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Linear and affine logics with temporal, spatial and epistemic operators

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Abstract

A temporal spatial epistemic intuitionistic linear logic (TSEILL) is introduced, and the completeness theorem for this logic is proved with respect to Kripke semantics. TSEILL has three temporal modal operators: $[F]$ (any time in the future), $[N]$ (next time) and $[P]$ (past), some spatial modal operators $[l_i]$ (locations), two epistemic modal operators; $[K]$ (know) and $\langle K\rangle$, and a linear modal operator ! (exponential). A basic normal modal intuitionistic affine logic (BIAL) and its normal extensions are also defined, and the completeness theorems for these logics are proved with respect to Kripke semantics. In the proposed semantic framework of these normal extensions, a simple correspondence can be given between frame conditions and Lemmon–Scott axioms. A dynamic intuitionistic affine logic (DIAL) is proposed as an affine version of (test-free) dynamic logic, and the completeness theorem for this logic is shown with respect to Kripke semantics. Finally, some intuitive interpretations, such as resource and informational interpretations, are given for the proposed logics and semantics. By using these logics, semantics and interpretations, various kinds of fine-grained resource-sensitive reasoning can be expressed.

Keywords: Completeness; Kripke-style semantics; Linear logic; Affine logic; Modal logic

1. Introduction

1.1. Resource-sensitive combined modal logics

In this paper, a number of combined modal logics, including and combining temporal, spatial, epistemic and dynamic logics, are studied based on linear and affine logics. The combined logics proposed can deal with fine-grained resource-sensitive reasoning appropriately. The logics can thus be viewed as resource-sensitive refinements of the traditional modal systems.

Why do we combine logics? The practical significance of this issue becomes apparent when working in knowledge representation within AI and in formal specification and verification within software engineering [45]. For example, in a knowledge representation issue, it may be necessary to work with both temporal (future, present, past) and resource-sensitive (e.g. costs of money) aspects, and in a software specification problem, it may be necessary to

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work with temporal, spatial (e.g. locations of processors in concurrent system) and resource-sensitive (e.g. memory management) aspects. These various aspects must be expressed precisely using a combined logical system to obtain concrete theoretical outcomes.

Why do we adopt linear and affine logics as the base logics? In this paper, the central motivation of introducing the logics is to represent a fine-grained form of resource-sensitive reasoning, such as time/space-dependent resource reasoning. It is known that linear and affine logics are good candidates for such resource logics [13]. [1] Thus, the linear and affine logics are adopted for the base logics in this paper.

The notion of "resource", encompassing concepts such as processor time, memory, cost of components and energy requirements, is fundamental to computational systems [42]. It is known that linear logics can elegantly represent the concept of "resource consumption" using the linear implication connective $\rightarrow$ and the fusion connective $*$, and can also represent the concept of "reusable resource" using the linear exponential operator !. A typical example is as follows:

$$coin * coin \rightarrow coffee * !\, water.$$

This example means "if we consume two coins, then we can take a cup of coffee and as much water as we like". More fine-grained and expressive resource descriptions are desired for discussing some real and practical examples. For example, the following expressions may be necessary for some practical situations:

$$[teashop](coin * coin * coin \rightarrow [N][N]coffee * [N]water),$$

$$[cafeteria](coin * coin \rightarrow [N]coffee).$$

These examples mean "in the teashop, if we consume three coins, then we can take a cup of coffee after two minutes and a cup of water after one minute", and "in the cafeteria, if we consume two coins, then we can take a cup of coffee after one minute", respectively. In these examples, the expressions $[teashop]$ and $[cafeteria]$ are regarded as spatial modal operators with spatial domain $S = \{teashop, cafeteria\}$, and the expression $[N][N]$ is regarded as the passage of two time units.

To express such fine-grained situations, further extensions (with various temporal, knowledge and spatial modal operators) based on the linear and affine logics are required.

1.2. Linear logic, temporal linear logic, spatial linear logic and epistemic linear logic

Girard's *linear logics* [13] are invaluable in the formalization of resource-conscious systems, concurrent systems and concurrent logic programming languages, and various extended linear logics have been proposed for these applications. The *temporal linear logics* posed by Hirai [15,16], Tanabe [46], Kanovich and Ito [31] are examples of such logics. In particular, Hirai's intuitionistic temporal linear logic (ITLL), which has two temporal modal operators $\Box$ (any time) and $\bigcirc$ (next time), is useful for describing timed Petri nets and timed linear logic programming languages. Dam's modal linear logic [11], which has a process algebraic interpretation, is also considered to be a version of temporal linear logic. Kobayashi et al. [32] introduced a higher-order multi-modal linear logic (MLL), which describes a distributed concurrent linear logic programming language. MLL is regarded as a version of *spatial linear logic*, since MLL has some spatial modalities $[l_i]$ (locations).

To date, a number of temporal and spatial linear logics have been developed quite successfully. In this paper, these two types of modal linear logics are integrated and combined with an epistemic modal linear logic in order to realize knowledge representation in resource-sensitive reasoning systems. Of course, while no *epistemic linear logic* has been presented, this approach believed to be very important in the discussion of security issues in concurrent systems. For example, BAN logic [7], which is a logic for cryptology and security protocols, is viewed as a modified version of epistemic logic.

1.3. Proposed approach

To establish a logical foundation for a general (or abstract) framework that can deal with various resource-conscious and concurrent systems, an interesting new linear logic TSEILL (*temporal spatial epistemic intuitionistic linear logic*) is introduced. TSEILL has various modal operators, including $[K]$ (know), $\langle K\rangle$, $[l_i]$ (locations), $[F]$ (any time in the future), $[N]$ (next time), $[P]$ (past) and ! (exponential). This logic is a natural extension of Girard's original

[1] Another example is the logic BI of bunched implications [42].

intuitionistic linear logic (ILL) [13], Hirai's ITLL [15,16], which has $[F]$, $[N]$ and !, and the propositional version of MLL by Kobayashi et al. [32], which has $[l_i]$ and !. A logic BIAL (*basic normal modal intuitionistic affine logic*) and its normal extensions are also introduced. BIAL is an extension of intuitionistic affine logic, which is also called BCK-logic or FL_{ew} in [40,17]. Modifying BIAL gives a useful logic DIAL (*dynamic intuitionistic affine logic*), which is an affine version of the (*test-free*) *propositional dynamic logic* by Pratt. The original propositional dynamic logic is known as "the logic of programs", and is very useful for verifying programs [14].

In this paper, *Kripke type semantics* (or also called *resource algebras*) are given for these logics. The Kripke models presented here are extensions of the models in [21,32]. The completeness theorems with respect to such extended models can be proved using an extended version of Ishihara's canonical model construction method [17]. The Kripke models in [21] are extensions of the established models (for the non-modal parts) by Došen [12], Ono–Komori [40] and Urquhart [48], and the non-modal part of the resource algebra in [32] is roughly the same model as in [12,40]. [2] The multiplicative part of the Kripke semantics for the logic BI of bunched implications [42] is also similar to the resource algebra. The established models by Došen and Ono–Komori are known as natural generalizations of Routley and Meyer's Kripke-type semantics for substructural logics (see e.g. [2,44]). An advantage of these models is that intuitive informational interpretations can be given for the corresponding logics. These interpretations have been presented by Wansing [50,51], Urquhart [48], Restall [43], Dam [11] and Pym et al. [42].

1.4. Overview of proposed logics

TSEILL can be summarized as follows:

- The epistemic modal operators $[K]$ (know) and $\langle K\rangle$ are defined as well-known S5-type modal axiom schemes. For example, we have $[K]\alpha\to\alpha$ (knowledge axiom), $[K]\alpha\to[K][K]\alpha$ (positive introspective axiom) and

$$\frac{\alpha}{[K]\alpha}\ \text{(epistemic necessitation).}$$

- The temporal modal operators $[F]$ (any time in the future) and $[N]$ (next time) are defined as S4-type modal axiom schemes, K-type modal axiom schemes and $[F]\alpha\to[N]\alpha$. This axiomatization was introduced in [16]. [3] The operator $[P]$ (past) is defined as $[P]\alpha\wedge[P]\beta\to[P](\alpha\wedge\beta)$, $\alpha\to[P]\alpha$, $[P][P]\alpha\to[P]\alpha$, and

$$\frac{\alpha\to\beta}{[P]\alpha\to[P]\beta}\ \text{(past regularity).}$$

This axiomatization for $[P]$ is similar to that in [11].

- The relationships between epistemic, temporal and linear modal operators are defined by $[K]\alpha\to!\alpha$ and $!\alpha\to[F]\alpha$ (thus we also have $[K]\alpha\to[F]\alpha$). The later axiom scheme is also adopted in [16], and means "if α is reusable at any time in the future, then α is usable at any time but only once". The former means "if an agent (or processor) knows of a resource α, then α is reusable at any time in the future". The former assumption may be justified in concurrent systems when the resource α is a password, command or public key for a computer system.
- The spatial modal operators $[l_i]$ (locations) are, for example, characterized as $[l_1][l_2]\alpha\leftrightarrow[l_2]\alpha$,

$$\frac{\alpha}{[l]\alpha}\ \text{(space necessitation)}\qquad\frac{\forall s\in S\ ([s]\alpha)}{\alpha}\ \text{(space induction)}$$

for any $l_1, l_2, l\in S$, where S (called a *spatial domain*) is a nonempty set of locations. This axiom scheme means that each location l_i is the absolute address of a location, that is, the location l_i refers to the same location anywhere. The rules indicate that α holds if and only if α holds at any location. Thus, the formula $[l]\alpha$ means "α is available at location l" or "execute process α at location l". These interpretations for $[l_i]$ are after Kobayashi et al. [32], but the Hilbert-style axiomatization is a new contribution. [4]

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[2] These established Kripke models are also called *groupoid models* by Došen. For a historical overview of these Kripke models for modal and non-modal substructural logics, see [20,21].

[3] Strictly speaking, Hirai's ITLL is a Gentzen-type sequent calculus.

[4] MLL by Kobayashi et al. is a Gentzen-type sequent calculus, and the modal operators $[l_i]$ in MLL are characterized using a structural congruence relation.

BIAL and its normal extensions are summarized below.

• BIAL is obtained from non-modal intuitionistic linear logic with a weakening axiom scheme $\alpha \to \beta \to \alpha$ by adding the spatial modal operators $[l_i]$ and new modal operators $\Box$ and $\Diamond$.

• The normal extensions of BIAL are obtained from BIAL by adding arbitrary combinations of the Lemmon–Scott axiom schemes such as T: $\Box\alpha \to \alpha$, 4: $\Box\alpha \to \Box\Box\alpha$, 5: $\Diamond\alpha \to \Box\Diamond\alpha$, D: $\Box\alpha \to \Diamond\alpha$ and B: $\alpha \to \Box\Diamond\alpha$.

By modifying BIAL, DIAL can be obtained, which has, for example, the axiom schemes $[a+b]\alpha \leftrightarrow [a]\alpha \wedge [b]\alpha$, $[a \times b]\alpha \leftrightarrow [a][b]\alpha$ and $[\mathbf{0}]\alpha \leftrightarrow \alpha$, where $+$ (nondeterministic choice) and $\times$ (composition) are binary operations on C (a set of programs) and $\mathbf{0} \in C$. The expression of the form $[c]\alpha$ (program necessity) for $c \in C$ is a formula, which intuitively states that "it is necessary that after executing the program c, the information α must be true". The program expressions $a + b$ and $a \times b$ mean "choose either a or b nondeterministically and execute it" and "execute a, then execute b", respectively.

1.5. Semantics and informational interpretation

An informational interpretation follows along with an outline of some of the advantages of the semantics for TSEILL, BIAL (and its normal extensions) and DIAL.

• The Kripke frame for TSEILL is a structure $\langle M, S, \dagger, K, J, F, N, P, \cap, \cdot, \varepsilon, \omega\rangle$ satisfying certain conditions. Here, the following informational interpretation by Wansing [50,51] and Urquhart [48] is adopted:
M is a set of information pieces,
$\cdot$ is the addition of information pieces,
$\cap$ is the intersection of information pieces,
ε is an empty piece of information,
ω is the greatest piece of information.

The following are new proposals.

S is a set of locations (the *spatial domain*),
$\dagger$, K, J, F, N and P are unary operations on M, corresponding to the modal operators !, $[K]$, $\langle K\rangle$, $[F]$, $[N]$ and $[P]$, respectively. $\dagger$ is the infinite addition of information pieces.

The forcing relation $(x, l) \models \alpha$ on the frame is interpreted as "α is obtained at location $l \in S$ using information piece (or resource) x".

There is also an alternative informational interpretation by Restall [43], presenting Barwise's account of *information flow*.[5] The process-algebraic interpretation by Dam [11] and the Petri net interpretation by Pym et al. [42] are also available with a few modifications.

• The Kripke frame for BIAL is a structure $\langle M, S, \sharp, \flat, \cap, \cdot, \varepsilon, \omega\rangle$ satisfying certain conditions. Here, $\sharp$ and $\flat$ are unary operations on M similar setting for the operation P in the frame for TSEILL. To formalize these operations, the weakening axiom scheme plays a crucial role in the completeness proof. The advantages of the present model for BIAL are that a simple correspondence can be given between frame conditions and Lemmon–Scotto axiom schemes, that is, a good correspondence theory is obtained analogous to that of usual normal modal logics based on the classical logic, and secondly, the valuation conditions for $\Box$ and $\Diamond$ in the models are very simple and intuitive:

$(x, l) \models \Box\alpha$ iff $(\sharp x, l) \models \alpha$,
$(x, l) \models \Diamond\alpha$ iff $(\flat x, l) \models \alpha$.

The idea of these valuation conditions is inspired by the condition in Dam's model for the temporal past operator in a modal intuitionistic linear logic [11], although the setting of the operations in this paper is different. Furthermore, a much wider class of normal modal affine logics can be dealt with uniformly, and the proof of the completeness theorem with respect to the model is considerably simpler than for other proposals.

[5] Strictly speaking, Restall showed that a large class of Routly–Meyer-style Kripke frames can be viewed as a model of information flow. The Kripke models (for non-modal part) used here are generalizations of the Routley–Meyer models (see [40]).

• The Kripke frame for DIAL is a two-sorted structure $\langle \mathbf{D}, \mathbf{M}, D_c\ (c \in C)\rangle$, where $\mathbf{D} := \langle C, +, \times, \cdot^*, \mathbf{0}\rangle$ and $\mathbf{M} := \langle M, S, \cap, \cdot, \varepsilon, \omega\rangle$, satisfying certain conditions. Here, D_c is a unary operation on M, and is a modification of the operators $\sharp$ and $\flat$. For this Kripke frame, the valuation condition for the program necessity operator is given by
$(x, l) \models [c]\alpha$ iff $(D_c x, l) \models \alpha$.

1.6. Organization of this paper

This paper is organized as follows. In Section 2, a Hilbert-style axiomatization of TSEILL is introduced, and the relationships between some sublogics of TSEILL and the established logics of Hirai [16] and Kobayashi et al. [32] are presented. In Section 3, a Kripke semantics is defined for TSEILL, and the soundness theorem is given. In Section 4, the completeness theorem for TSEILL is proved with respect to the Kripke model, representing the main result of this paper. In Section 5, BIAL and its normal extensions are introduced, considering a number of logics having Lemmon–Scott axiom schemes. Kripke models are then developed for these logics, and the correspondence between axiom schemes and conditions on the frame are examined. The completeness theorems are proved in a general setting, showing that any normal extension over BIAL is complete with respect to the corresponding model. This is also a main result of this paper. In Section 6, DIAL and its Kripke model are presented, and the corresponding completeness theorem is proved. In Section 7, further extensions and modifications of the results are discussed, and some extensions of TSEILL, by adding strong negation connective, soft exponential operator and mingle axiom, and a common knowledge affine logic are introduced with the corresponding completeness theorems. In Section 8, some resource and informational interpretations of the proposed logics and Kripke semantics are presented. In Section 9, some illustrative examples, such as medical and distributed systems, are given. In Section 10, some decidability and complexity issues on linear and affine logics are reviewed, and a comparison of the proposed logics and the existing logics is given.

2. Temporal spatial epistemic linear logic

In this section, we introduce a Hilbert-style axiomatization of the logic TSEILL. This axiomatization is rather complex, and hence considering the intuitive meaning of this logic may be difficult. The intuitive meaning and motivation are thus precisely discussed in Section 8. Only the formal definition of the logic is given in this section.

Prior to the precise discussion, we introduce the language used in this paper. *Formulas* are constructed from propositional variables, constants $\mathbf{1}$, $\top$ and $\bot$, $\rightarrow$ (implication), $\wedge$ (conjunction), $*$ (fusion), $\vee$ (disjunction), modal operators $[F]$ (any time in the future), $[N]$ (next time), $[P]$ (past), $[l_i]$ (locations), $[K]$ (know), $\langle K\rangle$ (the dual of $[K]$) and ! (exponential). We denote S for a nonempty set of locations, and the form $[l_i]\alpha$ is a formula where $l_i \in S$. We call the set S and its element l_i, *spatial domain* and *location* respectively. We will follow the notation for the constants $\mathbf{1}$, $\top$ and $\bot$ in [47], which differs from that in [13]. Furthermore, $\wedge$, $\vee$, and $*$ correspond to &, $\oplus$ and $\otimes$ in [13]. Lower case letters $p, q, \ldots$ are used as metavariables for propositional variables, lower case Greek letters $\alpha, \beta, \ldots$ are used as metavariables for formulas. The symbol $\equiv$ means equality of sequences of symbols. We adopt the convention of association to the right in order to omit parentheses. For example, $(\alpha\rightarrow\beta\rightarrow\gamma)\rightarrow\beta\rightarrow\alpha\rightarrow\gamma \equiv (\alpha\rightarrow(\beta\rightarrow\gamma))\rightarrow(\beta\rightarrow(\alpha\rightarrow\gamma))$. If a formula α is provable in a logic L, then we write $L\vdash \alpha$.

Definition 1 (*The logic TSEILL*). The axiom schemes and inference rules for the logic TSEILL are as follows.
Non-modal part[6]:
A1: $\alpha\rightarrow\top$,
A2: $\bot\rightarrow\alpha$,
A3: $\alpha\rightarrow\alpha$,
A4: $\alpha \wedge \beta\rightarrow\alpha$,
A5: $\alpha \wedge \beta\rightarrow\beta$,
A6: $(\gamma\rightarrow\alpha) \wedge (\gamma\rightarrow\beta)\rightarrow\gamma\rightarrow\alpha \wedge \beta$,
A7: $\alpha\rightarrow\alpha \vee \beta$,

[6] This axiomatization for non-modal part is due to Ishihara [17], which axiomatization corresponds to that for the non-modal propositional intuitionistic linear logic.

A8: $\beta \to \alpha \vee \beta$,
A9: $\alpha \to \beta \to \alpha * \beta$,
A10: $(\alpha \to \gamma) \wedge (\beta \to \gamma) \to \alpha \vee \beta \to \gamma$,
A11: $(\beta \to \gamma) \to (\alpha \to \beta) \to \alpha \to \gamma$,
A12: $(\mathbf{1} \to \alpha) \to \alpha$,
A13: $\alpha \to \bot \to \beta$,
A14: $\alpha \to \mathbf{1} \to \alpha$,
A15: $(\alpha \to \beta \to \gamma) \to \alpha * \beta \to \gamma$,
A16: $(\alpha \to \beta \to \gamma) \to \beta \to \alpha \to \gamma$,

$$\frac{\alpha \quad \alpha \to \beta}{\beta} \text{ (mp)}, \quad \frac{\alpha \quad \beta}{\alpha \wedge \beta} \text{ (adj)}.$$

Linear-modal part[7]:
L1: $!(\alpha \to \beta) \to !\alpha \to !\beta$,
L2: $!\alpha \to \alpha$,
L3: $!\alpha \to !!\alpha$,
L4: $(!\alpha \to !\alpha \to \beta) \to !\alpha \to \beta$,
L5: $\alpha \to !\beta \to \alpha$,

$$\frac{\alpha}{!\alpha} \text{ (!ness)}.$$

Epistemic-modal part:
E1: $[K](\alpha \to \beta) \to [K]\alpha \to [K]\beta$,
E2: $[K]\alpha \to \alpha$,
E3: $[K]\alpha \to [K][K]\alpha$,
E4: $[K](\alpha \to \beta) \to \langle K \rangle \alpha \to \langle K \rangle \beta$,
E5: $\alpha \to \langle K \rangle \alpha$,
E6: $\langle K \rangle \langle K \rangle \alpha \to \langle K \rangle \alpha$,
E7: $\langle K \rangle \alpha \to [K] \langle K \rangle \alpha$,

$$\frac{\alpha}{[K]\alpha} \text{ (}K\text{ness)}.$$

Temporal-modal part:
T1: $[F](\alpha \to \beta) \to [F]\alpha \to [F]\beta$,
T2: $[F]\alpha \to \alpha$,
T3: $[F]\alpha \to [F][F]\alpha$,

$$\frac{\alpha}{[F]\alpha} \text{ (}F\text{ness)},$$

T4: $[N](\alpha \to \beta) \to [N]\alpha \to [N]\beta$,
T5: $[F]\alpha \to [N]\alpha$,

$$\frac{\alpha}{[N]\alpha} \text{ (}N\text{ness)},$$

T6: $[P]\alpha \wedge [P]\beta \to [P](\alpha \wedge \beta)$,
T7: $\alpha \to [P]\alpha$,
T8: $[P][P]\alpha \to [P]\alpha$,

$$\frac{\alpha \to \beta}{[P]\alpha \to [P]\beta} \text{ (}P\text{regu)}.$$

[7] See [47].

Linear-epistemic-temporal part:

R1: $[K]\alpha \to !\alpha$,
R2: $!\alpha \to [F]\alpha$.

Spatial-modal part:

S1: $[l]\mathbf{1} \to \mathbf{1}$,
S2: $\mathbf{1} \to [l]\mathbf{1}$,
S3: $\top \to [l]\top$,
S4: $[l]\bot \to \bot$,
S5: $[l](\alpha \wedge \beta) \to [l]\alpha \wedge [l]\beta$,
S6: $[l]\alpha \wedge [l]\alpha \to [l](\alpha \wedge \beta)$,
S7: $[l](\alpha \vee \beta) \to [l]\alpha \vee [l]\beta$,
S8: $[l]\alpha \vee [l]\beta \to [l](\alpha \vee \beta)$,
S9: $[l](\alpha \to \beta) \to [l]\alpha \to [l]\beta$,
S10: $([l]\alpha \to [l]\beta) \to [l](\alpha \to \beta)$,
S11: $[l](\alpha * \beta) \to [l]\alpha * [l]\beta$,
S12: $[l]\alpha * [l]\beta \to [l](\alpha * \beta)$,
S13: $[l]!\alpha \to ![l]\alpha$,
S14: $![l]\alpha \to [l]!\alpha$,
S15: $[l][K]\alpha \to [K][l]\alpha$,
S16: $[K][l]\alpha \to [l][K]\alpha$,
S17: $[l]\langle K\rangle\alpha \to \langle K\rangle[l]\alpha$,
S18: $\langle K\rangle[l]\alpha \to [l]\langle K\rangle\alpha$,
S19: $[l][F]\alpha \to [F][l]\alpha$,
S20: $[F][l]\alpha \to [l][F]\alpha$,
S21: $[l][N]\alpha \to [N][l]\alpha$,
S22: $[N][l]\alpha \to [l][N]\alpha$,
S23: $[l][P]\alpha \to [P][l]\alpha$,
S24: $[P][l]\alpha \to [l][P]\alpha$,
S25: $[l_1][l_2]\alpha \to [l_2]\alpha$,
S26: $[l_2]\alpha \to [l_1][l_2]\alpha$,

$$\frac{\alpha}{[l]\alpha}\ (\text{space1}), \quad \frac{\forall s \in S\ ([s]\alpha)}{\alpha}\ (\text{space2})$$

for any $l, l_1, l_2 \in S$.

