}^{m} R_m(t_1) \leqslant (R_1(b))^{r-1} r R_m(b)$$

(b) *An explicit expression for the r-th moments of $x_m(t)$ can be obtained from Eq.* (7) *by taking the explicit representations of $x^{i_1}(t_1), \dots, x^{i_r}(t_r)$, multiplying them together, and taking expectations.*

(c) *If P and x_0 are independent of $A(t)$, then for $r \leqslant p$ the r-th moments of x_m depend only on the first r moments of $P(t)$ and x_0 and on the first rm moments of the components of $A(t)$. The r-th moments of $x(t)$ depend on the first r moments of x_0 and $P(t)$ and all moments of $A(t)$.*

Proof. (a) follows from the inequality, valid for any set of random variables $B_1, \dots, B_r$ and $C_1, \dots, C_r$,

$$| E(B_1 B_2 \cdots B_r) - E(C_1 C_2 \cdots C_r)| \leqslant \sum_{k=1}^{r} \| B_1 \|_r \cdots \| B_{k-1} \|_r \| B_k - C_k \|_r \| C_{k+1} \|_r \cdots \| C_r \|_r .$$

The other assertions follow from examining (7). In (b) one must remember that time and expectation integrals can be interchanged because the time integrals are $L(p, n)$ integrals. Q.E.D.

As an example of an application of this theory we consider the linear L^p problem with constant coefficients.

$$x'(t, \omega) = A(\omega) x(t, \omega) + P(t, \omega), \qquad x(a, \omega) = x_0(\omega), \tag{8}$$

where $A = (a_{ij}(\omega))$ is an $n \times n$ matrix of random variables with finite moments, $P : I \to L(p, n)$ is $L(p, n)$ integrable, and $x_0 \in L(p, n)$.

THEOREM 12. *In the linear L^p problem* (8), *suppose x_0 and $P(t)$ are* *independent of A. Then if the Laplace transforms $L_{ij}(s) = E(e^{sa_{ij}(\omega)})$ are analytic for $|s| < R$, then there exists a (unique) L^p solution of (5) on the interval $|t - a| < R/np$.*

Proof. Because the proof is quite lengthy and the result is only slightly better than an application of Corollary 7, we only sketch the proof, which is detailed in [6], p. 53–60.

Applying Corollary 10 we see we need to show

$$\sum_1^\infty \frac{\| A^n \|_p}{n!} t^n < \infty,$$

i.e., we need to bound the growth rate of $\| A^n \|_p$. Estimates reduce this to estimating the growth rate of the moments of the $a_{ij}(\omega)$. But since the transforms $L_{ij}(s)$ are essentially moment-generating functions, standard tests for the radius of convergence of a power series give just the desired bound.

Q.E.D.

It is easy to construct examples of linear equations where Theorem 9 gives better results than does Lemma 7. For example, consider

$$x'(t) = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} x(t), \qquad x(0) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

where a_i is independent of x_i but $x_1 a_2 \notin L^p$. Then if $x_1 \in L^p$ and a_1 has an analytic characteristic function, Theorem 9 but not Lemma 7 applies. However, Lemma 7 and its corollaries do seem to cover all important linear systems.

We now give two examples showing that in some cases at least Theorem 12 (and, hence, Theorem 9) are the best possible. We remark that no examples where Theorem 9 does not give the best possible result are known.

EXAMPLE 1. Consider $x'(t, \omega) = A(\omega)\, x(t, \omega)$, $x(0, \omega) = x_0(\omega)$, where $A(\omega) \geqslant 0$ is a positive random variable independent of x_0 and where $x_0 \in L^p$. The SP solution (and, hence, by Theorem 3(a) the only possible L^p solution) is $x(t, \omega) = x_0(\omega) \exp(tA(\omega))$. $x(t, \omega)$ is in L^p IFF $\| A^n \|_p\, t^n/n!$ converges. This in turn is true IFF $|t| < R/p$, where R is the radius of convergence of $E(\exp(sA(\omega)))$. This is just as Theorem 12 predicts.

For example, if $A(\omega)$ is exponentially distributed with density e^{-x}, then $R = 1$ and in this case an L^p solution exists on precisely $[0, 1/p]$.

EXAMPLE 2. Consider $x'(t, \omega) = A(\omega)\, x(t, \omega)$, $x(0, \omega) = x_0(\omega)$ where now $A(\omega) = (a(\omega))$ is an $n \times n$ matrix all of whose elements are the same positive random variable $a(\omega)$. If $x_0 \in L(p, n)$ is independent of $a(\omega)$, it is easy to see that an L^p solution exists precisely if $|t| < R/np$ where R is the radius of convergence of $E(\exp(sa(\omega)))$.

APPENDIX. Integration Theory

Throughout this appendix, $I = [a, b]$ is some finite interval, x maps $R \times \Omega \to R^n$, and $\hat{x} : R \to L(p, n)$ is equivalent to x (see Definition 2). B is a Banach space.