Here we remark that if $S = \{s_1, s_2, s_3\}$, then the rule (space2) is of the form

$$\frac{[s_1]\alpha \quad [s_2]\alpha \quad [s_3]\alpha}{\alpha}\ (\text{space2}).$$

We also remark that the following rules are derivable in TSEILL:

$$\frac{\beta \to \gamma}{(\alpha \to \beta) \to (\alpha \to \gamma)}\ (\text{pref}), \quad \frac{\alpha \to \beta}{(\beta \to \gamma) \to (\alpha \to \gamma)}\ (\text{suff}), \quad \frac{\alpha \to \beta \quad \beta \to \gamma}{\alpha \to \gamma}\ (\text{cut}),$$

$$\frac{\alpha \to \gamma \quad \beta \to \gamma}{\alpha \vee \beta \to \gamma}\ (\text{or}), \quad \frac{\alpha \to \beta \to \gamma}{\alpha * \beta \to \gamma}\ (\text{residu}), \quad \frac{\alpha}{\mathbf{1} \to \alpha}\ (\text{ness}),$$

$$\frac{\alpha \to \langle K\rangle\beta}{\langle K\rangle\alpha \to \langle K\rangle\beta}\ (K\text{left}), \quad \frac{\alpha \to \beta \quad \alpha \to \gamma}{\alpha \to \beta \wedge \gamma}\ (\wedge\text{right}).$$

For example, (Kleft) is derivable using E4, E6, (Kness), (mp) and (cut):

$$\dfrac{\dfrac{\dfrac{\alpha \to \langle K\rangle\beta}{[K](\alpha \to \langle K\rangle\beta)} \quad [K](\alpha \to \langle K\rangle\beta) \to \langle K\rangle\alpha \to \langle K\rangle\langle K\rangle\beta}{\langle K\rangle\alpha \to \langle K\rangle\langle K\rangle\beta} \quad \langle K\rangle\langle K\rangle\beta \to \langle K\rangle\beta}{\langle K\rangle\alpha \to \langle K\rangle\beta} \; .$$

Next we define the following sublogics of TSEILL:

ILL = Non-modal part + Linear-modal part,
EILL = ILL + Epistemic-modal part + R1,
TILL = ILL + Temporal-modal part + R2,
ITLL = TILL− (T6–T8, Pregu),
SILL = ILL + Spatial-modal part− (S15–S24),
SEILL = EILL + Spatial-modal part− (S19–S24),
TEILL = EILL + Temporal-modal part + R2,
TSILL = SILL+ (S19–S24) +R2.

The logic ILL is Girard's original intuitionistic linear logic. The logics EILL, TILL, SILL, SEILL, TEILL and TSILL are, respectively, called *epistemic* ILL, *temporal* ILL, *spatial* ILL, *spatial epistemic* ILL, *temporal epistemic* ILL and *temporal spatial* ILL. The logic ITLL is (a Hilbert-style version of) Hirai's intuitionisitic temporal linear logic [16], and the $\{\mathbf{1}, \to, *, \wedge, !, [l_i]\}$-part of SILL is, roughly speaking, (a Hilbert-style version of) propositional MLL by Kobayashi et al. [32]. [8] If the set S in SILL, SEILL, TSILL and TSEILL is empty, then SILL, SEILL, TSILL and TSEILL are equivalent to the non-spatial parts ILL, EILL, TILL and TEILL, respectively.

3. Semantics

In this section, we introduce the Kripke semantics for TSEILL and prove the soundness theorem. The intuitive meaning of this semantics and some useful interpretations, such as informational and Petri net interpretations, are discussed in Section 8.

Definition 2. A *Kripke frame* for TSEILL is a structure $\langle M, S, \dagger, K, J, F, N, P, \cap, \cdot, \varepsilon, \omega\rangle$ satisfying the following conditions:

1. S is a nonempty set.
2. $\langle M, \cap\rangle$ is a meet-semilattice with the greatest element ω.
3. $\cdot$ is a binary operation on M and $\varepsilon \in M$ such that [9]
   C1: $\varepsilon \cdot\ x = x$ for all $x \in M$,
   C2: $\omega \cdot\ x = \omega$ for all $x \in M$,
   C3: $x \leqslant y$ implies $z \cdot x \leqslant z \cdot y$ for all $x, y, z \in M$ (where the order relation $x \leqslant y$ is defined as $x \cap y = x$),
   C4: $(x \cap y) \cdot z = (x \cdot z) \cap (y \cdot z)$ for all $x, y, z \in M$,
   C5: $(x \cdot y) \cap (x \cdot z) \leqslant x \cdot (y \cap z)$ for all $x, y, z \in M$,
   C6: $x \cdot (y \cdot z) \leqslant (x \cdot y) \cdot z$ for all $x, y, z \in M$,
   C7: $x \cdot \varepsilon \leqslant x$ for all $x \in M$,
   C8: $\omega \leqslant x \cdot \omega$ for all $x \in M$,
   C9: $x \leqslant x \cdot \varepsilon$ for all $x \in M$,
   C10: $(x \cdot y) \cdot z \leqslant x \cdot (y \cdot z)$ for all $x, y, z \in M$,
   C11: $(x \cdot z) \cdot y \leqslant (x \cdot y) \cdot z$ for all $x, y, z \in M$.

[8] Strictly speaking, the axiom schemes of the modal operators $[l_i]$ in MLL are characterized by using structural congruence relation.

[9] For the frame conditions C1–C11, we can give more simple frame conditions than these conditions (i.e., these conditions are redundant), but, for the sake of compatibility of the completeness proofs, we follow the same manner of [17,21].

4. $\dagger$, K, J, F, N and P are unary operations on M such that
   C12: $\dagger(x \cdot y) \leqslant \dagger x \cdot \dagger y$ for all $x, y \in M$,
   C13: $x \leqslant \dagger x$ for all $x \in M$,
   C14: $\dagger\dagger x \leqslant \dagger x$ for all $x \in M$,
   C15: $(x \cdot \dagger y) \cdot \dagger y \leqslant x \cdot \dagger y$ for all $x, y \in M$,
   C16: $x \leqslant x \cdot \dagger y$ for all $x, y \in M$,
   C17: $\dagger\varepsilon \leqslant \varepsilon$,
   C18: $x \leqslant y$ implies $\dagger x \leqslant \dagger y$ for all $x, y \in M$,
   C19: $K(x \cdot y) \leqslant Kx \cdot Ky$ for all $x, y \in M$,
   C20: $x \leqslant Kx$ for all $x \in M$,
   C21: $KKx \leqslant Kx$ for all $x \in M$,
   C22: $J(x \cdot y) \leqslant Kx \cdot Jy$ for all $x, y \in M$,
   C23: $Jx \leqslant x$ for all $x \in M$,
   C24: $Jx \leqslant JJx$ for all $x \in M$,
   C25: $KJx \leqslant Jx$ for all $x \in M$,
   C26: $K\varepsilon \leqslant \varepsilon$,
   C27: $x \leqslant y$ implies $Kx \leqslant Ky$ for all $x, y \in M$,
   C28: $x \leqslant y$ implies $Jx \leqslant Jy$ for all $x, y \in M$,
   C29: $F(x \cdot y) \leqslant Fx \cdot Fy$ for all $x, y \in M$,
   C30: $x \leqslant Fx$ for all $x \in M$,
   C31: $FFx \leqslant Fx$ for all $x \in M$,
   C32: $F\varepsilon \leqslant \varepsilon$,
   C33: $x \leqslant y$ implies $Fx \leqslant Fy$ for all $x, y \in M$,
   C34: $N(x \cdot y) \leqslant Nx \cdot Ny$ for all $x, y \in M$,
   C35: $Nx \leqslant Fx$ for all $x \in M$,
   C36: $N\varepsilon \leqslant \varepsilon$,
   C37: $x \leqslant y$ implies $Nx \leqslant Ny$ for all $x, y \in M$,
   C38: $P(x \cap y) = Px \cap Py$ for all $x \in M$,
   C39: $x \leqslant Px$ for all $x \in M$,
   C40: $PPx \leqslant Px$ for all $x \in M$,
   C41: $\dagger x \leqslant Kx$ for all $x \in M$,
   C42: $Fx \leqslant \dagger x$ for all $x \in M$.

The non-modal part of this frame is the same as that in [40], that is, $\langle M, \cap, \cdot, \varepsilon, \omega \rangle$ is a *semilattice-ordered monoid* for the non-modal intuitionistic linear logic. A semilattice-ordered monoid $\langle M, \cap, \cdot, \varepsilon, \omega \rangle$ for the non-modal intuitionistic linear logic is defined as follows: (1) $\langle M, \cap \rangle$ is a meet-semilattice with the greatest element ω, (2) $\langle M, \cdot, \varepsilon \rangle$ is a *commutative* monoid with the identity ε that satisfies $x \cdot \omega = \omega$ for any $x \in M$, and (3) $z \cdot (x \cap y) \cdot w = (z \cdot x \cdot w) \cap (z \cdot y \cdot w)$ for any $x, y, z, w \in M$.

We remark that the condition C3′: $x \leqslant y$ implies $x \cdot z \leqslant y \cdot z$ for all $x, y, z \in M$ on the frame is derived from the condition C4. We also remark that on the frame, the conditions C18′: $\dagger(x \cap y) \leqslant \dagger x \cap \dagger y$, C27′: $K(x \cap y) \leqslant Kx \cap Ky$, C28′: $J(x \cap y) \leqslant Jx \cap Jy$, C33′: $F(x \cap y) \leqslant Fx \cap Fy$, C37′: $N(x \cap y) \leqslant Nx \cap Ny$ for all $x, y \in M$ are derived from the monotonicity conditions C18, C27, C28, C33, C37 and the fact that $\cap$ is the semilattice operation. These conditions C18′, C27′, C28′, C33′ and C37′ are used to prove the cases $\alpha \equiv [K]\beta$, $\alpha \equiv \langle K \rangle \beta$, $\alpha \equiv [F]\beta$ and $\alpha \equiv [N]\beta$ in Proposition 4.

Definition 3. A *valuation* $\models$ on a Kripke frame $\langle M, S, \dagger, K, J, F, N, P, \cap, \cdot, \varepsilon, \omega \rangle$ is a mapping from the set Ψ of all propositional variables to $2^{M \times S}$ (where $2^{M \times S}$ denotes the power set of the direct product $M \times S$) such that $(x, l), (y, l) \in \models(p)$ iff $(x \cap y, l) \in \models(p)$, that is, $\models(p) := X \times S' \subseteq M \times S$ where X is a filter of M,[10] and S' is nonempty. We will write $(x, l) \models p$ for $(x, l) \in \models(p)$. Each valuation $\models$ can be extended to a mapping from the set Φ

[10] Of course, X is nonempty.

of all formulas to $2^{M\times S}$ by

1. $(x, l) \models \mathbf{1}$ iff $\varepsilon \leqslant x$,
2. $(x, l) \models \top$ for all $x \in M$ and all $l \in S$,
3. $(x, l) \models \bot$ iff $x = \omega$,
4. $(x, l) \models \alpha \to \beta$ iff $x \cdot y \leqslant z$ and $(y, l) \models \alpha$ imply $(z, l) \models \beta$ for all $y, z \in M$,
5. $(x, l) \models \alpha \wedge \beta$ iff $(x, l) \models \alpha$ and $(x, l) \models \beta$,
6. $(x, l) \models \alpha \vee \beta$ iff $(y, l) \models \alpha$ or $(y, l) \models \beta$, and $(z, l) \models \alpha$ or $(z, l) \models \beta$ for some $y, z \in M$ with $y \cap z \leqslant x$,
7. $(x, l) \models \alpha * \beta$ iff $(y, l) \models \alpha$ and $(z, l) \models \beta$ for some $y, z \in M$ with $y \cdot z \leqslant x$,
8. $(x, l) \models\, !\alpha$ iff $(y, l) \models \alpha$ for some y with $\dagger y \leqslant x$,
9. $(x, l) \models [K]\alpha$ iff $(y, l) \models \alpha$ for some y with $Ky \leqslant x$,
10. $(x, l) \models \langle K\rangle\alpha$ iff $(y, l) \models \alpha$ for some y with $Jy \leqslant x$,
11. $(x, l) \models [F]\alpha$ iff $(y, l) \models \alpha$ for some y with $Fy \leqslant x$,
12. $(x, l) \models [N]\alpha$ iff $(y, l) \models \alpha$ for some y with $Ny \leqslant x$,
13. $(x, l) \models [P]\alpha$ iff $(Px, l) \models \alpha$,
14. $(x, l_1) \models [l_2]\alpha$ iff $(x, l_2) \models \alpha$.

We remark that the valuation condition 14 is from [32], and the valuation condition 13 is from [11].

Proposition 4. *Let* $\models$ *be a valuation on a Kripke frame* $\langle M, S, \dagger, K, J, F, N, P, \cap, \cdot, \varepsilon, \omega\rangle$. *Then,* $\models(\alpha)$ *is a mapping from the set* Φ *of all formulas to* $2^{M\times S}$ *such that* $(x, l), (y, l) \in\, \models(\alpha)$ *iff* $(x \cap y, l) \in\, \models(\alpha)$, *that is,* $\models(\alpha) := X \times S' \subseteq M \times S$ *where* X *is a filter of* M, *and* S' *is nonempty.*

Proof. We prove this proposition by induction on the complexity of α. The base step is obvious. For the induction step, we show only the cases for $\alpha \equiv [P]\beta$ and $\alpha \equiv [s]\beta$. The other cases are analogous to those in [17,21].

(*Case* $\alpha \equiv [P]\beta$): Suppose $(x, l), (y, l) \in\, \models([P]\beta)$. Then $(Px, l), (Py, l) \in\, \models(\beta)$ and hence $(Px \cap Py, l) \in\, \models(\beta)$ by the induction hypothesis. By C38, we obtain $(P(x \cap y), l) \in\, \models(\beta)$, and hence $(x \cap y, l) \in\, \models([P]\beta)$. Conversely, suppose $(x \cap y, l) \in\, \models([P]\beta)$. Then $(P(x \cap y), l) \in\, \models(\beta)$, and hence $(Px \cap Py, l) \in\, \models(\beta)$ by C38. We obtain $(Px, l), (Py, l) \in\, \models(\beta)$ by the induction hypothesis. Therefore $(x, l), (y, l) \in\, \models([P]\beta)$.

(*Case* $\alpha \equiv [s]\beta$): Suppose $(x, l), (y, l) \in\, \models([s]\alpha)$. Then $(x, s), (y, s) \in\, \models(\alpha)$, and hence $(x \cap y, s) \in\, \models(\alpha)$ by the induction hypothesis. Thus we have $(x \cap y, l) \in\, \models([s]\alpha)$. Conversely, suppose $(x \cap y, l) \in\, \models([s]\alpha)$. Then $(x \cap y, s) \in\, \models(\alpha)$, and hence $(x, s), (y, s) \in\, \models(\alpha)$ by the induction hypothesis. Therefore $(x, l), (y, l) \in\, \models([s]\alpha)$. □

In the following, we use Proposition 4 implicitly. Using Proposition 4, we can obtain the hereditary condition: $(x, l) \models \alpha$ and $x \leqslant y$ imply $(y, l) \models \alpha$.

Definition 5. A *Kripke model* for TSEILL is a structure $\langle M, S, \dagger, K, J, F, N, P, \cap, \cdot, \varepsilon, \omega, \models\rangle$ such that

1. $\langle M, S, \dagger, K, J, F, N, P, \cap, \cdot, \varepsilon, \omega\rangle$ is a Kripke frame for TSEILL,
2. $\models$ is a valuation on $\langle M, S, \dagger, K, J, F, N, P, \cap, \cdot, \varepsilon, \omega\rangle$.

A formula α is *true* in a Kripke model $\langle M, S, \dagger, K, J, F, N, P, \cap, \cdot, \varepsilon, \omega, \models\rangle$ if $(\varepsilon, l) \models \alpha$ for any $l \in S$, and *valid* in a Kripke frame $\langle M, S, \dagger, K, J, F, N, P, \cap, \cdot, \varepsilon, \omega\rangle$ if it is true for any valuation $\models$ on the Kripke frame.

Theorem 6 (*Soundness*). *Let* C *be the class of all Kripke frames for* TSEILL, $L := \{\gamma \mid \text{TSEILL} \vdash \gamma\}$ *and* $L(C) := \{\gamma \mid \gamma$ *is valid in all frames of* $C\}$. *Then* $L \subseteq L(C)$.

Proof. We prove this theorem by induction on the proof P of γ in TSEILL. Let $\models$ be a valuation on $\langle M, S, \dagger, K, J, F, N, P, \cap, \cdot, \varepsilon, \omega\rangle \in C$. The cases for the ILL-part are similar to the cases for ILL which cases are already proved in [17,21]. We show some cases.

(*Case* $\gamma \equiv [K](\alpha \to \beta) \to \langle K\rangle\alpha \to \langle K\rangle\beta$: E4): We show $(\varepsilon, s) \models [K](\alpha \to \beta) \to \langle K\rangle\alpha \to \langle K\rangle\beta$ for all $s \in S$. Let $x, y \in M$ be such that $(x, s) \models [K](\alpha \to \beta)$ and $(y, s) \models \langle K\rangle\alpha$ for all $s \in S$. We will show $(x \cdot y, s) \models \langle K\rangle\beta$. Then we have the following. There is $x' \in M$ such that (1) $Kx' \leqslant x$ and (2) $(x', s) \models \alpha \to \beta$. There is $y' \in M$ such that (3) $Jy' \leqslant y$ and (4) $(y', s) \models \alpha$. Then we get (5) $(x' \cdot y', s) \models \beta$ by (2) and (4). By (1), (3), C22, C3 and C3′, we obtain $J(x' \cdot y') \leqslant Kx' \cdot Jy' \leqslant x \cdot y$. Thus we have (6) $J(x' \cdot y') \leqslant x \cdot y$. Therefore, $(x \cdot y, s) \models \langle K \rangle \beta$ by (5) and (6).

(*Case* $\gamma \equiv [P]\alpha \wedge [P]\beta \to [P](\alpha \wedge \beta)$: T6): Let $x \in M$ and $s \in S$ be such that $(x, s) \models [P]\alpha \wedge [P]\beta$. Then we have $(x, s) \models [P]\alpha$ and $(x, s) \models [P]\beta$, and hence $(Px, s) \models \alpha$ and $(Px, s) \models \beta$. Thus we obtain $(Px, s) \models \alpha \wedge \beta$. This means $(x, s) \models [P](\alpha \wedge \beta)$. Therefore $(\varepsilon, s) \models [P]\alpha \wedge [P]\beta \to [P](\alpha \wedge \beta)$.

(*Case* $\gamma \equiv ([l]\alpha \to [l]\beta) \to [l](\alpha \to \beta)$: S10): We show $(\varepsilon, s) \models ([l]\alpha \to [l]\beta) \to [l](\alpha \to \beta)$ for all $s \in S$. Let $x \in M$ be such that (*) $(x, s) \models [l]\alpha \to [l]\beta$. We show that $(x, s) \models [l](\alpha \to \beta)$, i.e., $(x, l) \models \alpha \to \beta$. Suppose $(y, l) \models \alpha$, we will show $(x \cdot y, l) \models \beta$. By (*) and $(y, l) \models \alpha$ (i.e., $(y, s) \models [l]\alpha$), we obtain $(x \cdot y, s) \models [l]\beta$, and hence $(x \cdot y, l) \models \beta$.

(*Case* $\gamma \equiv [l][P]\alpha \to [P][l]\alpha$: S23): We show $(\varepsilon, s) \models [l][P]\alpha \to [P][l]\alpha$ for all $s \in S$. Let $x \in M$ be such that $(x, s) \models [l][P]\alpha$, i.e., $(Px, l) \models \alpha$. This means $(Px, s) \models [l]\alpha$. Therefore $(x, s) \models [P][l]\alpha$.

(*Case* $\gamma \equiv [l_1][l_2]\alpha \to [l_2]\alpha$: S25): We show $(\varepsilon, s) \models [l_1][l_2]\alpha \to [l_2]\alpha$ for all $s \in s$ Let $x \in M$ be such that $(x, s) \models [l_1][l_2]\alpha$. Then $(x, s) \models [l_1][l_2]\alpha$ iff $(x, l_1) \models [l_2]\alpha$ iff $(x, l_2) \models \alpha$ iff $(x, s) \models [l_2]\alpha$. □

4. Completeness

In this section, we prove the completeness theorem (with respect to the Kripke semantics) for TSEILL, which is one of the central results of this paper.

Definition 7 (*L-pretheory*). Let $L := \{\alpha \mid \text{TSEILL} \vdash \alpha\}$. An L-pretheory x [11] is a subset of the set Φ of all formulas such that

1. $\top \in x$,
2. if $\alpha \in x$ and $\alpha \to \beta \in L$, then $\beta \in x$,
3. if $\alpha, \beta \in x$, then $\alpha \wedge \beta \in x$.

The following lemma (Lemma 8) is proved in [21]. Lemma 9(11) is proved using Lemma 8.

Lemma 8. *Let M_L be the set of all L-pretheories. For all $\alpha \in \Phi$ and all $x \in M_L$, if $\alpha \in \dagger x$ then $!\alpha \in \dagger x$.*

Lemma 9. *Let M_L be the set of all L-pretheories. Then*

1. *if $x, y \in M_L$, then*
   $x \cap y \in M_L$,
   $x \cdot y := \{\beta \mid \exists \alpha \in y\ (\alpha \to \beta \in x)\} \in M_L$;
2. *if $x \in M_L$, then*
   $Jx := \{\beta \mid \exists \alpha \in x\ (\langle K \rangle \alpha \to \beta \in L)\} \in M_L$;
3. *if $x \in M_L$, then*
   $\dagger x := \{\beta \mid \exists \alpha \in x\ (!\alpha \to \beta \in L)\} \in M_L$,
   $Kx := \{\beta \mid \exists \alpha \in x([K]\alpha \to \beta \in L)\} \in M_L$,
   $Fx := \{\beta \mid \exists \alpha \in x([F]\alpha \to \beta \in L)\} \in M_L$,
   $Nx := \{\beta \mid \exists \alpha \in x([N]\alpha \to \beta \in L)\} \in M_L$;
4. *if $x \in M_L$, then*
   $Px := \{\beta \mid [P]\beta \in x\} \in M_L$;
5. $L \cdot \{\alpha\} \in M_L$;
6. *if $x, y, z \in M_L$, then $x \subseteq y$ implies $z \cdot x \subseteq z \cdot y$;*
7. *if $x, y, z \in M_L$, then $L \cdot x = x$, $(x \cap y) \cdot z = (x \cdot z) \cap (y \cdot z)$, $(x \cdot y) \cap (x \cdot z) \subseteq x \cdot (y \cap z)$, $x \cdot L \subseteq x$, $\Phi \subseteq x \cdot \Phi$, $\Phi \cdot x = \Phi$, $x \subseteq x \cdot L$, $(x \cdot y) \cdot z \subseteq x \cdot (y \cdot z)$, $x \cdot (y \cdot z) \subseteq (x \cdot y) \cdot z$ and $(x \cdot z) \cdot y \subseteq (x \cdot y) \cdot z$;*

[11] The notion of L-pretheory is from [40]. This notion can be viewed as a weak version of the notion of prime theory or saturated set used in the classical logic. Since the underlying substructural logics have no distributivity between $\vee$ and $\wedge$, the notion of prime theory is not adapted, and the notion of L-pretheory is used to prove the completeness theorems.

8. *if* $x, y \in M_L$, *then*
   $x \subseteq y$ *implies* $\dagger x \subseteq \dagger y$,
   $x \subseteq y$ *implies* $Kx \subseteq Ky$,
   $x \subseteq y$ *implies* $Jx \subseteq Jy$,
   $x \subseteq y$ *implies* $Fx \subseteq Fy$,
   $x \subseteq y$ *implies* $Nx \subseteq Ny$;
9. $\dagger L \subseteq L$, $KL \subseteq L$, $FL \subseteq L$ *and* $NL \subseteq L$;
10. *if* $x, y \in M_L$, *then* $\dagger(x \cdot y) \subseteq \dagger x \cdot \dagger y$, $x \subseteq \dagger x$, $\dagger\dagger x \subseteq \dagger x$ *and* $x \subseteq x \cdot \dagger y$;
11. *if* $x, y \in M_L$, *then* $(x \cdot \dagger y) \cdot \dagger y \subseteq x \cdot \dagger y$;
12. *if* $x, y \in M_L$, *then* $K(x \cdot y) \subseteq Kx \cdot Ky$, $x \subseteq Kx$ *and* $KKx \subseteq Kx$;
13. *if* $x, y \in M_L$, *then* $J(x \cdot y) \subseteq Kx \cdot Jy$, $Jx \subseteq x$, $Jx \subseteq JJx$ *and* $KJx \subseteq Jx$;
14. *if* $x, y \in M_L$, *then* $F(x \cdot y) \subseteq Fx \cdot Fy$, $x \subseteq Fx$ *and* $FFx \subseteq Fx$;
15. *if* $x, y \in M_L$, *then* $N(x \cdot y) \subseteq Nx \cdot Ny$ *and* $Nx \subseteq Fx$;
16. *if* $x, y \in M_L$, *then* $P(x \cap y) = Px \cap Py$, $x \subseteq Px$ *and* $PPx \subseteq Px$;
17. *if* $x \in M_L$, *then* $\dagger x \subseteq Kx$ *and* $Fx \subseteq \dagger x$.

Proof. The cases (1), (5), (6) and (7) are already proved in [17]. The cases (3), (8), (9) and (10)–(15) are analogous to the cases for ILL-modal part, which cases are already proved in [21]. Here we only show the cases (2), (4), (13), (16) and (17).

(2): Suppose $x \in M_L$. We show $Jx \in M_L$. First we show $\top \in Jx$. By A1: $\alpha \to \top$ and $x \in M_L$, we get that there is $\top \in x$ such that $\langle K\rangle\top \to \top \in L$. Second we show that $\alpha \in Jx$ and $\alpha \to \beta \in L$ imply $\beta \in Jx$. Suppose (1) $\alpha \in Jx$ and (2) $\alpha \to \beta \in L$. By (1), we have that there is $\gamma \in x$ such that (3) $\langle K\rangle\gamma \to \alpha \in L$. By (2), (3) and (cut), we have $\langle K\rangle\gamma \to \beta \in L$. Therefore we get that there is $\gamma \in x$ such that $\langle K\rangle\gamma \to \beta \in L$. This means $\beta \in Jx$. Third we show that $\alpha, \beta \in Jx$ implies $\alpha \wedge \beta \in Jx$. Suppose $\alpha, \beta \in Jx$. We have that (1) there is $\gamma \in x$ such that $\langle K\rangle\gamma \to \alpha \in L$, and (2) there is $\delta \in x$ such that $\langle K\rangle\delta \to \beta \in L$. Then we get (3) $\gamma \wedge \delta \in x$ by $x \in M_L$. By using the facts (*) $\langle K\rangle(\gamma \wedge \delta) \to \langle K\rangle\gamma \wedge \langle K\rangle\delta \in L$ and (**) $\langle K\rangle\gamma \wedge \langle K\rangle\delta \to \alpha \wedge \beta \in L$ (these are proved later), and (cut), we get (4) $\langle K\rangle(\gamma \wedge \delta) \to \alpha \wedge \beta \in L$. By (3) and (4), we have that there is $\gamma \wedge \delta \in x$ such that $\langle K\rangle(\gamma \wedge \delta) \to \alpha \wedge \beta \in L$. This means $\alpha \wedge \beta \in Jx$. We show the remained proofs of (*) and (**). First we show (*) by using E5, (Kleft), A4, A5, ($\wedge$right) and (cut)

$$\dfrac{\dfrac{\dfrac{\gamma \wedge \delta \to \gamma \quad \gamma \to \langle K\rangle\gamma}{\gamma \wedge \delta \to \langle K\rangle\gamma}}{\langle K\rangle(\gamma \wedge \delta) \to \langle K\rangle\gamma} \quad \dfrac{\dfrac{\gamma \wedge \delta \to \delta \quad \delta \to \langle K\rangle\delta}{\gamma \wedge \delta \to \langle K\rangle\delta}}{\langle K\rangle(\gamma \wedge \delta) \to \langle K\rangle\delta}}{\langle K\rangle(\gamma \wedge \delta) \to \langle K\rangle\gamma \wedge \langle K\rangle\delta}.$$

Next, (**) is proved using (1) $\langle K\rangle\gamma \to \alpha \in L$, (2) $\langle K\rangle\delta \to \beta \in L$, A4 and A5, (cut) and ($\wedge$right).

(4): First we show $\top \in Px$, i.e., $[P]\top \in x$. By T7, we have $\top \to [P]\top \in L$. Then we obtain $[P]\top \in x$ by $\top \in x \in M_L$ and $\top \to [P]\top \in L$. Second we show that $\alpha \in Px$ and $\alpha \to \beta \in L$ imply $\beta \in Px$. Suppose $\alpha \in Px$ and $\alpha \to \beta \in L$. Then we obtain $[P]\alpha \in x$ and $[P]\alpha \to [P]\beta \in L$ (by Pregu), and hence $[P]\beta \in x$ since $x \in M_L$. Therefore $\beta \in Px$. Third we show that $\alpha, \beta \in Px$ implies $\alpha \wedge \beta \in Px$. Suppose $\alpha, \beta \in Px$. Then we have $[P]\alpha \in x$ and $[P]\beta \in x$. Thus we obtain $[P]\alpha \wedge [P]\beta \in x$ since $x \in M_L$. By $[P]\alpha \wedge [P]\beta \to [P](\alpha \wedge \beta) \in L$ (T6), we have $[P](\alpha \wedge \beta) \in x$ since $x \in M_L$. Therefore $\alpha \wedge \beta \in Px$.

(13): We only show $J(x \cdot y) \subseteq Kx \cdot Jy$ if $x, y \in M_L$. Suppose $\gamma \in J(x \cdot y)$. Then there is $\alpha \in x \cdot y$ such that (1) $\beta \in y$, (2) $\beta \to \alpha \in x$ and (3) $\langle K\rangle\alpha \to \gamma \in L$. We will show $\gamma \in Kx \cdot Jy$, that is, there is $\delta \in Jy$ such that $\delta \to \gamma \in Kx$. Hence we will show that there is δ such that (*) there is $\omega \in y$ such that $\langle K\rangle\omega \to \delta \in L$, and (**) there is $\pi \in x$ such that $[K]\pi \to \delta \to \gamma \in L$. First we show (*). We take β and $\langle K\rangle\beta$ for ω and δ respectively. Then we have that $\beta \in y$ (by (1)) and $\langle K\rangle\beta \to \langle K\rangle\beta \in L$. Next we show (**). We take moreover $\beta \to \alpha$ for π. Then we have $\beta \to \alpha \in x$ by (2). By using (3), E4, (pref) and (cut), we have $[K](\beta \to \alpha) \to \langle K\rangle\beta \to \gamma \in L$.

(16): We only show $P(x \cap y) = Px \cap Py$. First we show $P(x \cap y) \subseteq Px \cap Py$. Suppose $\gamma \in P(x \cap y)$. Then $[P]\gamma \in x \cap y$ and hence $[P]\gamma \in x$ and $[P]\gamma \in y$. Thus $\gamma \in Px$ and $\gamma \in Py$, i.e., $\gamma \in Px \cap Py$. The converse is obvious.

(17): First we show $\dagger x \subseteq Kx$. Suppose $\gamma \in \dagger x$. Then there is $\alpha \in x$ such that $!\alpha \to \gamma \in L$. By $[K]\alpha \to !\alpha \in L$ (R1), $!\alpha \to \gamma \in L$ and (cut), we obtain $[K]\alpha \to \gamma \in L$. Thus we have that there is $\alpha \in x$ such that $[K]\alpha \to \gamma \in L$. This means $\gamma \in Kx$. The case $Fx \subseteq \dagger x$ is proved in a similar way by using R2. □

By using Lemma 9, we can conclude the following.

Proposition 10. *$F_L := \langle M_L, S, \dagger, K, J, F, N, P, \cap, \cdot, L, \Phi\rangle$ is a Kripke frame.*

Lemma 11. *Let $x \in M_L$. Then*

1. $\alpha \in x$ *iff* $L \cdot \{\alpha\} \subseteq x$,
2. $(L \cdot \{\alpha\}) \cap (L \cdot \{\beta\}) \subseteq L \cdot \{\alpha \vee \beta\}$,
3. $(L \cdot \{\alpha\}) \cdot (L \cdot \{\beta\}) \subseteq L \cdot \{\alpha * \beta\}$,
4. $\dagger(L \cdot \{\alpha\}) \subseteq L \cdot \{!\alpha\}$,
5. $K(L \cdot \{\alpha\}) \subseteq L \cdot \{[K]\alpha\}$,
6. $J(L \cdot \{\alpha\}) \subseteq L \cdot \{\langle K\rangle\alpha\}$,
7. $F(L \cdot \{\alpha\}) \subseteq L \cdot \{[F]\alpha\}$,
8. $N(L \cdot \{\alpha\}) \subseteq L \cdot \{[N]\alpha\}$.

Proof. We only show 6. The cases for 4, 5, 7 and 8 are proved in the same way of 6. The case 4 is proved in [21], and the cases 1–3 are also proved in [17].

(6): Suppose $\gamma \in J(L \cdot \{\alpha\})$. Then we have that there is $\beta \in L \cdot \{\alpha\}$ such that $\langle K\rangle\beta \to \gamma \in L$, and hence there is β such that (1) $\alpha \to \beta \in L$ and (2) $\langle K\rangle\beta \to \gamma \in L$. By (1), (2), ($K$ness), (mp), (cut) and E4, we get $\langle K\rangle\alpha \to \gamma \in L$, and hence $\gamma \in L \cdot \{\langle K\rangle\alpha\}$. □

By using Lemma 11, we can show the following.

Lemma 12 (*Key lemma*). *Let Ψ be the set of all propositional variables, and $\models_L$ be a mapping from Ψ to $2^{M_L \times S}$ defined by $\models_L (p) := \{(x, l) \in M_L \times S \mid [l]p \in x\}$. Then, $\models_L$ can be extended to a mapping from the set Φ of all formulas to $2^{M_L \times S}$, that is, we have the following: for any $\alpha \in \Phi$, any $l \in S$ and any $x \in M_L$, $[l]\alpha \in x$ iff $(x, l) \models_L \alpha$.*

Proof. We prove this lemma by induction on the complexity of α.

- The base step: Straightforward.
- The induction step:

(*Case* $\alpha \equiv [s]\beta$): Suppose $[l][s]\beta \in x$. Then we obtain $[s]\beta \in x$ by $[l][s]\beta \to [s]\beta \in L$ (S25) and $x \in M_L$. By the induction hypothesis, we have $(x, s) \models_L \beta$, and hence $(x, l) \models_L [s]\beta$. Conversely, suppose $(x, l) \models_L [s]\beta$. Then we have $(x, s) \models_L \beta$, and hence $[s]\beta \in x$ by the induction hypothesis. We obtain $[l][s]\beta \in x$ by $[s]\beta \to [l][s]\beta \in L$ (S26), $[s]\beta \in x$ and $x \in M_L$.

(*Case* $\alpha \equiv [P]\beta$): Suppose $[l][P]\beta \in x$. Then we have $[P][l]\beta \in x$ by $[l][P]\beta \to [P][l]\beta \in L$ (S23) and $x \in M_L$. This means $[l]\beta \in Px$, and hence $(Px, l) \models_L \beta$ by the induction hypothesis. Therefore $(x, l) \models [P]\beta$. Conversely, suppose $(x, l) \models_L [P]\beta$, i.e., $(Px, l) \models \beta$. Then we obtain $[l]\beta \in Px$ by the induction hypothesis. Thus $[P][l]\beta \in x$, and hence $[l][P]\beta \in x$ by $[P][l]\beta \to [l][P]\beta \in L$ (S24) and $x \in M_L$.

(*Case* $\alpha \equiv \langle K\rangle\beta$): Suppose $[l]\langle K\rangle\beta \in x$. Then we have $\langle K\rangle[l]\beta \in x$ by $[l]\langle K\rangle\beta \to \langle K\rangle[l]\beta \in L$ (S17) and $x \in M_L$. Now we have $L \cdot \{\langle K\rangle[l]\beta\} \subseteq x$ by Lemma 11(1), $\langle K\rangle[l]\beta \in x$ and $x \in M_L$. By Lemma 11(6), we have $J(L \cdot \{[l]\beta\}) \subseteq L \cdot \{\langle K\rangle[l]\beta\} \subseteq x$. Thus we obtain (*) $J(L \cdot \{[l]\beta\}) \subseteq x$ and $L \cdot \{[l]\beta\} \in M_L$ (by Lemma 9(5)). On the other hand, we have $[l]\beta \in L \cdot \{[l]\beta\}$, and hence (**) $(L \cdot \{[l]\beta\}, l) \models_L \beta$ by the induction hypothesis. By (*) and (**), we obtain $(x, l) \models_L \langle K\rangle\beta$. Conversely, suppose $(x, l) \models_L \langle K\rangle\beta$. Then there is $y \in M_L$ such that (1) $Jy \subseteq x$ and (2) $(y, l) \models_L \beta$. Applying the induction hypothesis to (2), we obtain (3) $[l]\beta \in y$. By $\langle K\rangle[l]\beta \to \langle K\rangle[l]\beta \in L$ (S18) and (3), we obtain (4) $\langle K\rangle[l]\beta \in Jy$. By (4) and (1), we have (5) $\langle K\rangle[l]\beta \in x$. By (5), $[\langle K\rangle[l]\beta \to [l]\langle K\rangle\beta \in L$ (S18) and $x \in M_L$, we obtain $[l]\langle K\rangle\beta \in x$.

(*Cases* $\alpha \equiv !\beta$, $\alpha \equiv [K]\beta$, $\alpha \equiv [F]\beta$ and $\alpha \equiv [N]\beta$): Similar to the case $\alpha \equiv \langle K\rangle\beta$.

(*Case* $\alpha \equiv \mathbf{1}$): Suppose $[l]\mathbf{1} \in x$. Then we have $\mathbf{1} \in x$ by $[l]\mathbf{1} \in x$, $[l]\mathbf{1} \to \mathbf{1} \in L$ (S1) and $x \in M_L$. We will show $(x, l) \models_L \mathbf{1}$, i.e., $L \subseteq x$. Suppose $\gamma \in L$. Then $\mathbf{1} \to \gamma \in L$ by (ness). By $\mathbf{1} \in x$, $\mathbf{1} \to \gamma \in L$ and $x \in M_L$, we obtain $\gamma \in x$. Conversely, suppose $(x, l) \models_L \mathbf{1}$, i.e., $L \subseteq x$. We have $\mathbf{1} \in L$ by A3 and A12. Then we have $\mathbf{1} \in x$ by $L \subseteq x$. By $\mathbf{1} \in x$, $\mathbf{1} \to [l]\mathbf{1} \in L$ (S2) and $x \in M_L$, we obtain $[l]\mathbf{1} \in x$.

(*Case* $\alpha \equiv \top$): Obvious. By using S3.

(*Case* $\alpha \equiv \bot$): Obvious. By using S4 and A2.

(*Case* $\alpha \equiv \beta \wedge \gamma$): Suppose $[l](\beta \wedge \gamma) \in x$. Then we obtain $[l]\beta \wedge [l]\gamma \in x$ by $[l](\beta \wedge \gamma) \to [l]\beta \wedge [l]\gamma \in L$ (S5) and $x \in M_L$. By $[l]\beta \wedge [l]\gamma \in x$, $[l]\beta \wedge [l]\gamma \to [l]\beta \in L$ (A4) and $x \in M_L$, we obtain (*) $[l]\beta \in x$. We also obtain (**) $[l]\gamma \in x$ in a similar way. By the induction hypothesis, (*) and (**), we obtain $(x, l) \vDash_L \beta$ and $(x, l) \vDash_L \gamma$. Therefore $(x, l) \vDash_L \beta \wedge \gamma$. Conversely, suppose $(x, l) \vDash_L \beta \wedge \gamma$, i.e., $(x, l) \vDash_L \beta$ and $(x, l) \vDash_L \gamma$. Then we have $[l]\beta \in x$ and $[l]\gamma \in x$ by the induction hypothesis. Thus we have $[l]\beta \wedge [l]\gamma \in x$ by $x \in M_L$. By $[l]\beta \wedge [l]\gamma \in x$, $[l]\beta \wedge [l]\gamma \to [l](\beta \wedge \gamma) \in L$ (S6) and $x \in M_L$, we obtain $[l](\beta \wedge \gamma) \in x$.

(*Case* $\alpha \equiv \beta \vee \gamma$): Suppose $[l](\beta \vee \gamma) \in x$. Then we have $[l]\beta \vee [l]\gamma \in x$ by $[l](\beta \vee \gamma) \to [l]\beta \vee [l]\gamma \in L$ (S7) and $x \in M_L$. By Lemma 11(1), we obtain $L \cdot \{[l]\beta \vee [l]\gamma\} \subseteq x$, and hence (*) $(L \cdot \{[l]\beta\}) \cap (L \cdot \{[l]\gamma\}) \subseteq x$ by Lemma 11(2). Of course we have $L \cdot \{[l]\beta\}$, $L \cdot \{l]\gamma\} \in M_L$ by Lemma 9(5). Moreover we have $[l]\beta \in L \cdot \{[l]\beta\}$ and $[l]\gamma \in L \cdot \{[l]\gamma\}$. Then, by the induction hypothesis, we obtain (**) $(L \cdot \{[l]\beta\}, l) \vDash_L \beta$ and $(L \cdot \{[l]\gamma\}, l) \vDash_L \gamma$. By (*) and (**), we have the claim that $(x, l) \vDash_L \beta \vee \gamma$. Conversely, suppose $(x, l) \vDash_L \beta \vee \gamma$, i.e., there are $y, z \in M_L$ such that (*) $y \cap z \subseteq x$, (**) $(y, l) \vDash_L \beta$ or $(y, l) \vDash_L \gamma$, and (***) $(z, l) \vDash_L \beta$ or $(z, l) \vDash_L \gamma$. Applying the induction hypothesis to (**) and (***), we have $[l]\beta \in y$ or $[l]\gamma \in y$, and $[l]\beta \in z$ or $[l]\gamma \in z$. Since $[l]\beta \to [l]\beta \vee [l]\gamma \in L$ (A7), $[l]\gamma \to [l]\beta \vee [l]\gamma \in L$ (A8) and $y, z \in M_L$, we have $[l]\beta \vee [l]\gamma \in y$ and $[l]\beta \vee [l]\gamma \in z$, and hence $[l]\beta \vee [l]\gamma \in y \cap z$. Thus we have $[l]\beta \vee [l]\gamma \in x$ by (*), and $[l]\beta \vee [l]\gamma \in y \cap z$. Therefore we obtain $[l](\beta \vee \gamma) \in x$ by $[l]\beta \vee [l]\gamma \in x$ and $[l]\beta \vee [l]\gamma \to [l](\beta \vee \gamma) \in L$ (S8).

(*Case* $\alpha \equiv \beta \to \gamma$): Suppose $[l](\beta \to \gamma) \in x$, $x \cdot y \subseteq z$ and $(y, l) \vDash_L \beta$ for any $y, z \in M_L$. Then we have (*) $[l]\beta \to [l]\gamma \in x$ by $[l](\beta \to \gamma) \to [l]\beta \to [l]\gamma \in L$ (S9) and $x \in M_L$. Applying the induction hypothesis to $(y, l) \vDash_L \beta$, we have (**) $[l]\beta \in y$. By (*),(**) and $x \cdot y \subseteq z$, we have $[l]\gamma \in x \cdot y \subseteq z$, and hence $[l]\gamma \in z$. Therefore $(z, l) \vDash_L \gamma$ by the induction hypothesis. Conversely, suppose $(x, l) \vDash_L \beta \to \gamma$. We have $(L \cdot \{[l]\beta\}, l) \vDash_L \beta$ by applying the induction hypothesis to $[l]\beta \in L \cdot \{[l]\beta\}$. Then we obtain $(x \cdot (L \cdot \{[l]\beta\}), l) \vDash_L \gamma$ by the hypothesis, and hence $[l]\gamma \in x \cdot (L \cdot \{[l]\beta\})$ by the induction hypothesis. Thus there is $\delta \in L \cdot \{[l]\beta\}$ such that $\delta \to [l]\gamma \in x$. Therefore we have $[l]\beta \to \delta \in L$, and hence $(\delta \to [l]\gamma) \to ([l]\beta \to [l]\gamma) \in L$ by (suff). Then we have $[l]\beta \to [l]\gamma \in x$ by $\delta \to [l]\gamma \in x$, $(\delta \to [l]\gamma) \to ([l]\beta \to [l]\gamma) \in L$ and $x \in M_L$. Therefore we obtain $[l](\beta \to \gamma) \in x$ by $[l]\beta \to [l]\gamma \in x$, $([l]\beta \to [l]\gamma) \to [l](\beta \to \gamma) \in L$ (S10) and $x \in M_L$.

(*Case* $\alpha \equiv \beta * \gamma$): Suppose $[l](\beta * \gamma) \in x$. Then we have $[l]\beta * [l]\gamma \in x$ by $[l](\beta * \gamma) \to [l]\beta * [l]\gamma \in L$ (S11) and $x \in M_L$. By Lemma 11(1), (3), we obtain $(L \cdot \{[l]\beta\}) \cdot (L \cdot \{[l]\gamma\}) \subseteq L \cdot \{[l]\beta * [l]\gamma\} \subseteq x$. Also we have $L \cdot \{[l]\beta\}$, $L \cdot \{[l]\gamma\} \in M_L$ by Lemma 9(5). We have $(L \cdot \{[l]\beta\}, l) \vDash_L \beta$ by applying the induction hypothesis to $[l]\beta \in L \cdot \{[l]\beta\}$. Similarly we have $(L \cdot \{[l]\gamma\}, l) \vDash_L \gamma$. Therefore $(x, l) \vDash_L \beta * \gamma$. Conversely, suppose $(x, l) \vDash_L \beta * \gamma$, i.e., there are $y, z \in M_L$ such that $y \cdot z \subseteq x$, $(y, l) \vDash_L \beta$ and $(z, l) \vDash_L \gamma$. By the induction hypothesis, we have $[l]\beta \in y$ and $[l]\gamma \in z$. Next we show $[l]\beta * [l]\gamma \in y \cdot z$. We have $[l]\gamma \to [l]\beta * [l]\gamma \in y$ by $[l]\beta \to [l]\gamma \to [l]\beta * [l]\gamma \in L$ (A9), $[l]\beta \in y$ and $y \in M_L$. By $[l]\gamma \in z$ and $[l]\gamma \to [l]\beta * [l]\gamma \in y$, we have $[l]\beta * [l]\gamma \in y \cdot z$. Therefore $[l]\beta * [l]\gamma \in y \cdot z \subseteq x$, and hence $[l]\beta * [l]\gamma \in x$. By $[l]\beta * [l]\gamma \to [l](\beta * \gamma) \in L$ (S12), $[l]\beta * [l]\gamma \in x$ and $x \in M_L$, we obtain $[l](\beta * \gamma) \in x$. □

By using Lemma 12, we can prove the following.

Lemma 13. $\mathbf{M}_L := \langle M_L, S, \dagger, K, J, F, N, P, \cap, \cdot, L, \Phi, \vDash_L \rangle$ *is a Kripke model such that* $\alpha \in L$ *iff* α *is true in* $\mathbf{M}_L$.

Proof. We can easily show that $\vDash_L$ is a valuation on a Kripke frame F_L. Thus $\mathbf{M}_L$ is a Kripke model. It remains to show that, for any $\alpha \in \Phi$, $\alpha \in L$ iff $(L, s) \vDash_L \alpha$ for any $s \in S$. We will prove this claim in the following. In Lemma 12, taking L for x, we have that

$\forall \alpha \in \Phi\ \forall s \in S\ ([s]\alpha \in L \text{ iff } (L, s) \vDash \alpha)$.