All undefined terms are defined in [7], Chap. 3.

Definition A-1. (a) The sample path integral (or "SP integral"), written $(SP) \int x(t, \omega)\, dt$, is just the Lebesque integral of the sample paths obtained by fixing ω. x is said to be SP integrable if this integral exists almost surely.

(b) The *L^p integral* of $\hat{x}$ is the Bochner integral of $\hat{x} : I \to L(p, n)$, if it exists. It is denoted $(L^p) \int \hat{x}(t)\, dt$. (See [7], p. 79.)

(c) The *W^p integral* of $\hat{x} : I \to L(p, n)$ is the Pettis integral of x. It is denoted $(w^p) \int \hat{x}(t)\, dt$. ([7], p. 77.)

Theorem A-2. (a) *x is SP integrable on I* iff *almost every sample path is Lebesque integrable.*

(b) *$\hat{x}$ is L^p integrable on I* iff *it is strongly measurable (in the $L(p, n)$ norm topology—*[7], *p.* 72) *and* $\int_a^b \| \hat{x}(t)\|_p\, dt < \infty$.

Proof. See [7], p. 78, 80.

Definition A-3. $\hat{x} : I \to L(p, n)$ is said to be

(a) *absolutely continuous* if for each $\epsilon > 0$ there exists a $\delta > 0$ such that for intervals $(a_1, b_1),..., (a_n, b_n)$ if $\sum | b_i - a_i | < \delta \, \| \sum(x(b_i) - x(a_i))\| < \epsilon$.

(b) *Strongly absolutely continuous* if, in addition, $\sum \| x(b_i) - x(a_i))\| < \epsilon$.

Definition A-4. (a) $y : I \times \Omega \to R^n$ is the SP *Derivative* of $x : I \times \Omega \to R^n$ iff almost all paths of x are absolutely continuous and $\partial x/\partial t(t, \omega) = y(t, \omega)$ almost surely along almost every sample path.

(b) $\hat{y} : I \to L(p, n)$ is the L^p derivative of $\hat{x}$ iff for a.e. t

$$\lim_{h \to 0} \|(\hat{x}(t + h) - \hat{x}(t))/h - \hat{y}(t)\|_p = 0.$$

(c) $\hat{y} : I \to L(p, n)$ is the W^p pseudoderivative of $\hat{x}$ iff for all $x^* \in (L(p, n))^*$, the dual space of $L(p, n)$, $x^*(\hat{x}(t))$ is differentiable almost surely, with derivative $x^*(\hat{y}(t))$.

Theorem A-5. (a) $\hat{x}(t) = \hat{x}(a) + (w^p) \int_a^t \hat{y}(s)\, ds$ iff *$\hat{y}$ is the W^p pseudo-derivative of $\hat{x}$, $\hat{y}$ is W^p integrable, and $\hat{x}$ is absolutely continuous.*

(b) $\hat{x}(t) = \hat{x}(a) + (L^p) \int_a^t \hat{y}(s)\, ds$ IFF $\hat{x}$ *is strongly absolutely continuous and almost surely* L^p *differentiable with* L^p *derivative* $\hat{y}$.

Proof. (a) follows from [1], p. 132 and [7], p. 78.

(b) follows from [7], p. 83–88 and [1], p. 132.

THEOREM A-6. (a) *If* $\hat{y} : I \to L(p, n)$ *is* L^p *integrable, then there exists* $y : I \times \Omega \to R^n$, *equivalent to* $\hat{y}$, *product integrable so that for each* t *in* $I (L^p) \int_a^t \hat{y}(s)\, ds$ *and* $(SP) \int_a^t \hat{y}(s, \omega)\, ds$ *are equivalent.*

(b) *If* $\hat{y} : I \to L(p, n)$ *is* W^p *integrable and there exists an equivalent product integrable* $y : I \times \Omega \to R^n$, *then* $(W^p) \int_a^t \hat{y}(s)\, ds$ *and* $(SP) \int_a^t y(s, \omega)\, ds$ *are equivalent.*

(c) *If* $y : I \times \Omega \to R^n$ *is product measurable and* $\int_a^b \| y(t, \omega)\|_p\, dt < \infty$, *then the equivalent* $\hat{y} : I \to L(p, n)$ *is* L^p *integrable.*

Proof. (a) follows from [8], p. 196–199,

(b) is proved in [6], p. 16,

(c) is proved by R. Edsinger in ([5], p. 8).

THEOREM A-7. *If* $x_n : I \to L(p, n)$ *and* $x : I \to L(p, n)$ *are integrable, if for each* $\epsilon > 0$ *the measure of* $\{t : \| x_n(t) - x(t)\|_p > \epsilon\} \to 0$, *and if there is an integrable* $f(t)$ *such that* $\| x_n(t)\|_p \leqslant f(t)$, *then* $\lim_{n\to\infty}(L^p) \int_a^t x_n(s)\, ds = \int_a^b x(s)\, ds$ *for* $t \in I$.

Proof. [7], p. 83.

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