This implies the fact that

$\forall \alpha \in \Phi\ (\forall s \in S([s]\alpha \in L) \text{ iff } \forall s \in S((L, s) \vDash \alpha))$.

In this fact, we have

$\forall s \in S([s]\alpha \in L) \text{ iff } \alpha \in L$,

by the rules (space1) and (space2). Therefore we obtain the claim. □

By using Lemma 13, we can prove the following theorem which is a main result of this paper.

Theorem 14 (*Completeness*). *Let C be the class of all Kripke frames for* TSEILL, $L(C) := \{\alpha \mid \alpha$ *is valid in all frames of* $C\}$ *and* $L := \{\alpha \mid$ TSEILL $\vdash \alpha\}$. *Then* $\mathrm{L} = \mathrm{L}(C)$.

We can also show the completeness theorems (w.r.t. the corresponding appropriate Kripke models [12]) for EILL, TILL, ITLL, SILL, SEILL, TEILL and TSILL, because the proof of the completeness theorem for TSEILL includes those for the logics. Moreover, we remark that we can introduce the logic TSEIAL (*temporal spatial epistemic intuitionistic affine logic*) which is obtained from TSEILL by adding the weakening axiom scheme: $\alpha \to \beta \to \alpha$, and can show the completeness theorem (w.r.t. the corresponding Kripke model) for TSEIAL in a similar way.

5. Normal modal affine logics

In this section, we introduce the logic BIAL and its normal extensions by Lemmon–Scott axiom schemes, and prove the completeness theorems together with a natural correspondence theory.

It is known that the minimal normal modal logic K can be extended by adding the Lemmon–Scott axiom schemes (or also called Sahlqvist formulas) which preserve the completeness theorems with the one-to-one correspondences between axiom schemes and frame conditions on Kripke semantics, i.e. these extended logics are uniformly characterized by such correspondences. This correspondence result is called the *correspondence theory*, and these logics are also called the *normal modal logics* in general. These normal modal logics include the modal logic S5, which is known as an epistemic logic, the modal logic S4, which is known as an epistemic or temporal logic, the modal logic K4, which is known as a logic of belief, and the modal logic KD, which is known as a deontic logic. KD and K4 are obtained from K by adding the axiom scheme D: $\Box\alpha \to \Diamond\alpha$ and the axiom scheme 4: $\Box\alpha \to \Box\Box\alpha$, respectively. S4 is obtained from K4 by adding the axiom scheme T: $\Box\alpha \to \alpha$, and S5 is obtained from S4 by adding the axiom scheme 5: $\Diamond\alpha \to \Box\Diamond\alpha$. These facts mean that the modal operators $\Box$ and $\Diamond$ can read as various aspects, such as epistemic and temporal, by assuming the axiom schemes D, 4, T and/or 5.

In the following discussion, we propose an affine analogue of the correspondence theory discussed above.

Prior to the discussion, we reconsider our language used in this section. We introduce new modal operators $\Box$ and $\Diamond$ instead of !, $[K]$, $\langle K\rangle$, $[F]$, $[N]$ and $[P]$. We remark that, in BIAL, we can remove the multiplicative constant **1** as the language, because the constants $\top$ and **1** are logically equivalent in this logic. Thus the axiom schemes A12: $(\mathbf{1} \to \alpha) \to \alpha$ and A14: $\alpha \to \mathbf{1} \to \alpha$ are redundant. Moreover, the axiom scheme A13: $\alpha \to \bot \to \beta$ is also redundant, because A13 is provable using the weakening axiom scheme and A2: $\bot \to \alpha$. In the following discussion, we can also add the linear modality ! (here the axiom scheme $\alpha \to !\beta \to \alpha$ is redundant) in a similar way, but for the sake of brevity, we omit ! and focus the new modalities $\Box$ and $\Diamond$. Moreover, we remark that we can remove the spatial modal operators $[l_i]$, but for the sake of compatibility of the completeness proof, we consider here the language including $[l_i]$.

Definition 15 (*The logic BIAL*). The axiom schemes and inference rules for the logic BIAL are as follows.

Non-modal part:

The non-modal part of TSEILL (i.e., the non-modal part of ILL) by adding A17: $\alpha \to \beta \to \alpha$,

$\Box\Diamond$-modal part:

B1: $\Box\alpha \wedge \Box\beta \to \Box(\alpha \wedge \beta)$,
B2: $\Box(\alpha \to \beta) \to \Box\alpha \to \Box\beta$,
B3: $\Diamond\alpha \wedge \Diamond\beta \to \Diamond(\alpha \wedge \beta)$,
B4: $\Diamond(\alpha \to \beta) \to \Diamond\alpha \to \Diamond\beta$,
B5: $\Box(\alpha \to \beta) \to \Diamond\alpha \to \Diamond\beta$,

$$\frac{\alpha}{\Box\alpha}\ (\Box\text{ness}), \quad \frac{\alpha}{\Diamond\alpha}\ (\Diamond\text{ness}).$$

Spatial-modal part:

S3–S12, S25, S26, (space1) and (space2) in Definition 1,
S27: $[l]\Box\alpha \to \Box[l]\alpha$,
S28: $\Box[l]\alpha \to [l]\Box\alpha$,
S29: $[l]\Diamond\alpha \to \Diamond[l]\alpha$,
S30: $\Diamond[l]\alpha \to [l]\Diamond\alpha$
for any $l \in S$.

[12] These Kripke models are obtained from the model for TSEILL by deleting the appropriate frame and valuation conditions.

We remark that the rules

$$\frac{\alpha \rightarrow \beta}{\Box\alpha \rightarrow \Box\beta}\ (\Box\text{regu}), \quad \frac{\alpha \rightarrow \beta}{\Diamond\alpha \rightarrow \Diamond\beta}\ (\Diamond\text{regu})$$

are derivable in BIAL.

We denote $\Box^n\alpha$ and $\Diamond^n\alpha$ $(1 \leqslant n)$ for the formulas $\overbrace{\Box\cdots\Box}^{n}\alpha$ and $\overbrace{\Diamond\cdots\Diamond}^{n}\alpha$, respectively.

Definition 16 (*Axiom schemes*). Let k, l, m and n be fixed positive integers. We define the following axiom schemes:

M1: $\Box\alpha \rightarrow \alpha$,
M2: $\alpha \rightarrow \Diamond\alpha$,
M3: $\Box\alpha \rightarrow \Box\Box\alpha$,
M4: $\Diamond\Diamond\alpha \rightarrow \Diamond\alpha$,
M5: $\Diamond\alpha \rightarrow \Box\Diamond\alpha$,
M6: $\Diamond\Box\alpha \rightarrow \Box\alpha$,
M7: $\alpha \rightarrow \Box\Diamond\alpha$,
M8: $\Diamond\Box\alpha \rightarrow \alpha$,
M9: $\Box\alpha \rightarrow \Diamond\alpha$,
M10: $\Box\Diamond\alpha \rightarrow \Diamond\Box\alpha$,
M11: $\Diamond\Box\alpha \rightarrow \Box\Diamond\alpha$,
M12: $\Diamond^k\alpha \rightarrow \Box^m\Diamond^n\alpha$,
M13: $\Box^k\Diamond^l\alpha \rightarrow \Diamond^m\Box^n\alpha$,
M14: $\Diamond^k\Box^l\alpha \rightarrow \Box^m\Diamond^n\alpha$.

We call the logics obtained from BIAL by adding arbitrary combinations of the axiom schemes introduced above *normal modal affine logics*.

We remark that the axiom schemes M1–14 are well-known axiom schemes which include Lemmon–Scott axiom schemes. Thus, the normal modal affine logics can be viewed as affine logic versions of the standard normal modal logics, and these logics are also characterized uniformly as a general theory of correspondence (see Proposition 20). From this fact, it can be seen that BIAL has the role of the minimal normal modal logic K. We also remark that the general setting proposed is not compatible to that of the linear logic base in some purely technical reasons.

Next we give Kripke models for normal modal affine logics.

Definition 17. A *Kripke frame* for normal modal affine logics is a structure $\langle M, S, \sharp, \flat, \cap, \cdot, \varepsilon, \omega\rangle$ satisfying the following conditions:

1. S is a nonempty set,
2. $\langle M, \cap\rangle$ is a meet-semilattice with the greatest element ω,
3. $\cdot$ is a binary operation on M and $\varepsilon \in M$ such that
   C1–C11 in Definition 2,
   C43: $x \leqslant x \cdot y$ for all $x, y \in M$,
4. $\sharp$ and $\flat$ are unary operations on M such that
   C44: $\sharp(x \cap y) = \sharp x \cap \sharp y$ for all $x, y \in M$,
   C45: $\sharp x \cdot \sharp y \leqslant \sharp(x \cdot y)$ for all $x, y \in M$,
   C46: $\varepsilon \leqslant \sharp\varepsilon$,
   C47: $\flat(x \cap y) = \flat x \cap \flat y$ for all $x, y \in M$,
   C48: $\flat x \cdot \flat y \leqslant \flat(x \cdot y)$ for all $x, y \in M$,
   C49: $\varepsilon \leqslant \flat\varepsilon$,
   C50: $\sharp x \cdot \flat y \leqslant \flat(x \cdot y)$ for all $x, y \in M$.

Definition 18. A *valuation* $\models$ on a Kripke frame $\langle M, S, \sharp, \flat, \cap, \cdot, \varepsilon, \omega\rangle$ is a mapping from the set Ψ of all propositional variables to $2^{M \times S}$ such that $(x, l), (y, l) \in\ \models(p)$ iff $(x \cap y, l) \in\ \models(p)$, that is, $\models(p) := X \times S' \subseteq M \times S$ where

X is a filter of M, and S' is nonempty. We write $(x,l) \models p$ for $(x,l) \in \models(p)$. Each valuation $\models$ can be extended to a mapping from the set Φ of all formulas to $2^{M\times S}$ by

the conditions 1–7 and 14 in Definition 3,

15. $(x,l) \models \Box\alpha$ iff $(\sharp x, l) \models \alpha$,
16. $(x,l) \models \Diamond\alpha$ iff $(\flat x, l) \models \alpha$.

We remark that the conditions 15 and 16 in this definition are the same as the condition 13 in Definition 3.

We have a similar proposition for Proposition 4 and a similar definition for Definition 5. Then we obtain the following theorem.

Theorem 19 (*Soundness*). *Let C be the class of all Kripke frames for* BIAL, $L := \{\gamma \mid \text{BIAL} \vdash \gamma\}$ *and* $L(C) := \{\gamma \mid \gamma$ *is valid in all frames of* $C\}$. *Then* $L \subseteq L(C)$.

Proof. We prove this theorem by induction on the proof P of γ in BIAL. Let $\models$ be a valuation on $\langle M, S, \sharp, \flat, \cap, \cdot, \varepsilon, \omega\rangle \in C$. We only show the following cases.

(*Case* $\gamma \equiv \Box(\alpha\to\beta)\to\Box\alpha\to\Box\beta$: B2): Let $x, y \in M$ and $l \in S$ be such that $(x,l) \models \Box(\alpha\to\beta)$ and $(y,l) \models \Box\alpha$. Then we obtain $(\sharp x, l) \models \alpha\to\beta$ and $(\sharp y, l) \models \alpha$, and hence $(\sharp x \cdot \sharp y, l) \models \beta$. By C45, we obtain $(\sharp(x\cdot y), l) \models \beta$, and hence $(x \cdot y, l) \models \Box\beta$. Therefore $(\varepsilon, l) \models \Box(\alpha\to\beta)\to\Box\alpha\to\Box\beta$.

(*Case* $\gamma \equiv \Box(\alpha\to\beta)\to\Diamond\alpha\to\Diamond\beta$: B5): Let $x, y \in M$ and $l \in S$ be such that $(x,l) \models \Box(\alpha\to\beta)$ and $(y,l) \models \Diamond\alpha$. Then we obtain $(\sharp x, l) \models \alpha\to\beta$ and $(\flat y, l) \models \alpha$, and hence $(\sharp x \cdot \flat y, l) \models \beta$. By C50, we obtain $(\flat(x\cdot y), l) \models \beta$, and hence $(x \cdot y, l) \models \Diamond\beta$. Therefore $(\varepsilon, l) \models \Box(\alpha\to\beta)\to\Diamond\alpha\to\Diamond\beta$.

(*Case* ($\Box$ness)): We show that $L(C)$ is closed under ($\Box$ness). Let $(\varepsilon, l) \models \alpha$ for $l \in S$. Then $(\sharp\varepsilon, l) \models \alpha$ by C46. Thus $(\varepsilon, l) \models \Box\alpha$. □

We denote $\sharp^n\alpha$ and $\flat^n\alpha$ ($1 \leqslant n$ and $x \in M$) for the forms $\overbrace{\sharp\cdots\sharp}^{n} x$ and $\overbrace{\flat\cdots\flat}^{n} x$, respectively.

Proposition 20 (*Correspondence*). *Let* $F := \langle M, S, \sharp, \flat, \cap, \cdot, \varepsilon, \omega\rangle$ *be a Kripke frame, and* k, l, m, n *be fixed positive integers. Then*:

1. M1 *is valid in* F *iff* $\sharp x \leqslant x$ *for all* $x \in M$,
2. M2 *is valid in* F *iff* $x \leqslant \flat x$ *for all* $x \in M$,
3. M3 *is valid in* F *iff* $\sharp x \leqslant \sharp\sharp x$ *for all* $x \in M$,
4. M4 *is valid in* F *iff* $\flat\flat x \leqslant \flat x$ *for all* $x \in M$,
5. M5 *is valid in* F *iff* $\flat x \leqslant \flat\sharp x$ *for all* $x \in M$,
6. M6 *is valid in* F *iff* $\sharp\flat x \leqslant \sharp x$ *for all* $x \in M$,
7. M7 *is valid in* F *iff* $x \leqslant \flat\sharp x$ *for all* $x \in M$,
8. M8 *is valid in* F *iff* $\sharp\flat x \leqslant x$ *for all* $x \in M$,
9. M9 *is valid in* F *iff* $\sharp x \leqslant \flat x$ *for all* $x \in M$,
10. M10 *is valid in* F *iff* $\flat\sharp x \leqslant \sharp\flat x$ *for all* $x \in M$,
11. M11 *is valid in* F *iff* $\sharp\flat x \leqslant \flat\sharp x$ *for all* $x \in M$,
12. M12 *is valid in* F *iff* $\flat^k x \leqslant \flat^n\sharp^m x$ *for all* $x \in M$,
13. M13 *is valid in* F *iff* $\flat^l\sharp^k x \leqslant \sharp^n\flat^m x$ *for all* $x \in M$,
14. M14 *is valid in* F *iff* $\sharp^l\flat^k x \leqslant \flat^n\sharp^m x$ *for all* $x \in M$.

Proof. We show only the case 5. The other cases are similar to the case 5.

(5). Suppose $(x,l) \models \Diamond\alpha$. Then $(\flat x, l) \models \alpha$ and hence $(\flat\sharp x, l) \models \alpha$ by the frame condition $\flat x \leqslant \flat\sharp x$ and the fact that $\models(\alpha)$ satisfy the property as discussed in Proposition 4. Therefore $(x,l) \models \Box\Diamond\alpha$ and hence $(\varepsilon, l) \models \Diamond\alpha\to\Box\Diamond\alpha$. Conversely suppose that $\Diamond\alpha\to\Box\Diamond\alpha$ is valid in F. Let $\models(\alpha) := \{z \in M \mid \flat x \leqslant z\} \times S$. Then by the hypothesis, $(x,l) \models \Box\Diamond\alpha$ for $l \in S$ and hence $(\flat\sharp x, l) \models \alpha$. Therefore $\flat x \leqslant \flat\sharp x$. □

By using this proposition, we can show the soundness theorem for any logic obtained from BIAL by adding arbitrary combinations of the axiom schemes M1–M14.

Next we prove the completeness theorem. We use Definition 7, and have (the same corresponding) Lemmas 11, 12, Proposition 10 and the following lemma, which corresponds to Lemma 9.

Lemma 21. *Let M_L be the set of all L-pretheories. Then*

1. *if* $x, y \in M_L$, *then*
   $x \cap y \in M_L$,
   $x \cdot y := \{\beta \mid \exists \alpha \in y\ (\alpha \to \beta \in x)\} \in M_L$;
2. $L \cdot \{\alpha\} \in M_L$;
3. *if* $x, y, z \in M_L$, *then* $L \cdot x = x$, $(x \cap y) \cdot z = (x \cdot z) \cap (y \cdot z)$, $(x \cdot y) \cap (x \cdot z) \subseteq x \cdot (y \cap z)$, $x \cdot L \subseteq x$, $\Phi \subseteq x \cdot \Phi$, $x \subseteq x \cdot L$, $(x \cdot y) \cdot z \subseteq x \cdot (y \cdot z)$ *and* $(x \cdot z) \cdot y \subseteq (x \cdot y) \cdot z$;
4. *if* $x, y, z \in M_L$, *then* $x \subseteq y$ *implies* $z \cdot x \subseteq z \cdot y$;
5. *if* $x \in M_L$, *then*
   $\sharp x := \{\beta \mid \Box\beta \in x\}$, $\flat x := \{\beta \mid \Diamond\beta \in x\} \in M_L$;
6. *if* $x, y \in M_L$, *then* $x \subseteq x \cdot y$;
7. $L \subseteq \sharp L$ *and* $L \subseteq \flat L$;
8. *if* $x, y \in M_L$, *then* $\sharp(x \cap y) = \sharp x \cap \sharp y$, $\flat(x \cap y) = \flat x \cap \flat y$;
9. *if* $x, y \in M_L$, *then* $\sharp x \cdot \sharp y \subseteq \sharp(x \cdot y)$, $\flat x \cdot \flat y \subseteq \flat(x \cdot y)$, $\sharp x \cdot \flat y \subseteq \flat(x \cdot y)$.

Proof. We only show the cases 5–9. The cases 1–4 are already proved in Lemma 9.

(5): We only show the case for $\sharp x$. First, we show $\top \in \sharp x$, i.e., $\Box\top \in x$. By the weakening axiom scheme A17, A1, A3, ($\Box$ness) and (mp), we obtain $\top \to \Box\top \in L$:

$$\dfrac{\dfrac{\dfrac{\alpha\to\alpha \quad (\alpha\to\alpha)\to\top}{\top}}{\Box\top} \qquad \Box\top\to\top\to\Box\top}{\top\to\Box\top}\ .$$

Then we have $\Box\top \in x$ by $\top \in x \in M_L$.

Second, we show that $\alpha \in \sharp x$ and $\alpha \to \beta \in L$ imply $\beta \in \sharp x$. Suppose $\alpha \in \sharp x$ and $\alpha \to \beta \in L$. Then we obtain $\Box\alpha \in x$ and $\Box\alpha \to \Box\beta \in L$ (by $\Box$regu), and hence $\Box\beta \in x$ since $x \in M_L$. Therefore $\beta \in \sharp x$.

Third, we show that $\alpha, \beta \in \sharp x$ implies $\alpha \wedge \beta \in \sharp x$. Suppose $\alpha, \beta \in \sharp x$. Then we have $\Box\alpha \in x$ and $\Box\beta \in x$. Thus we obtain $\Box\alpha \wedge \Box\beta \in x$ since $x \in M_L$. By $\Box\alpha \wedge \Box\beta \to \Box(\alpha \wedge \beta) \in L$ (B1), we have $\Box(\alpha \wedge \beta) \in x$ since $x \in M_L$. Therefore $\alpha \wedge \beta \in \sharp x$.

(6): By using the weakening axiom scheme A17.

(7): We only show $L \subseteq \sharp L$. Suppose $\gamma \in L$. Then $\Box\gamma \in L$ by ($\Box$ness). Therefore $\gamma \in \sharp L$.

(8): Similar to the case (16) in Lemma 9.

(9): First, we show $\sharp x \cdot \sharp y \subseteq \sharp(x \cdot y)$. Suppose $\gamma \in \sharp x \cdot \sharp y$. Then there is $\alpha \in \sharp y$ such that $\alpha \to \gamma \in \sharp x$, and hence $\Box\alpha \in y$ and $\Box(\alpha \to \gamma) \in x$. By $\Box(\alpha \to \gamma) \to \Box\alpha \to \Box\gamma \in L$ (B2) and $\Box(\alpha \to \gamma) \in x$, we have $\Box\alpha \to \Box\gamma \in x$ since $x \in M_L$. Thus we have that there is $\Box\alpha \in y$ such that $\Box\alpha \to \Box\gamma \in x$. This means $\Box\gamma \in x \cdot y$ and hence $\gamma \in \sharp(x \cdot y)$.

Second, we show $\sharp x \cdot \flat y \subseteq \flat(x \cdot y)$. Suppose $\gamma \in \sharp x \cdot \flat y$. Then we have that there is $\beta \in \flat y$ such that $\beta \to \gamma \in \sharp x$, and hence $\Diamond\beta \in y$ and $\Box(\beta \to \gamma) \in x$. By $\Box(\beta \to \gamma) \to \Diamond\beta \to \Diamond\gamma \in L$ (B5) and $\Box(\beta \to \gamma) \in x$, we obtain $\Diamond\beta \to \Diamond\gamma \in x$ since $x \in M_L$. Thus we have that there is $\Diamond\beta \in y$ such that $\Diamond\beta \to \Diamond\gamma \in x$. This means $\Diamond\gamma \in x \cdot y$, and hence $\gamma \in \flat(x \cdot y)$.

The case $\flat x \cdot \flat y \subseteq \flat(x \cdot y)$ is similar to the case $\sharp x \cdot \sharp y \subseteq \sharp(x \cdot y)$. $\Box$

We remark that, in the proof of (5) in Lemma 21, we must use the weakening axiom scheme A17 instead of T7: $\alpha \to [P]\alpha$ in the proof of (4) in Lemma 9.

Proposition 22. *Let k, l, m and n be fixed positive integers. Then*:

1. *if* M1 $\in L$, *then* $\sharp x \subseteq x$ *for all* $x \in M_L$,
2. *if* M2 $\in L$, *then* $x \subseteq \flat x$ *for all* $x \in M_L$,
3. *if* M3 $\in L$, *then* $\sharp x \subseteq \sharp\sharp x$ *for all* $x \in M_L$,
4. *if* M4 $\in L$, *then* $\flat\flat x \subseteq \flat x$ *for all* $x \in M_L$,

5. *if* M5 $\in L$, *then* $\flat x \subseteq \flat\sharp x$ *for all* $x \in M_L$,
6. *if* M6 $\in L$, *then* $\sharp\flat x \subseteq \sharp x$ *for all* $x \in M_L$,
7. *if* M7 $\in L$, *then* $x \subseteq \flat\sharp x$ *for all* $x \in M_L$,
8. *if* M8 $\in L$, *then* $\sharp\flat x \subseteq x$ *for all* $x \in M_L$,
9. *if* M9 $\in L$, *then* $\sharp x \subseteq \flat x$ *for all* $x \in M_L$,
10. *if* M10 $\in L$, *then* $\flat\sharp x \subseteq \sharp\flat x$ *for all* $x \in M_L$,
11. *if* M11 $\in L$, *then* $\sharp\flat x \subseteq \flat\sharp x$ *for all* $x \in M_L$,
12. *if* M12 $\in L$, *then* $\flat^k x \subseteq \flat^n \sharp^m x$ *for all* $x \in M_L$,
13. *if* M13 $\in L$, *then* $\flat^l \sharp^k x \subseteq \sharp^n \flat^m x$ *for all* $x \in M_L$,
14. *if* M14 $\in L$, *then* $\sharp^l \flat^k x \subseteq \flat^n \sharp^m x$ *for all* $x \in M_L$.

Proof. We only show 5.

(5): Suppose $\gamma \in \flat x$ and hence $\Diamond\gamma \in x$. By $\Diamond\gamma \rightarrow \Box\Diamond\gamma \in L$ (M5) and $\Diamond\gamma \in x$, we have $\Box\Diamond\gamma \in x$ since $x \in M_L$. Therefore $\gamma \in \flat\sharp x$. $\Box$

We have the lemma corresponding to Lemma 13, and then obtain the following general result which is also a main result of this paper.

Theorem 23 (*Completeness*). *Let S be any logic obtained from* BIAL *by adding arbitrary combinations of the axiom schemes* M1–M14, *and C be the class of all Kripke frames for S*, $L := \{\gamma \mid \gamma$ *is provable in S*$\}$ *and* $L(C) := \{\gamma \mid \gamma$ *is valid in all frames of C*$\}$. *Then* $L = L(C)$.

6. Dynamic affine logic

It is known that propositional dynamic logic (PDL) is a useful tool for facilitating the process of producing correct programs [14]. For this reason, various versions of dynamic logics have been proposed and studied by many researchers. Concurrent PDL, converse PDL and automata PDL are examples of such versions.

In this section, we introduce a dynamic intuitionistic affine logic (DIAL), which is regarded as an affine version of the (test-free) PDL, and prove the completeness theorem (w.r.t. Kripke model) for this logic.

Prior to the precise discussion, we reconsider the language used in this section. The language of DIAL has expressions of two sorts: *formulas* $\alpha, \beta, \gamma, \ldots$ and *programs* $a, b, c, \ldots$. We denote C for a set of programs. The language of the non-modal part of DIAL is the same as that of BIAL. For the modal part, we add the spatial operators $[l_i]$ ($l_i \in S$) and new modal operators $[c]$ (program necessity) where $c \in C$. We introduce the program constant $\mathbf{0}$ and some operations on C: $+$ (nondeterministic choice), $\times$ (composition) and $\cdot^*$ (iteration) where $+$ and $\times$ are binary operations, and $\cdot^*$ is a unary operation. Then the expression of the form $[c]\alpha$ for $c \in C$ is a formula.

Intuitive interpretations for formulas of DIAL are as follows.

- $[c]\alpha$: "If c halts when started in current state, then it does so in a state satisfying α."
- $a \times b$: "Do a, then do b."
- $a + b$: "Nondeterministically choose one of a or b and execute it."
- a^*: "Execute a some nondeterministically chosen finite number of times."

An atomic program a means "one step of program a". The expression $\alpha \rightarrow [c]\beta$ means "if the proposition α holds before executing the program c, then by executing c, the proposition β holds if c halts". Thus, roughly speaking, α and β respectively mean the input and the output of the program c. This expression corresponds to the expression $\{\alpha\}c\{\beta\}$ in Hoar logic. Although DIAL has no test operator ?, this operator is useful for describing the "while program", e.g. the program: "if α then b else c" is described as "$\alpha? \times (b + \neg\alpha?) \times c$", and the program: "while α do b" is described as "$(\alpha? \times b)^* \times \neg\alpha$".

The program necessity operators can be interpreted as some temporal modal operators. For example, the following encoding of the temporal operators $[F]$ (future), $[N]$ (next) and U (until) can be obtained: $[F]\alpha$ is interpreted as $[a^*]\alpha$, $[N]\alpha$ is interpreted as $[a]\alpha$, and $\alpha U \beta$ is interpreted as $\langle (a \times \alpha?)^* \times a \rangle \beta$, where a is an atomic program. Some temporal interpretations of a kind of program operators based on linear logic are adopted by Tanabe [46] and Dam [11].

Definition 24 (*The logic DIAL*). The axiom schemes and inference rules for the logic DIAL are as follows.

Non-modal part:

The same as the non-modal part of BIAL.

Dynamic-modal part:

D1: $[c](\alpha\to\beta)\to[c]\alpha\to[c]\beta$,
D2: $[c]\alpha\wedge[c]\beta\to[c](\alpha\wedge\beta)$,
D3: $[a+b]\alpha\to[a]\alpha\wedge[b]\alpha$,
D4: $[a]\alpha\wedge[b]\alpha\to[a+b]\alpha$,
D5: $[a\times b]\alpha\to[a][b]\alpha$,
D6: $[a][b]\alpha\to[a\times b]\alpha$,
D7: $[c^*]\alpha\to[c][c^*]\alpha$,
D8: $[c][c^*]\alpha\to[c^*]\alpha$,
D9: $[c^*](\alpha\to[c]\beta)\to\alpha\to[c^*]\beta$,
D10: $[\mathbf{0}]\alpha\to\alpha$,
D11: $\alpha\to[\mathbf{0}]\alpha$,

$$\frac{\alpha}{[c]\alpha}\ (D\text{ness}),$$

for any $a, b, c \in C$.

Spatial-modal part:

S3–S12, S25, S26, (space1) and (space2) in Definition 1,
S31: $[l][c]\alpha\to[c][l]\alpha$,
S32: $[c][l]\alpha\to[l][c]\alpha$
for any $c \in C$ and any $l \in S$.

We remark that the axiom schemes D7–D9 for the iteration operator are different from that in the original dynamic logic [14]. We also remark that we can add the axiom schemes and the rule for the dynamic-modal part with the addition of the axiom scheme: $\alpha\to[c]\alpha$ to TSEILL, and can prove the completeness theorem (w.r.t. the corresponding model) for the extended TSEILL with the dynamic-modal axiom schemes and rule, in a similar way as that in this section.

Definition 25. A *Kripke frame* for DIAL is a two-sorted structure $\langle \mathbf{D}, \mathbf{M}, D_c\ (c \in C)\rangle$ satisfying the following conditions:

1. $\mathbf{D} := \langle C, +, \times, \cdot^*, \mathbf{0}\rangle$ is the absolutely free algebra, that is, the binary operations $+$ and $\times$ on C and the unary operation $\cdot^*$ on C have no condition, and $\mathbf{0} \in C$.
2. $\mathbf{M} := \langle M, S, \cap, \cdot, \varepsilon, \omega\rangle$ satisfying the following conditions:
   2.1. S is a nonempty set,
   2.2. $\langle M, \cap\rangle$ is a meet-semilattice with the greatest element ω,
   2.3. $\cdot$ is a binary operation on M and $\varepsilon \in M$ such that
   C1–C11 in Definition 2 and C43 in Definition 17.
3. D_c is a unary operation on M such that
   C51: $D_c x \cdot D_c y \leqslant D_c(x\cdot y)$ for all $x, y \in M$,
   C52: $\varepsilon \leqslant D_c\varepsilon$,
   C53: $D_c(x\cap y) = D_c x \cap D_c y$ for all $x, y \in M$,
   C54: $D_{a+b}x = D_a x \cap D_b x$ for all $x \in M$,
   C55: $D_{a\times b}x = D_b D_a x$ for all $x \in M$,
   C56: $D_{c^*}x = D_{c^*}D_c x$ for all $x \in M$,
   C57: $D_c((D_{c^*}x)\cdot y) \leqslant D_{c^*}(x\cdot y)$ for all $x, y \in M$,
   C58: $D_{\mathbf{0}}x = x$ for all $x \in M$.

Some conditions for D_c in Definition 25 are from the dynamic algebras by Pratt [41].

Definition 26. A *valuation* $\models$ on a Kripke frame $\langle \mathbf{D}, \mathbf{M}, D_c\ (c \in C)\rangle$ is a mapping from the set Ψ of all propositional variables to $2^{M\times S}$ such that $(x, l), (y, l) \in \models(p)$ iff $(x \cap y, l) \in \models(p)$, that is, $\models(p) := X \times S' \subseteq M \times S$ where X is a filter of M, and S' is nonempty. We write $(x, l) \models p$ for $(x, l) \in \models(p)$. Each valuation $\models$ can be extended to a mapping from the set Φ of all formulas to $2^{M\times S}$ by

the conditions 1–7 and 14 in Definition 3,

17. $(x, l) \models [c]\alpha$ iff $(D_c x, l) \models \alpha$.

We have a similar proposition for Proposition 4 and a similar definition for Definition 5. We then obtain the following theorem.

Theorem 27 (*Soundness*). *Let K be the class of all Kripke frames for* DIAL, $L := \{\gamma \mid \text{DIAL} \vdash \gamma\}$ *and* $L(K) := \{\gamma \mid \gamma$ *is valid in all frames of* $K\}$. *Then* $L \subseteq L(K)$.

Proof. We prove this theorem by induction on the proof P of γ in DIAL. Let $\models$ be a valuation on $\langle \mathbf{D}, \mathbf{M}, D_c\ (c \in C)\rangle \in K$. We only show the following cases.

(*Case* $\gamma \equiv [a+b]\alpha \to [a]\alpha \wedge [b]\alpha$: D3): Let $x \in M$ and $l \in S$ be such that $(x, l) \models [a+b]\alpha$. Then $(x, l) \models [a+b]\alpha$ iff $(D_{a+b}x, l) \models \alpha$ iff $(D_a x \cap D_b x, l) \models \alpha$ (by C54) iff $(D_a x, l) \models \alpha$ and $(D_b x, l) \models \alpha$ iff $(x, l) \models [a]\alpha$ and $(x, l) \models [b]\alpha$ iff $(x, l) \models [a]\alpha \wedge [b]\alpha$.

(*Case* $\gamma \equiv [a \times b]\alpha \to [a][b]\alpha$: D5): Let $x \in M$ and $l \in S$ be such that $(x, l) \models [a \times b]\alpha$. Then $(x, l) \models [a \times b]\alpha$ iff $(D_{a\times b}x, l) \models \alpha$ iff $(D_b D_a x, l) \models \alpha$ (by C55) iff $(D_a x, l) \models [b]\alpha$ iff $(x, l) \models [a][b]\alpha$.

(*Case* $\gamma \equiv [c][c^*]\alpha \to [c^*]\alpha$: D8): Let $x \in M$ and $l \in S$ be such that $(x, l) \models [c][c^*]\alpha$. Then $(x, l) \models [c][c^*]\alpha$ iff $(D_{c^*} D_c x, l) \models \alpha$ iff $(D_{c^*} x, l) \models \alpha$ (by C56) iff $(x, l) \models [c^*]\alpha$.

(*Case* $[c^*](\alpha \to [c]\beta) \to \alpha \to [c^*]\beta$: D9): Let $x, y \in M$ and $l \in S$ be such that $(x, l) \models [c^*](\alpha \to [c]\beta)$ and $(y, l) \models \alpha$. Then we have $(D_{c^*} x, l) \models \alpha \to [c]\beta$, and hence $((D_{c^*} x) \cdot y, l) \models [c]\beta$. Thus we obtain $(D_c((D_{c^*} x) \cdot y), l) \models \beta$. By C57, we obtain $(D_{c^*}(x \cdot y), l) \models \beta$, and hence $x \cdot y \models [c^*]\beta$.

(*Case* $[\mathbf{0}]\alpha \to \alpha$: D10): Let $x \in M$ and $l \in S$ be such that $(x, l) \models [\mathbf{0}]\alpha$. Then $(x, l) \models [\mathbf{0}]\alpha$ iff $(D_{\mathbf{0}} x, l) \models \alpha$ iff $(x, l) \models \alpha$ (by C58). □

Next we prove the completeness theorem. We use Definition 7, and have (the same corresponding) Lemmas 11, 12, Proposition 10 (we take the canonical frame $\mathbf{F_L} := \langle \mathbf{D_L}, \mathbf{M_L}, D_c\ (c \in C)\rangle$ where $\mathbf{D_L}$ is the same free algebra $\mathbf{D} := \langle C, +, \times, \cdot^*, \mathbf{0}\rangle$) and the following lemma, which corresponds to Lemma 9.

Lemma 28. *Let M_L be the set of all L-pretheories. Then*

1. *The conditions* 1–4 *and* 6 *in Lemma* 21;
2. *if* $x \in M_L$ *and* $c \in C$, *then*
   $D_c x := \{\beta \mid [c]\beta \in x\} \in M_L$;
3. *if* $x, y \in M_L$ *and* $c \in C$, *then* $D_c x \cdot D_c y \subseteq D_c(x \cdot y)$, $D_c(x \cap y) = D_c x \cap D_c y$, $L \subseteq D_c L$;
4. *if* $x \in M_L$ *and* $a, b \in C$, *then* $D_{a+b}x = D_a x \cap D_b x$;
5. *if* $x \in M_L$ *and* $a, b \in C$, *then* $D_{a\times b}x = D_b D_a x$;
6. *if* $x \in M_L$ *and* $c \in C$, *then* $D_{c^*}x = D_{c^*} D_c x$;
7. *if* $x, y \in M_L$ *and* $c \in C$, *then* $D_c((D_{c^*}x) \cdot y) \subseteq D_{c^*}(x \cdot y)$;
8. *if* $x \in M_L$, *then* $D_{\mathbf{0}} x = x$.

Proof. We only show cases 4–8. Case 1 is already proved in Lemma 21, and cases 2 and 3 are also proved in a similar way in Lemma 21.

(4): First, we show $D_{a+b}x \subseteq D_a x \cap D_b x$. Suppose $\gamma \in D_{a+b}x$. Then we have $[a+b]\gamma \in x$. By $[a+b]\gamma \to [a]\gamma \wedge [b]\gamma \in L$ (D3) and $x \in M_L$, we obtain $[a]\gamma \wedge [b]\gamma \in x$. Moreover, by A4 and $x \in M$, we obtain $[a]\gamma \in x$. Also we can obtain $[b]\gamma \in x$ using A5. Thus we have $\gamma \in D_a x$ and $\gamma \in D_b x$. Therefore $\gamma \in D_a x \cap D_b x$. Next we show $D_a x \cap D_b x \subseteq D_{a+b}x$. Suppose $\gamma \in D_a x \cap D_b x$. Then we have $\gamma \in D_a x$ and $\gamma \in D_b x$, and hence $[a]\gamma \in x$ and $[b]\gamma \in x$. By $x \in M_L$, we obtain $[a]\gamma \wedge [b]\gamma \in x$. By $[a]\gamma \wedge [b]\gamma \to [a+b]\gamma \in L$ (D4) and $x \in M_L$, we obtain $[a+b]\gamma \in x$. Therefore $\gamma \in D_{a+b}x$.

(5): First, we show $D_{a\times b}x \subseteq D_b D_a x$. Suppose $\gamma \in D_{a\times b}x$, i.e., $[a \times b]\gamma \in x$. By $[a \times b]\gamma \to [a][b]\gamma \in L$ (D5) and $x \in M_L$, we obtain $[a][b]\gamma \in x$, and hence $[b]\gamma \in D_a x$. Therefore $\gamma \in D_b D_a x$. Next we show $D_b D_a x \subseteq D_{a\times b}x$. Suppose $\gamma \in D_b D_a x$, i.e., $[a][b]\gamma \in x$. By $[a][b]\gamma \to [a \times b]\gamma \in L$ (D6) and $x \in M_L$, we obtain $[a \times b]\gamma \in x$. Therefore $\gamma \in D_{a\times b}x$.

(6): By using D7 and D8.

(7): Suppose $\gamma \in D_c((D_{c^*}x) \cdot y)$, i.e., $[c]\gamma \in (D_{c^*}x) \cdot y$. Then we have that there is $\alpha \in y$ such that $\alpha \to [c]\gamma \in D_{c^*}x$. Thus we obtain that there is $\alpha \in y$ such that $[c^*](\alpha \to [c]\gamma) \in x$. By $[c^*](\alpha \to [c]\gamma) \to \alpha \to [c^*]\gamma \in L$ (D9) and $x \in M_L$, we obtain $\alpha \to [c^*]\gamma \in x$. Hence we have that there is $\alpha \in y$ such that $\alpha \to [c^*]\gamma \in x$, i.e., $[c^*]\gamma \in x \cdot y$. Therefore $\gamma \in D_{c^*}(x \cdot y)$.

(8): By using D10 and D11. □

Then we can give the canonical Kripke model $\langle \mathbf{D_L}, \mathbf{M_L}, D_c\ (c \in C), \models_L \rangle$, and hence obtain the following main theorem.

Theorem 29 (*Completeness*). *Let K be the class of all Kripke frames for* DIAL, $L := \{\gamma \mid \gamma$ *is provable in* DIAL$\}$ *and* $L(K) := \{\gamma \mid \gamma$ *is valid in all frames of* $K\}$. *Then* $L = L(K)$.

7. Extensions and modifications

7.1. Adding strong negation to TSEILL

We can introduce a logic TSEILLs which is obtained from TSEILL by adding strong negation connective $\sim$, and can prove, by using the technique introduced in [20,23], the completeness theorem for this logic. Using the strong negation connective $\sim$, we can formalize the negative introspective axiom scheme $\sim[K]\alpha \to [K]\sim[K]\alpha$ in the epistemic part of TSEILLs, and can describe inexact predicates and exceptions in knowledge representation, and stop action of processes in concurrent systems. The intuitive meaning and motivation for TSEILLs are discussed in Section 8. Only the formal definition of TSEILLs is presented in this section. For more information on applications of logics with strong negation, see [22,49,51].

Historically, strong negation was introduced by Nelson in 1949, and linear logic with strong negation was presented by Wansing [50,51] in 1993. Moreover, Wansing's linear logic (WILL) is recently extended by Kamide [22,23]. These logics are defined as Gentzen-style sequent calculi with the cut-elimination properties. TSEILLs is an extension of these logics.

First, we give a definition of TSEILLs. Prior to the discussion, we extend our language of TSEILLs by adding the strong negation connective $\sim$.

Definition 30 (*The logic TSEILLs*). The logic TSEILLs is obtained from TSEILL by adding the following axiom schemes:

N1: $\alpha \to \sim\sim\alpha$,
N2: $\sim\sim\alpha \to \alpha$,
N3: $\sim\mathbf{1} \to \alpha$,
N4: $\sim\top \to \alpha$,
N5: $\alpha \to \sim\bot$,
N6: $\sim(\alpha \to \beta) \to \alpha * \sim\beta$,
N7: $\alpha * \sim\beta \to \sim(\alpha \to \beta)$,
N8: $\sim(\alpha \wedge \beta) \to \sim\alpha \vee \sim\beta$,
N9: $\sim\alpha \vee \sim\beta \to \sim(\alpha \wedge \beta)$,
N10: $\sim(\alpha \vee \beta) \to \sim\alpha \wedge \sim\beta$,
N11: $\sim\alpha \wedge \sim\beta \to \sim(\alpha \vee \beta)$,
N12: $\sim(\alpha * \beta) \to \sim\alpha * \sim\beta$,
N13: $\sim\alpha * \sim\beta \to \sim(\alpha * \beta)$,
N14: $\sim!\alpha \to !\sim\alpha$,

N15: $!{\sim}\alpha \to {\sim}!\alpha$,
N16: ${\sim}[K]\alpha \to \langle K\rangle{\sim}\alpha$,
N17: $\langle K\rangle{\sim}\alpha \to {\sim}[K]\alpha$,
N18: ${\sim}\langle K\rangle\alpha \to [K]{\sim}\alpha$,
N19: $[K]{\sim}\alpha \to {\sim}\langle K\rangle\alpha$,
N20: ${\sim}[F]\alpha \to [F]{\sim}\alpha$,
N21: $[F]{\sim}\alpha \to {\sim}[F]\alpha$,
N22: ${\sim}[N]\alpha \to [N]{\sim}\alpha$,
N23: $[N]{\sim}\alpha \to {\sim}[N]\alpha$,
N24: ${\sim}[P]\alpha \to [P]{\sim}\alpha$,
N25: $[P]{\sim}\alpha \to {\sim}[P]\alpha$,
N26: ${\sim}[l]\alpha \to [l]{\sim}\alpha$,
N27: $[l]{\sim}\alpha \to {\sim}[l]\alpha$,
N28: $[l]{\sim}\mathbf{1} \to {\sim}\mathbf{1}$,
N29: $[l]{\sim}\top \to {\sim}\top$,
N30: $\alpha \to [l]{\sim}\bot$

for any $l \in S$.

We remark that the following rules are derivable in TSEILLs:

$$\frac{\alpha \to \beta}{\gamma * \alpha \to \gamma * \beta}\ (*\text{regu}),\quad \frac{\alpha \to \beta}{!\alpha \to !\beta}\ (!\text{regu}),\quad \frac{\alpha \to \beta}{[K]\alpha \to [K]\beta}\ (K\text{regu}),$$

$$\frac{\alpha \to \beta}{\langle K\rangle\alpha \to \langle K\rangle\beta}\ (J\text{regu}),\quad \frac{\alpha \to \beta}{[F]\alpha \to [F]\beta}\ (F\text{regu}),$$

$$\frac{\alpha \to \beta}{[N]\alpha \to [N]\beta}\ (N\text{regu}),\quad \frac{\alpha \to \beta}{[l]\alpha \to [l]\beta}\ (\text{space3})$$

for any $l \in S$. For example, ($*$regu) is derivable using A9, (residu) and (suff), and (Jregu) is derivable using E5 and (Kleft).

We define the following sublogics of TSEILLs:

ILLs = ILL + (N1–N15),
EILLs = EILL + (N1–N19),
TILLs = TILL + (N1–N15) + (N20–N25),
ITLLs = ITLL + (N1–N15) + (N20–N23),
SILLs = SILL + (N1–N15) + (N26–N30),
SEILLs = SEILL + (N1–N19) + (N26–N30),
TEILLs = TEILL + (N1–N25),
TSILLs = TSILL + (N1–N15) + (N20–N30).

We remark that ILLs is Kamide's ILL$\sim$ [22], and the modal-free part of ILLs is Wansing's WILL [50,51].

Next, we mention that the negative introspective axiom ${\sim}[K]\alpha \to [K]{\sim}[K]\alpha$, the duality principles $\langle K\rangle\alpha \to {\sim}[K]{\sim}\alpha$, ${\sim}[K]{\sim}\alpha \to \langle K\rangle\alpha$, $[K]\alpha \to {\sim}\langle K\rangle{\sim}\alpha$ and ${\sim}\langle K\rangle{\sim}\alpha \to [K]\alpha$ are provable in EILLs.

Proposition 31. *Let* $L := \{\alpha \mid \text{EILLs} \vdash \alpha\}$. *Then*

1. ${\sim}[K]\alpha \to [K]{\sim}[K]\alpha \in L$,
2. $\langle K\rangle\alpha \to {\sim}[K]{\sim}\alpha$, ${\sim}[K]{\sim}\alpha \to \langle K\rangle\alpha, \in L$,
3. $[K]\alpha \to {\sim}\langle K\rangle{\sim}\alpha$, ${\sim}\langle K\rangle{\sim}\alpha \to [K]\alpha \in L$.

Proof. We only show (1). By E7, N17, N16, (Kregu) and (cut), we obtain

$$\dfrac{\sim[K]\alpha\to\langle K\rangle\sim\alpha \quad \dfrac{\langle K\rangle\sim\alpha\to[K]\langle K\rangle\sim\alpha \quad \dfrac{\langle K\rangle\sim\alpha\to\sim[K]\alpha}{[K]\langle K\rangle\sim\alpha\to[K]\sim[K]\alpha}}{\langle K\rangle\sim\alpha\to[K]\sim[K]\alpha}}{\sim[K]\alpha\to[K]\sim[K]\alpha}\ .$$

□

In the proof of Lemma 37, we will use the following proposition.

Proposition 32. *Let* $L := \{\alpha \mid \text{TSEILLs} \vdash \alpha\}$. *Then*:

1. $\sim[l](\beta\to\gamma)\to[l]\beta*\sim[l]\gamma,\ [l]\beta*\sim[l]\gamma\to\sim[l](\beta\to\gamma)\in L$,
2. $\sim[l](\beta\wedge\gamma)\to\sim[l]\beta\vee\sim[l]\gamma,\ \sim[l]\beta\vee\sim[l]\gamma\to\sim[l](\beta\wedge\gamma)\in L$,
3. $\sim[l](\beta\vee\gamma)\to\sim[l]\beta\wedge\sim[l]\gamma,\ \sim[l]\beta\wedge\sim[l]\gamma\to\sim[l](\beta\vee\gamma)\in L$,
4. $\sim[l](\beta*\gamma)\to\sim[l]\beta*\sim[l]\gamma,\ \sim[l]\beta*\sim[l]\gamma\to\sim[l](\beta*\gamma)\in L$,
5. $\sim[l]!\beta\to!\sim[l]\beta,\ !\sim[l]\beta\to\sim[l]!\beta\in L$,
6. $\sim[l][K]\beta\to\langle K\rangle\sim[l]\beta,\ \langle K\rangle\sim[l]\beta\to\sim[l][K]\beta\in L$,
7. $\sim[l]\langle K\rangle\beta\to[K]\sim[l]\beta,\ [K]\sim[l]\beta\to\sim[l]\langle K\rangle\beta\in L$,
8. $\sim[l][F]\beta\to[F]\sim[l]\beta,\ [F]\sim[l]\beta\to\sim[l][F]\beta\in L$,
9. $\sim[l][N]\beta\to[N]\sim[l]\beta,\ [N]\sim[l]\beta\to\sim[l][N]\beta\in L$,
10. $\sim[l][P]\beta\to[P]\sim[l]\beta,\ [P]\sim[l]\beta\to\sim[l][P]\beta\in L$,
11. $\sim[l][s]\beta\to\sim[s]\beta,\ \sim[s]\beta\to\sim[l][s]\beta\in L$,
12. $\sim[l]\sim\beta\to[l]\beta,\ [l]\beta\to\sim[l]\sim\beta\in L$

for any formulas β,γ *and any* $l, s\in S$.

Proof. We only show cases (1), (2), (5) and (11). Case (3) is similar to case (2). Case (4) is similar to case (1). Cases (6)–(10) are similar to case (5). Case (12) is straightforward.

(1): We prove $\sim[l](\beta\to\gamma)\to[l]\beta*\sim[l]\gamma\in L$ by using N26, N27, N6, N7, ($*$regu), (space3) and (cut):

$$\dfrac{\sim[l](\beta\to\gamma)\to[l]\sim(\beta\to\gamma) \quad \dfrac{\dfrac{\sim(\beta\to\gamma)\to\beta*\sim\gamma}{[l]\sim(\beta\to\gamma)\to[l](\beta*\sim\gamma)} \quad C}{[l]\sim(\beta\to\gamma)\to[l]\beta*\sim[l]\gamma}}{\sim[l](\beta\to\gamma)\to[l]\beta*\sim[l]\gamma}\ ,$$

where

$$\dfrac{[l](\beta*\sim\gamma)\to[l]\beta*[l]\sim\gamma \quad \dfrac{[l]\sim\gamma\to\sim[l]\gamma}{[l]\beta*[l]\sim\gamma\to[l]\beta*\sim[l]\gamma}}{C : [l](\beta*\sim\gamma)\to[l]\beta*\sim[l]\gamma}\ .$$

Next we prove $[l]\beta*\sim[l]\gamma\to\sim[l](\beta\to\gamma)\in L$ by using N26, N27, N7, S11, (space3), ($*$regu) and (cut):

$$\dfrac{\dfrac{\sim[l]\gamma\to[l]\sim\gamma}{[l]\beta*\sim[l]\gamma\to[l]\beta*[l]\sim\gamma} \quad \dfrac{D \quad [l]\sim(\beta\to\gamma)\to\sim[l](\beta\to\gamma)}{[l]\beta*[l]\sim\gamma\to\sim[l](\beta\to\gamma)}}{[l]\beta*\sim[l]\gamma\to\sim[l](\beta\to\gamma)}\ ,$$

where

$$\dfrac{[l]\beta*[l]\sim\gamma\to[l](\beta*\sim\gamma) \quad \dfrac{(\beta*\sim\gamma)\to\sim(\beta\to\gamma)}{[l](\beta*\sim\gamma)\to[l]\sim(\beta\to\gamma)}}{D : [l]\beta*[l]\sim\gamma\to[l]\sim(\beta\to\gamma)}\ .$$

(2): We prove $\sim[l](\beta\wedge\gamma)\rightarrow\sim[l]\beta\vee\sim[l]\gamma$ by using N7, N8, N26, N27, A7, A8, (space3), (or) and (cut):

$$\dfrac{\sim[l](\beta\wedge\gamma)\rightarrow[l]\sim(\beta\wedge\gamma) \qquad \dfrac{\dfrac{\sim(\beta\wedge\gamma)\rightarrow\sim\beta\vee\sim\gamma}{[l]\sim(\beta\wedge\gamma)\rightarrow[l](\sim\beta\vee\sim\gamma)} \quad C}{[l]\sim(\beta\wedge\gamma)\rightarrow\sim[l]\beta\vee\sim[l]\gamma}}{\sim[l](\beta\wedge\gamma)\rightarrow\sim[l]\beta\vee\sim[l]\gamma}\ ,$$

where

$$\dfrac{[l](\sim\beta\vee\sim\gamma)\rightarrow[l]\sim\beta\vee[l]\sim\gamma \qquad \dfrac{\dfrac{[l]\sim\beta\rightarrow\sim[l]\beta \quad \sim[l]\beta\rightarrow\sim[l]\beta\vee\sim[l]\gamma}{[l]\sim\beta\rightarrow\sim[l]\beta\vee\sim[l]\gamma} \quad D}{[l]\sim\beta\vee[l]\sim\gamma\rightarrow\sim[l]\beta\vee\sim[l]\gamma}}{C:[l](\sim\beta\vee\sim\gamma)\rightarrow\sim[l]\beta\vee\sim[l]\gamma}\ ,$$

where

$$\dfrac{[l]\sim\gamma\rightarrow\sim[l]\gamma \quad \sim[l]\gamma\rightarrow\sim[l]\beta\vee\sim[l]\gamma}{D:[l]\sim\gamma\rightarrow\sim[l]\beta\vee\sim[l]\gamma}\ .$$

Next we prove $\sim[l]\beta\vee\sim[l]\gamma\rightarrow\sim[l](\beta\wedge\gamma)\in L$ by using N26, N27, A7, A8, N9, S7, (or), (space3), and (cut):

$$\dfrac{\dfrac{C \quad \dfrac{\sim\beta\vee\sim\gamma\rightarrow\sim(\beta\wedge\gamma)}{[l](\sim\beta\vee\sim\gamma)\rightarrow[l]\sim(\beta\wedge\gamma)}}{\sim[l]\beta\vee\sim[l]\gamma\rightarrow[l]\sim(\beta\wedge\gamma)} \qquad [l]\sim(\beta\wedge\gamma)\rightarrow\sim[l](\beta\wedge\gamma)}{\sim[l]\beta\vee\sim[l]\gamma\rightarrow\sim[l](\beta\wedge\gamma)}\ ,$$

where

$$\dfrac{\dfrac{D \quad E}{\sim[l]\beta\vee\sim[l]\gamma\rightarrow[l]\sim\beta\vee[l]\sim\gamma} \qquad [l]\sim\beta\vee[l]\sim\gamma\rightarrow[l](\sim\beta\vee\sim\gamma)}{C:\sim[l]\beta\vee\sim[l]\gamma\rightarrow[l](\sim\beta\vee\sim\gamma)}\ ,$$

where

$$\dfrac{\sim[l]\beta\rightarrow[l]\sim\beta \quad [l]\sim\beta\rightarrow[l]\sim\beta\vee[l]\sim\gamma}{D:\sim[l]\beta\rightarrow[l]\sim\beta\vee[l]\sim\gamma}$$

and

$$\dfrac{\sim[l]\gamma\rightarrow[l]\sim\gamma \quad [l]\sim\gamma\rightarrow[l]\sim\beta\vee[l]\sim\gamma}{E:\sim[l]\gamma\rightarrow[l]\sim\beta\vee[l]\sim\gamma}\ .$$

(5): We prove $\sim[l]!\beta\rightarrow!\sim[l]\beta\in L$ by using N26, N27, N14, S13, (space3), (!regu) and (cut):

$$\dfrac{\sim[l]!\beta\rightarrow[l]\sim!\beta \qquad \dfrac{\dfrac{\sim!\beta\rightarrow!\sim\beta}{[l]\sim!\beta\rightarrow[l]!\sim\beta} \qquad \dfrac{[l]!\sim\beta\rightarrow![l]\sim\beta \quad \dfrac{[l]\sim\beta\rightarrow\sim[l]\beta}{![l]\sim\beta\rightarrow!\sim[l]\beta}}{[l]!\sim\beta\rightarrow!\sim[l]\beta}}{[l]\sim!\beta\rightarrow!\sim[l]\beta}}{\sim[l]!\beta\rightarrow!\sim[l]\beta}\ .$$

Next we prove $!\sim[l]\beta\rightarrow\sim[l]!\beta\in L$ by using N26, N27, S14, N15, (space3), (!regu) and (cut):

$$\dfrac{\dfrac{\dfrac{\sim[l]\beta\rightarrow[l]\sim\beta}{!\sim[l]\beta\rightarrow![l]\sim\beta} \qquad \dfrac{![l]\sim\beta\rightarrow[l]!\sim\beta \quad \dfrac{!\sim\beta\rightarrow\sim!\beta}{[l]!\sim\beta\rightarrow[l]\sim!\beta}}{![l]\sim\beta\rightarrow[l]\sim!\beta}}{!\sim[l]\beta\rightarrow[l]\sim!\beta} \qquad [l]\sim!\beta\rightarrow\sim[l]!\beta}{!\sim[l]\beta\rightarrow\sim[l]!\beta}\ .$$

(11): We prove $\sim[l][s]\beta\to\sim[s]\beta \in L$ by using N26, N27, S25, (space) and (cut):

$$\dfrac{\sim[l][s]\beta\to[l]\sim[s]\beta \qquad \dfrac{\dfrac{\sim[s]\beta\to[s]\sim\beta}{[l]\sim[s]\beta\to[l][s]\sim\beta} \qquad \dfrac{[l][s]\sim\beta\to[s]\sim\beta \quad [s]\sim\beta\to\sim[s]\beta}{[l][s]\sim\beta\to\sim[s]\beta}}{[l]\sim[s]\beta\to\sim[s]\beta}}{\sim[l][s]\beta\to\sim[s]\beta}\ .$$

Next we prove $\sim[s]\beta\to\sim[l][s]\beta \in L$ by using N26, N27, S26, (space3) and (cut):

$$\dfrac{\sim[s]\beta\to[s]\sim\beta \qquad \dfrac{\dfrac{[s]\sim\beta\to[l][s]\sim\beta \quad \dfrac{[s]\sim\beta\to\sim[s]\beta}{[l][s]\sim\beta\to[l]\sim[s]\beta}}{[s]\sim\beta\to[l]\sim[s]\beta} \qquad [l]\sim[s]\beta\to\sim[l][s]\beta}{[s]\sim\beta\to\sim[l][s]\beta}}{\sim[s]\beta\to\sim[l][s]\beta}\ .$$

□

The definition of Kripke frame for TSEILLs is the same as that for TSEILL (in Definition 2).

Definition 33. A *valuation* $\models^+$ (and $\models^-$) on a Kripke frame $\langle M, S, \dagger, K, J, F, N, P, \cap, \cdot, \varepsilon, \omega\rangle$ is a mapping from the set Ψ of all propositional variables to $2^{M\times S}$ such that $(x,l), (y,l) \in \models^+(p)$ iff $(x\cap y, l) \in \models^+(p)$ ($(x,l),(y,l) \in \models^-(p)$ iff $(x\cap y, l) \in \models^-(p)$), that is, $\models^+(p) := X\times S' \subseteq M\times S$ ($\models^-(p) := X'\times S'' \subseteq M\times S$) where X (and X') is a filter of M, and S' (and S'') is nonempty. We will write $(x,l)\models^+ p$ (and $(x,l)\models^- p$) for $(x,l)\in\models^+(p)$ (and $(x,l)\models^- p$). Each valuation $\models^+$ (and $\models^-$) can be extended to a mapping from the set Φ of all formulas to $2^{M\times S}$ by

1. $(x,l)\models^+ \mathbf{1}$ iff $\varepsilon\leqslant x$,
2. $(x,l)\models^+ \top$ for all $x\in M$ and all $l\in S$,
3. $(x,l)\models^+ \bot$ iff $x=\omega$,
4. $(x,l)\models^+ \alpha\to\beta$ iff $x\cdot y\leqslant z$ and $(y,l)\models^+\alpha$ imply $(z,l)\models^+\beta$ for all $y,z\in M$,
5. $(x,l)\models^+ \alpha\wedge\beta$ iff $(x,l)\models^+\alpha$ and $(x,l)\models^+\beta$,
6. $(x,l)\models^+ \alpha\vee\beta$ iff $(y,l)\models^+\alpha$ or $(y,l)\models^+\beta$, and $(z,l)\models^+\alpha$ or $(z,l)\models^+\beta$ for some $y,z\in M$ with $y\cap z\leqslant x$,
7. $(x,l)\models^+ \alpha*\beta$ iff $(y,l)\models^+\alpha$ and $(z,l)\models^+\beta$ for some $y,z\in M$ with $y\cdot z\leqslant x$,
8. $(x,l)\models^+ !\alpha$ iff $(y,l)\models^+\alpha$ for some y with $\dagger y\leqslant x$,
9. $(x,l)\models^+ [K]\alpha$ iff $(y,l)\models^+\alpha$ for some y with $Ky\leqslant x$,
10. $(x,l)\models^+ \langle K\rangle\alpha$ iff $(y,l)\models^+\alpha$ for some y with $Jy\leqslant x$,
11. $(x,l)\models^+ [F]\alpha$ iff $(y,l)\models^+\alpha$ for some y with $Fy\leqslant x$,
12. $(x,l)\models^+ [N]\alpha$ iff $(y,l)\models^+\alpha$ for some y with $Ny\leqslant x$,
13. $(x,l)\models^+ [P]\alpha$ iff $(Px,l)\models^+\alpha$,
14. $(x,l_1)\models^+ [l_2]\alpha$ iff $(x,l_2)\models^+\alpha$,
15. $(x,l)\models^+ \sim\alpha$ iff $(x,l)\models^-\alpha$,
16. $(x,l)\models^- \sim\alpha$ iff $(x,l)\models^+\alpha$,
17. $(x,l)\models^- \mathbf{1}$ iff $x=\omega$,
18. $(x,l)\models^- \top$ iff $x=\omega$,
19. $(x,l)\models^- \bot$ for all $x\in M$ and all $l\in S$,
20. $(x,l)\models^- \alpha\to\beta$ iff $(y,l)\models^+\alpha$ and $(z,l)\models^-\beta$ for some $y,z\in M$ with $y\cdot z\leqslant x$,
21. $(x,l)\models^- \alpha\wedge\beta$ iff $(y,l)\models^-\alpha$ or $(y,l)\models^-\beta$, and $(z,l)\models^-\alpha$ or $(z,l)\models^-\beta$ for some $y,z\in M$ with $y\cap z\leqslant x$,
22. $(x,l)\models^- \alpha\vee\beta$ iff $(x,l)\models^-\alpha$ and $(x,l)\models^-\beta$,
23. $(x,l)\models^- \alpha*\beta$ iff $(y,l)\models^-\alpha$ and $(z,l)\models^-\beta$ for some $y,z\in M$ with $y\cdot z\leqslant x$,
24. $(x,l)\models^- !\alpha$ iff $(y,l)\models^-\alpha$ for some y with $\dagger y\leqslant x$,
25. $(x,l)\models^- [K]\alpha$ iff $(y,l)\models^-\alpha$ for some y with $Jy\leqslant x$,
26. $(x,l)\models^- \langle K\rangle\alpha$ iff $(y,l)\models^-\alpha$ for some y with $Ky\leqslant x$,
27. $(x,l)\models^- [F]\alpha$ iff $(y,l)\models^-\alpha$ for some y with $Fy\leqslant x$,
28. $(x,l)\models^- [N]\alpha$ iff $(y,l)\models^-\alpha$ for some y with $Ny\leqslant x$,
29. $(x,l)\models^- [P]\alpha$ iff $(Px,l)\models^-\alpha$,
30. $(x,l_1)\models^- [l_2]\alpha$ iff $(x,l_2)\models^-\alpha$.

The valuation conditions 15–23 are from [50,51].

We remark that $(x, l) \models^- \alpha$ means "α is refutable at $l \in S$ by using the information piece x".

Proposition 34. *Let* $\models^+$ (*and* $\models^-$) *be a valuation on a Kripke frame* $\langle M, S, \dagger, K, J, F, N, P, \cap, \cdot, \varepsilon, \omega\rangle$. *Then* $\models^+ (\alpha)$ (*and* $\models^-$) *is a mapping from the set* Φ *of all formulas to* $2^{M\times S}$ *such that* $(x, l), (y, l) \in \models^+ (\alpha)$ *iff* $(x \cap y, l) \in \models^+ (\alpha)$ $((x, l), (y, l) \in \models^- (\alpha)$ *iff* $(x \cap y, l) \in \models^- (\alpha))$, *that is,* $\models^+ (\alpha) := X \times S' \subseteq M \times S$ $(\models^- (\alpha) := X' \times S'' \subseteq M \times S)$ *where* X (*and* X') *is a filter of* M, *and* S' (*and* S'') *is nonempty.*

Proof. We prove this proposition by induction on the complexity of α. We only show the following.

(*Case* $\alpha \equiv \sim\beta$): Suppose $(x, l), (y, l) \in \models^+ \sim\beta$. Then we have $(x, l), (y, l) \in \models^- \beta$, and hence $(x \cap y, l) \in \models^- \beta$ by the induction hypothesis (for $\models^-$). Thus we obtain $(x \cap y, l) \in \models^+ \sim\beta$. The converse is obvious. □

Definition 35. A *Kripke model* for TSEILLs is a structure $\langle M, S, \dagger, K, J, F, N, P, \cap, \cdot, \varepsilon, \omega, \models^+, \models^-\rangle$ such that

1. $\langle M, S, \dagger, K, J, F, N, P, \cap, \cdot, \varepsilon, \omega\rangle$ is a Kripke frame for TSEILLs,
2. $\models^+$ and $\models^-$ are valuations on $\langle M, S, \dagger, K, J, F, N, P, \cap, \cdot, \varepsilon, \omega\rangle$.

A formula α is *true* in a Kripke model $\langle M, S, \dagger, K, J, F, N, P, \cap, \cdot, \varepsilon, \omega, \models^+, \models^-\rangle$ if $(\varepsilon, l) \models^+ \alpha$ for any $l \in S$, and *valid* in a Kripke frame $\langle M, S, \dagger, K, J, F, N, P, \cap, \cdot, \varepsilon, \omega\rangle$ if it is true for any valuations $\models^+$ and $\models^-$ on the Kripke frame.

Theorem 36 (*Soundness*). *Let* C *be the class of all Kripke frames for* TSEILLs, $L := \{\gamma \mid \text{TSEILLs} \vdash \gamma\}$ *and* $L(C) := \{\gamma \mid \gamma$ *is valid in all frames of* $C\}$. *Then* $L \subseteq L(C)$.

Proof. We prove this theorem by induction on the proof P of γ in TSEILLs. Let $\models$ be a valuation on $\langle M, S, \dagger, K, J, F, N, P, \cap, \cdot, \varepsilon, \omega\rangle \in C$. We only show the following case.

(*Case* $\gamma \equiv \sim[K]\alpha \rightarrow \langle K\rangle\sim\alpha$: N16): We show $(\varepsilon, s) \models^+ \sim[K]\alpha \rightarrow \langle K\rangle\sim\alpha$ for all $s \in S$. Let $x \in M$ be such that $(x, s) \models^+ \sim[K]\alpha$, i.e., $(x, s) \models^- [K]\alpha$. Then there is $y \in M$ such that $Jy \leqslant x$ and $(y, s) \models^- \alpha$ (i.e., $(y, s) \models^+ \sim\alpha$). Therefore $(x, s) \models^+ \langle K\rangle\sim\alpha$. □

Next we prove the completeness theorem. We use Definition 7, and have (the same corresponding) Lemmas 9, 8, 11, 12, and Proposition 10. Moreover, we need the following lemma.

Lemma 37 (*Key lemma*). *Let* Ψ *be the set of all propositional variables,* $\models^+{}_L$ *be a mapping from* Ψ to $2^{M_L\times S}$ *defined by* $\models^+{}_L(p) := \{(x, l) \in M_L \times S \mid [l]p \in x\}$ *and* $\models^-{}_L$ *be a mapping from* Ψ to $2^{M_L\times S}$ *defined by* $\models^-{}_L(p) := \{(x, l) \in M_L \times S \mid \sim[l]p \in x\}$. *Then,* $\models^+{}_L$ (*and* $\models^-$) *can be extended to a mapping from the set* Φ *of all formulas to* $2^{M_L\times S}$, *that is, we have the following*: *for any* $\alpha \in \Phi$, *any* $l \in S$ *and any* $x \in M_L$, (1): $[l]\alpha \in x$ *iff* $(x, l) \models^+{}_L \alpha$ *and* (2): $\sim[l]\alpha \in x$ *iff* $(x, l) \models^-{}_L \alpha$.

Proof. We prove this lemma by (simultaneous) induction on the complexity of α.

- The base step: Straightforward.
- The induction step for (1): We only show the following case. The other cases are the same as those in Lemma 12. (*Case* $\alpha \equiv \sim\beta$): Suppose $[l]\sim\beta \in x$. Then we have $\sim[l]\beta \in x$ by $[l]\sim\beta \rightarrow \sim[l]\beta \in L$ (N27) and $x \in M_L$. Thus we obtain $(x, l) \models^-{}_L \beta$ by the induction hypothesis for (2). Therefore $(x, l) \models^-{}_L \beta$. Conversely suppose $(x, l) \models^+{}_L \sim\beta$, i.e., $(x, l) \models^-{}_L \beta$. Then we obtain $\sim[l]\beta \in x$ by the induction hypothesis for (2). Thus we have $[l]\sim\beta \in x$ by $\sim[l]\beta \rightarrow [l]\sim\beta \in L$ (N26) and $x \in M_L$.
- The induction step for (2):

  (*Case* $\alpha \equiv \sim\beta$): Suppose $\sim[l]\sim\beta \in x$. Then we have $[l]\beta \in x$ by $\sim[l]\sim\beta \rightarrow [l]\beta \in L$ (Proposition 32(12)) and $x \in M_L$. Thus we obtain $(x, l) \models^+{}_L \beta$ by the induction hypothesis (for (1)). Therefore $(x, l) \models^-{}_L \sim\beta$. Conversely suppose $(x, l) \models^-{}_L \sim\beta$, i.e., $(x, l) \models^+{}_L \beta$. Then we have $[l]\beta \in x$ by the induction hypothesis (for (1)). Thus we obtain $\sim[l]\sim\beta \in x$ by $[l]\beta \rightarrow \sim[l]\sim\beta \in L$ (Proposition 32(12)) and $x \in M_L$.

(*Case* $\alpha \equiv [s]\beta$): Suppose $\sim[l][s]\beta \in x$. Then we have $\sim[s]\beta \in x$ by $\sim[l][s]\beta \rightarrow \sim[s]\beta \in L$ (Proposition 32(11)) and $x \in M_L$. Thus we have $(x, s) \models^-{}_L \beta$ by the induction hypothesis. Therefore $(x, l) \models^-{}_L [s]\beta$. Conversely suppose $(x, l) \models^-{}_L [s]\beta$, i.e., $(x, s) \models^-{}_L \beta$. Then we have $\sim[s]\beta \in x$ by the induction hypothesis. By $\sim[s]\beta \rightarrow \sim[l][s]\beta \in L$ (Proposition 32(11)) and $x \in M_L$, we have $\sim[l][s]\beta \in x$.

(*Case* $\alpha \equiv [P]\beta$): Suppose $\sim[l][P]\beta \in x$. Then we have $[P]\sim[l]\beta \in x$ by $\sim[l][P]\beta \rightarrow [P]\sim[l]\beta \in L$ (Proposition 32(10)) and $x \in M_L$. Thus we have $\sim[l]\beta \in Px$, and hence $(Px, l) \models^-{}_L \beta$ by the induction hypothesis. Therefore $(x, l) \models^-{}_L [P]\beta$. Conversely suppose $(x, l) \models^-{}_L [P]\beta$, i.e., $(Px, l) \models^-{}_L \beta$. Then we have $\sim[l]\beta \in Px$ by the induction hypothesis, and hence $[P]\sim[l]\beta \in x$. Thus we have $\sim[l][P]\beta \in x$ by $[P]\sim[l]\beta \rightarrow \sim[l][P]\beta \in L$ (Proposition 32(10)) and $x \in M_L$.

(*Case* $\alpha \equiv \langle K\rangle\beta$): Suppose $\sim[l]\langle K\rangle\beta \in x$. Then we have $[K]\sim[l]\beta \in x$ by $\sim[l]\langle K\rangle\beta \rightarrow [K]\sim[l]\beta \in L$ (Proposition 32(7)) and $x \in M_L$. By Lemma 11(1), (5), we obtain (*) $K(L \cdot \{\sim[l]\beta\}) \subseteq L \cdot \{[K]\sim[l]\beta\} \subseteq x$ and also have $L \cdot \{\sim[l]\beta\} \in M_L$. Moreover, we have $\sim[l]\beta \in L \cdot \{\sim[l]\beta\}$, and hence (**) $(L \cdot \{\sim[l]\beta\}, l) \models^-{}_L \beta$ by the induction hypothesis. Therefore we have $(x, l) \models^-{}_L \langle K\rangle\beta$ by (*) and (**). Conversely, suppose $(x, l) \models^-{}_L \langle K\rangle\beta$. Then there is $y \in M_L$ such that $Ky \subseteq x$ and $(y, l) \models^-{}_L \beta$. By the induction hypothesis, we have $\sim[l]\beta \in y$. Also we have $[K]\sim[l]\beta \rightarrow \sim[l]\langle K\rangle\beta \in L$ by Proposition 32 (7). Thus we have $\sim[l]\langle K\rangle\beta \in Ky \subseteq x$.

(*Cases* $\alpha \equiv !\beta$, $\alpha \equiv [K]\beta$, $\alpha \equiv [F]\beta$ and $\alpha \equiv [N]\beta$): Similar to the case $\alpha \equiv \langle K\rangle\beta$.

(*Case* $\alpha \equiv \mathbf{1}$): Suppose $\sim[l]\mathbf{1} \in x$. Then we have $\sim\mathbf{1} \in x$ by $[l]\sim\mathbf{1} \rightarrow \sim\mathbf{1} \in L$ (N28) and $x \in M_L$. Now we show $(x, l) \models^-{}_L \mathbf{1}$, i.e., $x = \Phi$. $x \subseteq \Phi$ is obvious. We show $\Phi \subseteq x$. Suppose $\gamma \in \Phi$. We have $\gamma \in x$ by $\sim\mathbf{1} \in x$, $\sim\mathbf{1} \rightarrow \gamma \in L$ (N3) and $x \in M_L$. The converse is obvious.

(*Case* $\alpha \equiv \top$): By using N29.

(*Case* $\alpha \equiv \bot$): By using N30.

(*Case* $\alpha \equiv \beta \wedge \gamma$): Suppose $\sim[l](\beta \wedge \gamma) \in x$. Then we have $\sim[l]\beta \vee \sim[l]\gamma \in x$ by $\sim[l](\beta \wedge \gamma) \rightarrow \sim[l]\beta \vee \sim[l]\gamma \in L$ (Proposition 32(2)) and $x \in M_L$. By Lemma 11(1), (2), we obtain (*) $(L \cdot \{\sim[l]\beta\}) \cap (L \cdot \{\sim[l]\gamma\}) \subseteq L \cdot \{\sim[l]\beta \vee \sim[l]\gamma\} \subseteq x$. Moreover, we have $\sim[l]\beta \in L \cdot \{\sim[l]\beta\} \in M_L$ and $\sim[l]\gamma \in L \cdot \{\sim[l]\gamma\} \in M_L$, and hence (**) $(L \cdot \{\sim[l]\beta\}, l) \models^-{}_L \beta$ and $(L \cdot \{\sim[l]\gamma\}, l) \models^-{}_L \gamma$ by the induction hypothesis. By (*) and (**), we have $(x, l) \models^-{}_L \beta \wedge \gamma$. Conversely, suppose $(x, l) \models^-{}_L \beta \wedge \gamma$. Then there are $y, z \in M_L$ such that $y \cap z \subseteq x$, $(y, l) \models^-{}_L \beta$ or $(y, l) \models^-{}_L \gamma$, and $(z, l) \models^-{}_L \beta$ or $(z, l) \models^-{}_L \gamma$. By the induction hypothesis, we obtain $\sim[l]\beta \in y$ or $\sim[l]\gamma \in y$, and $\sim[l]\beta \in z$ or $\sim[l]\gamma \in z$. By A7, A8 and the hypothesis $y \cap z \subseteq x$, we obtain $\sim[l]\beta \vee \sim[l]\gamma \in x$. By Proposition 32(2), we obtain $\sim[l](\beta \wedge \gamma) \in x$.

(*Case* $\alpha \equiv \beta \vee \gamma$): Straightforward by using Proposition 32(3).

(*Case* $\alpha \equiv \beta \rightarrow \gamma$): Suppose $\sim[l](\beta \rightarrow \gamma) \in x$. Then we have $[l]\beta * \sim[l]\gamma \in x$ by Proposition 32(1) and $x \in M_L$. By Lemma 11(1), (3), we obtain (*) $(L \cdot \{[l]\beta\}) \cdot (L \cdot \{\sim[l]\gamma\} \subseteq L \cdot \{[l]\beta * \sim[l]\gamma\} \subseteq x$. Moreover, we have $[l]\beta \in L \cdot \{[l]\beta\} \in M_L$ and $\sim[l]\gamma \in L \cdot \{\sim[l]\gamma\} \in M_L$, and hence (**) $(L \cdot \{[l]\beta\}, l) \models^+{}_L \beta$ and $(L \cdot \{\sim[l]\gamma\}, l) \models^-{}_L \gamma$ by the induction hypothesis. Therefore $(x, l) \models^-{}_L \beta \rightarrow \gamma$ by (*) and (**). Conversely, suppose $(x, l) \models^-{}_L \beta \rightarrow \gamma$, i.e., there are $y, z \in M_L$ such that $y \cdot z \subseteq x$, $(y, l) \models^+ \beta$ and $(z, l) \models^- \gamma$. By the induction hypothesis, we have (*) $[l]\beta \in y$ and (**) $\sim[l]\gamma \in z$. Now we show $[l]\beta * \sim[l]\gamma \in y \cdot z$. By (*) and A9, we have $\sim[l]\gamma \rightarrow [l]\beta * \sim[l]\gamma \in y$. By this and (**), we obtain the claim. Thus $[l]\beta * \sim[l]\gamma \in x$ by the hypothesis $y \cdot z \subseteq x$. By Proposition 32(1), we obtain $\sim[l](\beta \rightarrow \gamma) \in x$.

(Case $\alpha \equiv \beta * \gamma$): Similar to the case $\alpha \equiv \beta \rightarrow \gamma$. We use Proposition 32(4). □

We can prove the following theorem by using the same way used in Section 4.

Theorem 38 (*Completeness*). *Let C be the class of all Kripke frames for* TSEILLs, $L(C) := \{\alpha \mid \alpha$ *is valid in all frames of* $C\}$ *and* $L := \{\alpha \mid \text{TSEILLs} \vdash \alpha\}$. *Then* $L = L(C)$.

We can also show the completeness theorems (w.r.t. the corresponding appropriate Kripke models) for ILLs, EILLs, TILLs, ITLLs, SILLs, SEILLs, TEILLs and TSILLs.

7.2. Adding mingle to TSEILL

We can extend TSEILL by adding the mingle axiom scheme $\alpha \rightarrow \alpha \rightarrow \alpha$, and can prove the completeness theorem by assuming the additional frame condition: $x \cap y \leqslant x \cdot y$ for any $x, y \in M$. This axiom scheme corresponds to the following inference rule in a sequent calculus:

$$\frac{\Gamma_1, \Sigma \Rightarrow \gamma \quad \Gamma_2, \Sigma \Rightarrow \gamma}{\Gamma_1, \Gamma_2, \Sigma \Rightarrow \gamma},$$

and also corresponds to the following transition rule, called a *communication-merge rule* by Milner, in an operational semantics of process algebra:

$$\frac{x \xrightarrow{v} \surd \quad y \xrightarrow{w} \surd}{x||y \xrightarrow{\gamma(v,w)} \surd},$$

where (1) || is the merge operator and represents the parallel execution of two process terms x and y, (2) γ is a communication function for each pair of atomic actions v and w and (3) $x \xrightarrow{v} \surd$ expresses the ability of the process term x to terminate successfully by the execution of action v.

It can be observed in [27] that a sequent calculus for an intuitionistic linear logic with the mingle axiom scheme and some axiom schemes can encode an operational semantics for a process algebra with communication-merge.

7.3. Adding soft exponential to TSEILL

We can extend TSEILL by adding the soft exponential operator $!_s$ introduced by Lafont [35]. The operator $!_s$ is characterized by the multiplexing rule (in a sequent calculus)

$$\frac{\overbrace{\alpha, \ldots, \alpha}^{n}, \Gamma \Rightarrow \gamma}{!_s\alpha, \Gamma \Rightarrow \gamma},$$

where n can be any natural number. By reading this rule from the bottom up, this gives us the useful interpretation: "The expression $!_s\alpha$ means that the resource α is usable in any number, but only once" [24].

We propose that the following axiom schemes are added to TSEILL:

E1: $!_s(\alpha \rightarrow \beta) \rightarrow !_s\alpha \rightarrow !_s\beta$,
E2: $!_s\alpha \rightarrow \alpha$,
E3: $\alpha \rightarrow !_s\beta \rightarrow \alpha$,
E4: $(\overbrace{\alpha \rightarrow \cdots \rightarrow \alpha}^{n} \rightarrow \beta) \rightarrow !_s\alpha \rightarrow \beta \quad \text{for } 2 \leqslant n$,
E5: $!\alpha \rightarrow !_s\alpha$,
E6: $!_s\alpha \rightarrow [N]\alpha$,
E7: $[l]!_s\alpha \rightarrow !_s[l]\alpha$,
E8: $!_s[l]\alpha \rightarrow [l]!_s\alpha$.

The axiom schemes E1–4 are the definition of $!_s$, and E5–8 are additional axiom schemes to represent the relationships between $!$, $!_s$, $[N]$ and $[l]$. Since the modality of $!_s$ (or $[N]$) is weaker than that of $!$ (or $!_s$, respectively), the axiom schemes E5–6 may be a natural choice. For example, the intuitive meaning of E6 is "if the resource α is usable in any number, but only once at the next moment, then α is usable only once at the next moment". It is known that the operator ! represents an infinitely reusable resource, i.e. it is reusable not only for any number, but also many times. On the other hand, $!_s$ represents a consumable resource, i.e. not reusable any number. The frame condition for E4 is

$$\overbrace{x \cdot \cdots \cdot x}^{n} \leqslant \dagger_s x \quad \text{for any } x \in M \text{ and } 2 \leqslant n,$$

where $\dagger_s$ is the operation corresponding to $!_s$. Other frame conditions for the axiom schemes for $!_s$, $!$ and $[N]$ are obtained in a straightforward way. Then, we can prove the completeness theorem for this extended logic by using a similar technique introduced in [23].

7.4. Common knowledge affine logic

The framework presented can be extended to a multi-agent logic with a common knowledge operator $[C]$. We demonstrate a common knowledge intuitionistic affine logic CIAL as an example. The language of CIAL is the language of BIAL by replacing $\Box, \Diamond$ and $[l]$ by $[C]$ (common knowledge) and $[B_i]$ (the agent i knows) for $i \in \{1, 2, \ldots, n\}$. CIAL is obtained from BIAL by replacing the axiom schemes and rules for $\Box, \Diamond$ and $[l]$ by the following axiom schemes and rules: $[B_i]\alpha \wedge [B_i]\beta \rightarrow [B_i](\alpha \wedge \beta)$, $[B_i](\alpha \rightarrow \beta) \rightarrow [B_i]\alpha \rightarrow [B_i]\beta$, $[B_i]\alpha \rightarrow [B_i][B_i]\alpha$, $[C]\alpha \rightarrow \alpha$, $[C]\alpha \rightarrow [B_i]\alpha$, $[C]\alpha \rightarrow [B_i][C]\alpha$, $[C]\alpha \wedge [C]\beta \rightarrow [C](\alpha \wedge \beta)$, $[C](\alpha \rightarrow \beta) \rightarrow [C]\alpha \rightarrow [C]\beta$,

$$\frac{\alpha}{[B_i]\alpha}\ (B_i\text{ness}) \quad \frac{\alpha}{[C]\alpha}\ (C\text{ness})$$

for any $i \in \{1, 2, \ldots, n\}$. Here $[B_i]\alpha$ means "the agent i knows (or believes) α", and $[C]\alpha$ means "the knowledge α is a common knowledge".

We can also construct and define the appropriate Kripke model for CIAL, in which valuation conditions for $[B_i]$ and $[C]$ are similar to that for $\Box$ and $\Diamond$, and the notions "true" and "valid" for the frame and model. Then we can show the completeness theorem using a similar way as in Section 5.

The resource-sensitive interpretations of the operators $[C]$ and $[B_i]$ used in CIAL are not clear, i.e. the relationships between the concepts of "common knowledge", "belief" and "resource" cannot be determined in a general way. Further considerations are thus necessary on this point.

The belief operator $[B_i]$ corresponds approximately to the modal logic K4-type operator, and hence the knowledge axiom scheme $[B_i]\alpha \rightarrow \alpha$ is not assumed. It is known that this axiom scheme is not compatible to "belief" because the statement "if person i believes fact α, then fact α is true" is known not to be true. The common knowledge operator $[C]$ is also a K4-type operator, and adopts the axiom schemes $[C]\alpha \rightarrow [B_i]\alpha$ and $[C]\alpha \rightarrow [B_i][C]\alpha$, which roughly mean that "common knowledge" can be viewed as "belief".

Some examples for the expression of the BAN logic-based authentication protocol verification in [7] are as follows. Suppose that K represents a key for communication between two persons P_1 and P_2. The expression $[C]K$ means that "K is a shared key (i.e. common knowledge) for the two persons", and the expression $[B_i]\alpha$ means that "the information α is believed by P_i". Then, the four completeness conditions of authentication protocols posed in [7] are simply expressed as (1) $[B_1][C]K$, (2) $[B_2][C]K$, (3) $[B_1][B_2][C]K$, and (4) $[B_2][B_1][C]K$, which, respectively, correspond to the BAN logic type expressions of the forms (1) P_1 believes $P_1 \leftrightarrow^K P_2$, (2) P_2 believes $P_2 \leftrightarrow^K P_2$, (3) P_1 believes P_2 believes $P_1 \leftrightarrow^K P_2$, and (4) P_2 believes P_1 believes $P_1 \leftrightarrow^K P_2$.

7.5. Alternative base logics

In this paper, we focused only linear and affine logics as bases of temporal, spatial, epistemic and dynamic logics. However, for these bases, we can adopt other (non-modal) substructural logics not only some positive (i.e. negationless) relevant logics, intuitionistic logic, full Lambek logic FL and Corsi's logic F, but also Visser's logic BPL. These non-modal substructural logics and their Kripke models are discussed in [12,17,20,40,50], and these results are also extended to the present framework by imposing certain modifications. Hence, the method presented in this paper is applicable to a wider class of substructural logics.

8. Resource interpretations

8.1. Interpretations of the proposed logics

In the following, the basic motivation and some intuitive interpretations of the formulations of the proposed logics and semantics are explained from a resource-sensitive and concurrent computational point of view, including an interpretation of the relationships among the operators !, $[K]$, and $[F]$. In the following explanation, however, the question of whether the various spatial, temporal and epistemic aspects can be modularized is not completely solved because of the difficulties of the question.

Resource and time. The temporal operator $[F]$ (any time) used in the proposed logics is the modal logic S4-type operator, and the linear exponential operator ! is an extension of the S4-type operator with the addition of some axiom schemes. This means that $[F]$ has a weaker modality than !, and hence the axiom scheme $!\alpha \rightarrow [F]\alpha$ in the proposed logics may be a natural choice. Although the temporal operator $[N]$ (next time) is the modal logic K-type operator, the modality of $[N]$ is weaker than that of $[F]$, and hence the axiom scheme $[F]\alpha \rightarrow [N]\alpha$ may also be a natural choice. The axiom scheme $[F]\alpha \rightarrow [N]\alpha$ has a natural resource interpretation: "If the resource α is usable once at any time in the future, then α is usable only once, at the nearest time in the future".

The idea of combining !, $[F]$, and $[N]$ explained above was originally proposed by Hirai, and the idea leads to practical application to a timed Petri net specification [15,16]. In the proposed logics, the formulation of the temporal operator $[P]$ (past) is independent of $[F]$ and $[N]$. This means that "future" and "past" are independent. However, this interpretation may be questionable, and further considerations on this point may be needed.

The intuitive meaning of the axiom scheme $!\alpha \rightarrow [F]\alpha$, which describes the relationship between "resource" and "time", is "if the resource α is reusable as many times as needed at any time in the future, then α is usable once at any time in the future". In this interpretation, "*time*" *is regarded as a* "*resource*". This intuitive meaning may be justified in that the concept of "time" in computer systems, such as CPU-time in process scheduling, is considered to be a "resource". Similarly, in the real world, "time is money".

Knowledge and resource. The knowledge operator $[K]$ (knows) used in the proposed logics is the modal logic S5-type operator, and it is not compared with !, i.e. the modality of $[K]$ is not weaker than that of !, and vice versa. The axiom scheme $[K]\alpha \rightarrow !\alpha$ in the proposed logics is thus not a natural choice, but this axiom scheme is useful in some cases. The meaning of the axiom scheme $[K]\alpha \rightarrow !\alpha$, which describes the relationship between "knowledge" and "resource", is explained as follows. For this axiom scheme, if α represents an infinitely reusable resource in computer systems, such as a login password and network protocol, then the interpretation "if α is known, then α is reusable" may be sound, e.g. the following expression is appropriate:

$$[K]\mathit{login\text{-}password}(\mathit{john}) \rightarrow !\ \mathit{login\text{-}password}(\mathit{john}).$$

This means "if John knows the login password for the personal computer used, then John can use the password as many times as needed to use the computer". On the other hand, if α represents a finitely usable (or consumable) resource, such as money, then such an interpretation is not valid. Thus, in the latter case, the axiom scheme must be removed from the underlying logics. Of course, the resulting logics have the complete Kripke semantics with appropriate modifications.

Knowledge, *information and resource.* Although it is difficult to determine the relationship between "knowledge" and "information", in the proposed semantic framework, "knowledge" is dealt with as an operator of "information", i.e. "*knowledge*" *is regarded as a special case of* "*information*". In fact, in the proposed Kripke frames, the knowledge operator K is a unary function on the set M of information pieces. Thus, a reasonable description of "information" may be "integrated data", whereas "knowledge" is "meaningful or internalized information".

Using this description, an appropriate use of $[K]$ is that $[K]$ is only applicable to "information", "proposition" or "fact". For example, suppose that

$$\mathit{medicine}(\mathit{john}) * \mathit{medicine}(\mathit{john}) \rightarrow \mathit{recover}(\mathit{john})$$

means "if John takes two medicines in one day, then John can recover from the disease". Then, the following expression may be sound:

$$[K](\mathit{medicine}(\mathit{john}) * \mathit{medicine}(\mathit{john}) \rightarrow \mathit{recover}(\mathit{john})).$$

This means "John knows the effect of the medicine", i.e. John knows the information about the medicine. Assuming the axiom scheme $[K]\alpha \rightarrow !\alpha$, the following formula may also be sound:

$$!\ (\mathit{medicine}(\mathit{john}) * \mathit{medicine}(\mathit{john}) \rightarrow \mathit{recover}(\mathit{john})).$$

This means "John can reuse the information about the medicine".

Space and resource. The difficulty in determining the relationships among "time", "knowledge" and "resource" has been discussed, but it is also difficult to determine the relationships among "space", "time", "knowledge" and "resource". In the proposed framework, the following practical description is adopted: "*Space*" *is almost independent of* "*time*", "*knowledge*" *and* "*resource*". However, this description may be problematic in a wide range of applications in computer science. For example, it is possible to isolate the temporal aspects from the spatial aspects? Further consideration is thus necessary for such a problem, but the spatial operators $[l_i]$ are nevertheless useful in some restricted situations such as in distributed systems.

The proposed setting of the spatial operators $[l_i]$ represents the discrete space (or location) interpretation in which a location is regarded as a point and is independent of other locations. The validity of axiom schemes related to the other connectives, such as $[l](\alpha \wedge \beta) \leftrightarrow [l]\alpha \wedge [l]\beta$, $[l](\alpha \rightarrow \beta) \leftrightarrow [l]\alpha \rightarrow [l]\beta$ and $[l]!\alpha \leftrightarrow ![l]\alpha$, means that the location operators are not influenced by other connectives. For example, $[l](\alpha \wedge \beta) \leftrightarrow [l]\alpha \wedge [l]\beta$ means "if the fact $\alpha \wedge \beta$ holds at location l, then both α and β also hold at l, and vice versa". Mathematically speaking, the setting for $[l]$ is the same as that of the involution operator, which appears in the algebraic structure called the "involutive quantale" by Mulvey and Pelletier (for detailed information, see e.g. [25]).

A good application of $[l_i]$ may be distributed concurrent systems. For example, let the space domain S be $\{computer\_1, computer\_2, computer\_3\}$. Then, the following expression can be used:

$[computer\_i][F]!\,password$ for any $i \in \{1, 2, 3\}$.

This means "the password can be entered as needed at any computer in the office at any time".

An application to distributed concurrent linear logic programming language was proposed by Kobayashi et al. [32]. Distributed concurrent linear logic programming aims to introduce the spatial operators into the underlying logic in order to express location-dependent computation. From a computational point of view, the expression $[l]\alpha$ means "execute process α at location l". A range of issues in distributed computation, such as location-dependence/independence of names, movement of computation and locations as first-class data, have been comprehensively discussed [32].

Program and resource. The proposed construction of program operators in DIAL is a slight modification of standard dynamic logic (without the test operator). In general, the expression $[c]\alpha$ intuitively states "it is necessary that after executing the program c, the information α must be true". In this statement, the part "the information α must be true" is replaced by "the resource α is usable". In this modified reading, the term "resource" may be consistent with the concept of resources in computer systems, such as CPU-time and memory.

Resource interpretations for modal operators. Using the intuitive meanings of the modal operators $[K]$, !, $[F]$, $[N]$, $[P]$, $[c]$ and $[l]$, the following useful interpretations for some formulas can be provided:

- $[K]\alpha$: "The information (or fact) α is reusable as many times as needed (i.e. known) at any time".
- $!\alpha$: "The resource (or program) α is reusable (or executable) at any time in the future (i.e. it is reusable as many times as needed at any time)".
- $[F]\alpha$: "The resource α is usable exactly once at any time (i.e. it is consumed after use)".
- $[N]\alpha$: "The resource α is usable only once at the next moment (i.e. it is usable only once at the nearest time in the future)".
- $[P]\alpha$: "The resource α was usable in the past".
- $[c]\alpha$: "It is necessary that after executing the program c, the resource α must be usable".
- $[l]\alpha$: "The resource α is usable at the location l".
- α: "The resource α is usable once at the present moment (i.e. it is usable only once, and must be used only at the present time)".

Resource interpretation for strong negation. The construction of the strong negation axiom schemes of TSEILLs may be regarded as a natural extension of Wansing's WILL [50]. Indeed, the modal operator-free part of TSEILLs corresponds to WILL. The additional self dual axiom scheme $\sim!\alpha \leftrightarrow !\sim\alpha$ (i.e. N14 and N15) is a natural analogue of the axiom scheme $\sim(\alpha * \beta) \leftrightarrow \sim\alpha * \sim\beta$ (i.e. N12 and N13), since ! is known as the infinite version of $*$. In the same policy, the axiom schemes for $[F]$, $[N]$, $[P]$ concerning $\sim$ are constructed, and the axiom schemes for $[K]$ and $\langle K\rangle$ concerning $\sim$ are formulated as the De Morgan duality and satisfying the negative introspective axiom scheme.

An intuitive meaning of the strong negation connective $\sim$ can then be explained based on "refutability interpretation". The following resource-sensitive interpretation is proposed.

- $\sim\alpha$: "We can refute the fact that the resource α is available at the present time (i.e. α is not available at the present time)".

An alternative interpretation of $\sim$ is the "stop-action" of processes in concurrency theory. Following [22], such a concurrency theoretic meaning is as follows. A sequent system for WILL is formulated using the following inference rules with respect to $\sim$ and $*$:

$$\frac{\alpha, \Gamma \Rightarrow \gamma}{\sim\sim\alpha, \Gamma \Rightarrow \gamma}\ \text{(restoration)} \quad \frac{\sim\alpha, \sim\beta, \Gamma \Rightarrow \gamma}{\sim(\alpha * \beta), \Gamma \Rightarrow \gamma}\ \text{(parallel suspension)}.$$

The following interpretations can be obtained by reading each of the rules from the bottom up.

- (restoration): stop-action $\sim\alpha$ is removed, i.e. suspended process α is restored.
- (parallel suspension): stop-action $\sim(\alpha * \beta)$ halts two processes α and β simultaneously.

It is known that $\sim$ can express *inexact predicates* and can allow *paraconsistency* [49,50]. An inexact predicate is an incomplete predicate in an empirical domain. An example of an inexact predicate is a color predicate such as "$green(x)$" which means that "x is green". This predicate is incomplete in the sense that we cannot determine exactly that the formula $\sim green(x) \vee green(x)$ is true. An alternative example is a disease (or symptom) predicate such as $melancholia(x)$, which means that "a person x suffers from the first-stage melancholia". This type of disease predicate will be used in an illustrative example in the latter section. It is also known that logics with paraconsistency can handle inconsistency-tolerant reasoning more appropriately. Since the paraconsistency ensures the fact that for some formulas α and β, the formula $\sim \alpha \wedge \alpha \rightarrow \beta$ is not provable, an inconsistent large medical knowledge-base can be described naturally.

8.2. Interpretations of the proposed semantics

In the following, three interpretations of the proposed semantics for the underlying logics are briefly explained.

Informational interpretation. An informational interpretation for the proposed Kripke semantics for TSEILLs is explained below. The Kripke model $\langle M, S, \dagger, K, J, F, N, P, \cap, \cdot, \varepsilon, \omega, \models^+, \models^- \rangle$ for TSEILLs provides the following informational interpretation:

- M is a set of information pieces,
- S is a set of locations (or spaces),
- $\dagger$ is the infinite addition of information pieces,
- K, J, F, N and P are unary operations on M, corresponding to the modal operators $[K]$, $\langle K\rangle$, $[F]$, $[N]$ and $[P]$, respectively,
- $\cap$ is the intersection of information pieces,
- $\cdot$ is the addition of information pieces,
- ε is an empty piece of information,
- ω is the greatest piece of information,
- $(x, l) \models^+ \alpha$ is interpreted as "α is obtained at the location $l \in S$ using the information piece x",
- $(x, l) \models^- \alpha$ is interpreted as "we can refute the fact that α is obtained at the location $l \in S$ using the information piece x, i.e. α is not obtained at l using x".

In this setting, using the fact $F\varepsilon = \varepsilon$ and the hereditary condition on $\models^+$, $(F\varepsilon, l) \models^+ \alpha$ for any $l \in S$ means $(\varepsilon, l) \models^+ [F]\alpha$ for any $l \in S$. Then, this gives the following intuitive understanding of the "eternal truth" in the real-world:

> "If the proposition α is obtained by using zero-information at any time in the future for all spaces in the world, then α is the eternal truth in the world".

This interpretation is called in this paper the "spatio-temporal informational truth assertion". This means that the eternal truth is time-, space- and information-independent. This fact may be instructive in some natural scientists and engineers. Scientists desire to discover a scientific eternal truth, i.e. an invaluable truth in the real world. Such an invaluable truth may be time- and space-independent in the following sense: the truth may be accepted for centuries (i.e. it is time-independent), and may also be accepted in a large circle of scientists (i.e. it is space-independent).

Petri net interpretation. Following [24], a Petri net interpretation of the underlying Kripke semantics is explained below. Roughly speaking, a substructure $\langle M, \cap, \cdot, \varepsilon\rangle$ of the Kripke frame for TSEILLs is regarded as a partially ordered commutative monoid $\langle M, \geqslant, \cdot, \varepsilon\rangle$ where $\geqslant$ is a partial order defined by $\cap$. It can be shown that the partially ordered commutative monoid determines a Petri net structure $\langle N, \gg, +, \emptyset\rangle$ where

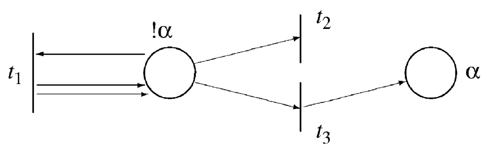
1. N is the set of all markings,
2. $\gg$ is a reachability relation on N,
3. $+$ is the multiset union operation on N,
4. $\emptyset$ is an empty multiset.

It is shown in [25] that the Petri net structure $\langle N, \gg, +, \emptyset\rangle$ just corresponds to a canonical Kripke structure $\langle M, \geqslant, \cdot, \varepsilon\rangle$ based on the sequent calculus for the non-modal $\vee$-free intuitionistic linear logic ($\mathrm{ILL}^{\vee}$), i.e.

1. $M := \{\Gamma \mid \Gamma$ is a finite multiset of formulas$\}$,
2. $\Gamma \geqslant \Delta$ is defined by $\mathrm{ILL}^{\vee} \vdash \Gamma \Rightarrow \Delta^{\star}$ where $\Delta^{\star} \equiv \gamma_1 * \cdots * \gamma_n$ if $\Delta \equiv \{\gamma_1, \ldots, \gamma_n\}$ $(0 < n)$ and $\Delta^{\star} \equiv \mathbf{1}$ if Δ is empty, [13]
3. $\Gamma \cdot \Delta := \Gamma \cup \Delta$ (the multiset union),
4. ε is an empty multiset.

Using this basic correspondence, the reachability in Petri net can be viewed as the provability of the logic, i.e. $\Gamma \gg \Delta$ corresponds to $\mathrm{ILL}^{\vee} \vdash \Gamma \Rightarrow \Delta^{\star}$. Moreover, using some extended Kripke frames with the operators $\dagger$, F and N, which correspond to !, $[F]$ and $[N]$, respectively, the Petri net interpretations of !, $[F]$ and $[N]$ may be naturally obtained, and the Petri net interpretation of $[l_i]$ may also be closely related to algebraic high-level net. The strong negation is roughly interpreted as inhibitor arc [29].

The following example for Petri net interpretation of ! is from [18] and the Kripke semantics-based interpretation is from [24]. Suppose that $N := \langle P, T, (\cdot)^{\bullet}, (\cdot)_{\bullet}\rangle$ is a Petri net such that $P := \{!\alpha, \alpha\}$ (a set of places), $T := \{t_1, t_2, t_3\}$ (a set of transitions), $t_1^{\bullet} := \{!\alpha\}$ (the pre-multiset of t_1), $t_{1_{\bullet}} := \{!\alpha\}$ (the post-multiset of t_1), $t_2^{\bullet} := \{!\alpha\}$ (the pre-multiset of t_1), $t_{2_{\bullet}} := \{\}$ (the post-multiset of t_2), $t_3^{\bullet} := \{!\alpha\}$ (the pre-multiset of t_3) and $t_{3_{\bullet}} := \{\alpha\}$ (the post-multiset of t_3). Then, graphically this becomes the following:



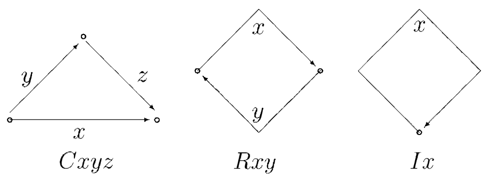
This net corresponds to the facts $\vdash !\alpha \rightarrow !\alpha * !\alpha$, $\vdash !\alpha \rightarrow \mathbf{1}$ and $\vdash !\alpha \rightarrow \alpha$. In this net, the place $!\alpha$ (if $!\alpha$ has a token) can produce a number of tokens in many times (i.e. as many as needed). Thus, this interpretation elegantly expresses the computational role of !.

In conclusion, the proposed Kripke model produces an intuitive Petri net interpretation for the underlying logics. It is remarked that such a Kripke semantics-based Petri net interpretation using the partially ordered (or pre-ordered) commutative monoid is also provided by Pym et al. [42] for the logic BI of bunched implications.

Arrow interpretation. Arrow logics were introduced by van Benthem in order to express a wide range of "transition mechanism or action dynamics", and have been widely studied by many logicians (see e.g. [5,6]). In arrow logics, the notion of arrow means an abstraction of a basic transition mechanism. A Kripke frame for these logics has three kinds of relations C, R and I on a set W of arrows, which, respectively, mean

- $Cxyz$: "x is the composition of y and z",
- Rxy: "x is the converse arrow of y",
- Ix: "x is an identity arrow".

Graphically, this becomes the following:



A language of an arrow logic is constructed by using some constant and operators: $\hat{\mathbf{1}}$ (identity), which intuitively means "skip", $\otimes\alpha$ (converse), which means "α conversely", and $\alpha \circ \beta$ (composition), which means "first α, then β".

[13] $\mathrm{ILL}^{\vee} \vdash \Gamma \Rightarrow \Delta^{\star}$ means that the sequent $\Gamma \Rightarrow \Delta^{\star}$ is provable in $\mathrm{ILL}^{\vee}$.

These constant $\hat{\mathbf{1}}$, operators $\otimes$ and $\circ$ are respectively analogous to $\mathbf{1}$, ! and $*$ in the linear logic language. Such a correspondence is explained below. An arrow model is a structure $\langle W, C, R, I, \models \rangle$ where W is a non-empty set of arrows, C, R and I are a ternary, a binary and a unary relation on W, respectively, $\models$ is a valuation, and satisfying

1. $x \models \hat{\mathbf{1}}$ iff Ix,
2. $x \models \otimes\alpha$ iff $y \models \alpha$ for some y with Rxy,
3. $x \models \alpha \circ \beta$ iff $y \models \alpha$ and $z \models \beta$ for some y and z with $Cxyz$.

This arrow model corresponds to a substructure $\langle M, \leqslant, \cdot, \dagger, \varepsilon, \models \rangle$ of the proposed Kripke model, e.g. the valuation $\models$ of this model is defined as

1. $x \models \mathbf{1}$ iff $\varepsilon \leqslant x$,
2. $x \models\ !\alpha$ iff $y \models \alpha$ for some y with $\dagger y \leqslant x$,
3. $x \models \alpha * \beta$ iff $y \models \alpha$ and $z \models \beta$ for some y and z with $y \cdot z \leqslant x$.

Indeed, this substructure is regarded as a Kripke model for the $\{*, !, \mathbf{1}\}$-fragment of the intuitionistic linear logic. It is remarked that this correspondence is not precise, because the exact frame conditions are not assumed in this discussion. But, considering these similarities may be valuable for a further research.

9. Illustrative examples

9.1. Medical system

Constructing a good medical reasoning system is one of the main objectives of AI. For example, various expert (and database) systems for medical diagnosis have been proposed based on a number of logical systems. In the following, two examples of medical reasoning based on TSEILLs are presented as a prototype of such applications. In these examples, the concept of resource-sensitivity is useful for describing the consumption of medicines, and the linear exponential, temporal, spatial and strong negation operators are also useful for representing reusable medicines, the passage of time, the locations of doctors (or hospitals), and inexact symptoms, respectively.

Medical diagnosis. The following criterion for symptoms are provided. If more than two doctors or pathologists answer that a person x has (or do not have) a symptom s, then this judgment is accepted. This majority decision rule is expressed formally as follows:

$\vdash\ !\,ans\_has\_s(x) * (ans\_has\_s(x))^3 \rightarrow s(x)$,

$\vdash\ !\,ans\_has\_no\_s(x) * (ans\_has\_no\_s(x))^3 \rightarrow {\sim} s(x)$,

where an expression α^n for any formula α means $\overbrace{\alpha * \cdots * \alpha}^{n}$, and "$\vdash$" means the provability (as an axiom) of the underlying logic.[14]

Suppose that the judgement explained above is valid. A very simple example for the diagnosis of a disease d based on three symptoms s_1, s_2 and s_3 is presented below as a modified version of an example posed in [38]. Suppose that $[N]^m$ denotes $\overbrace{[N] \cdots [N]}^{m}$. The opinions of a doctor are then expressed as follows:

O1: $\vdash [F]s_1(x) * [F]s_2(x) * s_3(x) \rightarrow [N]^3 d(x)$,

O2: $\vdash s_1(x) * s_2(x) \rightarrow [N]^3 {\sim} d(x)$.

These expressions mean "if a person x always has two symptoms s_1 and s_2, and also has a symptom s_3 at the present time, then x will have the disease d after 3 years", and "if a person x has two symptoms s_1 and s_2 only at the present

---

[14] By using !, we can express a number of realistic situations. An expression $!\alpha$ means "we can use α in any number including zero." or "we can use α infinitely." Thus, α is approximately expressed as $\overbrace{\alpha * \cdots * \alpha}^{n}$ where n can be any natural number. Then, $!\alpha * \alpha$ means "we can use the positive number of α (i.e. we can use α more than zero)." Also, $!(\alpha * \alpha)$ means "we can use the even number of α." Thus, as presented in the majority decision example, $!\alpha * (\alpha * \alpha * \alpha)$ may express "we can use α more than two."

time, then x will not have the disease d after 3 years", respectively. It is assumed that the following information is obtained by pathologists through tests conducted for an individual "John":

$\vdash [F]s_1(john), \quad \vdash [F]s_2(john), \quad \vdash s_3(john).$

We then ask the question: "Will John have d after 5 years?", i.e. "$\vdash [N]^5 d(john)$?". The answer is "Yes", i.e. the following derivation can be obtained using the axiom scheme A9: $\alpha \rightarrow (\beta \rightarrow (\alpha * \beta))$, the rule ($N$ness) and the opinion O1:

$$\dfrac{\dfrac{\dfrac{s_3(john) \quad \dfrac{\dfrac{[F]s_2(john) \quad \dfrac{[F]s_1(john) \quad \text{A9}}{[F]s_2(john) \rightarrow [F]s_1(john) * [F]s_2(john)}}{[F]s_1(john) * [F]s_2(john) \qquad \text{A9}}}{s_3(john) \rightarrow [F]s_1(john) * [F]s_2(john) * s_3(john)}}{[F]s_1(john) * [F]s_2(john) * s_3(john) \qquad \text{O1}}}{\dfrac{\dfrac{[N]^3 d(john)}{[N]^4 d(john)}}{[N]^5 d(john)}}.$$

On the other hand, if the following information is assumed:

$\vdash s_1(john), \quad \vdash s_2(john),$

then we obtain the answer "No" for the question, i.e. John will not have the disease:

$$\dfrac{\dfrac{s_2(john) \quad \dfrac{s_1(john) \quad \text{A9}}{s_2(john) \rightarrow s_1(john) * s_2(john)}}{s_1(john) * s_2(john) \qquad \text{O2}}}{\dfrac{\dfrac{[N]^3 {\sim} d(john)}{[N]^4 {\sim} d(john)}}{[N]^5 {\sim} d(john)}}.$$

If the person dies, the state of death may also be judged by a number of doctors. This "death diagnosis" problem can be handled using the spatial operators in an axiomatic expression: for $S = \{doctor\_1, doctor\_2, doctor\_3\}$,

$[doctor\_1]death(x) * [doctor\_2]death(x) * [doctor\_3]death(x) \rightarrow death(x).$

This means "if three doctors in the hospital say that a person x is dead, then x is determined by the doctors to be dead". In this case, the following alternative expression can be obtained using the space induction rule:

$$\frac{[doctor\_1]death(x) \quad [doctor\_2]death(x) \quad [doctor\_3]death(x)}{death(x)} \ \text{(space2)}$$

and by the space necessitation rule

$$\frac{death(x)}{[doctor\_i]death(x)} \ \text{(space1)}$$

for the inverse situation.

Medical treatment. Reasoning on medical treatment based on the underlying logic may be achieved as follows. It is known that the linear implication $\rightarrow$ means the consumption of resources, and medicine m can be considered to be a resource. Then, the expression $m(x) \rightarrow recover(x)$ means "if a person x use medicine m to recover from a disease, then x makes a recovery from the disease". Then, the following useful expressions can be obtained:

- $[N]^{30} medicine(x)$: "The medicine can be taken 30 minutes after a meal".
- $dinner(x) * [N]^{30} medicine(x) \rightarrow [N]^{60} recover\_symptom(x)$: "If dinner is taken, and the medicine is taken 30 minutes after the dinner, then the symptom will be alleviated 60 minutes after the dinner".

The following situation can be considered: John has medicines m_1 (water, which can be used as many times as needed), m_2 (one pill) and m_3 (two pills and one powder, which have the same ingredient). This situation is expressed formally as follows:

$\vdash !\, m_1(john)$, $\vdash m_2(john)$, $\vdash m_3(john)^2$, $\vdash m_3(john)$.

We also have the following hypothesis for recovery to recover from disease d:

H: $\vdash !\, m_1(x) * m_2(x) * m_3(x)^2 \rightarrow recover\_d(x)$.

Then, the following question arises: Can John recover from d? Our answer is "Yes", i.e. the fact $\vdash recover\_d(john)$ is derived using the hypothesis H and the axiom scheme A9: $\alpha \rightarrow (\beta \rightarrow (\alpha * \beta))$ as

$$\dfrac{\dfrac{m_3(john)^2 \qquad \dfrac{\dfrac{m_2(john) \qquad \dfrac{!m_1(john) \quad \text{A9}}{m_2(john) \rightarrow !m_1(john) * m_2(john)}}{!m_1(john) * m_2(john)} \qquad \text{A9}}{m_3(john)^2 \rightarrow !m_1(john) * m_2(john) * m_3(john)^2}}{!m_1(john) * m_2(john) * m_3(john)^2} \qquad \text{H}}{recover\_d(john)} .$$

9.2. Distributed system

By modifying the context discussed in [24], the following new example is addressed. A network system consists of three distinct computers: Computer-1, Computer-2 and Computer-3, which cannot be used now because the computers are down, and thus, the network system is also unavailable. Then, we hope that the computers are up, and we desire to use the resource programs which are distributed on the network system. The resource programs are called here a net-resource. Moreover, the following facts are assumed:

(1) we know a login-password for any-time at Computer-i for any $i \in \{1, 2, 3\}$;
(2) if we can use as many login-password as needed, then we can use the net-work oriented operating system as many as needed after 2-time units;
(3) if we can use the operating system as many as needed after m-time units, then we can use the net-resource after $(m + 2)$-time units.

We then try to solve the question: "Can we use the net-resource after 6-time units at Computer-2 ?"

Using TSEILL-formulas, the situation above is described formally as follows. The spatial domain S is $\{co_1, co_2, co_3\}$ where co_i means Computer-i ($i \in \{1, 2, 3\}$). In the following discussion, "*pas*" means the login-password, "*os*" means the operating system and "*net*" means the net-resource. The assumptions presented are described as follows.

(1) $\vdash [co_i][F][K]pas$ (for any $i \in \{1, 2, 3\}$),
(2) $\vdash !pas \rightarrow ![N]^2 os$,
(3) $\vdash ![N]^m os \rightarrow [N]^{m+2} net$.

Then, our question is expressed as follows.

$\vdash [co_2][N]^6 net$?

We can answer this question in the following, using assumptions (1)–(3), rules (space1), (space2), (Nness) and the axiom schemes T2: $[F]\alpha \rightarrow \alpha$ and R1: $[K]\alpha \rightarrow !\alpha$.

$$\dfrac{\dfrac{\dfrac{[co_i][F][K]pas \quad (i \in \{1,2,3\})}{[F][K]pas} \qquad [F][K]pas \rightarrow [K]pas}{[K]pas} \qquad [K]pas \rightarrow !pas}{!pas} ,$$

and then

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\begin{array}{c}\vdots\\ !pas\end{array} \quad !pas\to ![N]^2 os}{![N]^2 os} \qquad ![N]^2 os\to [N]^4 net}{[N]^4 net}}{[N]^5 net}}{[N]^6 net}}{[co_2][N]^6 net.}\ .$$

We thus obtain the answer, “Yes”.

9.3. Secret sharing system

Suppose that three bank clerks b_1, b_2 and b_3 in a bank have the privilege of opening the safe of the bank. In a point of view of safety, the safe can be opened by the agreement of all the bank clerks, i.e. the key of the safe is both sharing in the bank clerks and splitting them into three parts. [15] The following conditions are then assumed:

(1) $\vdash [b_i]key$ $(i \in \{1, 2, 3\})$,
(2) $\vdash [K]pas$,
(3) $\vdash ID$,
(4) $\vdash pas * key * ID \to open$,

where (1) means that the bank clerk b_i has a part of the shared key, $[b_i]key$, (2) means that the three bank clerks know the password of the safe, (3) means that three bank clerks have the identification cards, and (4) means that if the password, the identification cards and the complete shared key, key, are completely and simultaneously obtained, then the safe can be opened.

Then, a situation of opening of the safe is expressed formally as follows:

$$\cfrac{\cfrac{(3) \qquad \cfrac{\cfrac{\cfrac{(1)}{key}\,(*) \qquad \cfrac{\cfrac{\cfrac{(2) \quad [K]pas\to !pas}{!pas} \qquad !pas\to pas}{pas}\ \text{A9}}{key\to pas * key}}{pas * key}\ \text{A9}}{ID\to pas * key * ID}}{pas * key * ID} \qquad (4)}{open}\ ,$$

where (*) represents the situation of the secret sharing, by using the rule (space2). In this example, the agreement between the clerks is expressed by the rule (space2) and assumption (1). For example, if we have no $\vdash [b_1]key$, then we cannot apply (space2). Then, the agreement is failed, and hence the safe cannot be opened.

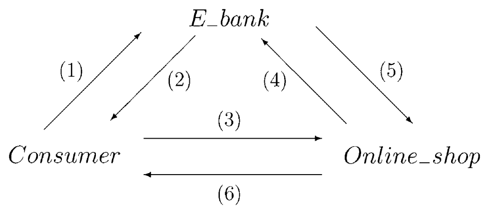
9.4. Electronic money system

In the following, a usual net/coin-type electronic money system is considered. Typical problems of such an electronic money system are regarded as (P1): the problem of the double use of electronic coins, i.e. electronic coins are duplicated by fakers, and (P2): the counterfeit coin problem. To prohibit such situations, the electronic coins, which are really electronic data such as prime numbers, are issued by an electronic bank by imposing the discrimination number to each coin. Suppose that the space domain $S = \{1, \ldots, n\}$ is a set of discrimination numbers of electronic coins, and

[15] Various similar (but more complex) situations have been discussed in a theory of secret sharing systems, such as multiparty protocols in cryptology.

the same discrimination number is, of course, not allowed for such coins. The expression $[i]$*e-coin* means that the electronic coin *e-coin* has discrimination number i. A real coin, i.e. cash, is also expressed as *coin*.

We then consider the following situation: (1) a consumer sends some real coins to an electronic bank to get some electronic coins, (2) the electronic bank issues some electronic coins with the discrimination numbers, (3) the consumer sends by e-mail some electronic coins to a online shop and enters the password in the web site of the shop to get a goods such as web contents, (4) the online shop sends the electronic coins received by the consumer to the electronic bank, (5) the electronic bank sends the corresponding cash to the online shop, and (6) the online shop sends the goods to the consumer.



Such a situation is expressed formally as follows:

$$(1), (2): \quad \vdash coin * coin \rightarrow [i]e\text{-}coin * [j]e\text{-}coin, \quad i \neq j \text{ and } i, j \in S,$$
$$(3), (6): \quad \vdash [i]e\text{-}coin * [j]e\text{-}coin * pas \rightarrow goods, \quad i \neq j \text{ and } i, j \in S,$$
$$(4), (5): \quad \vdash [i]e\text{-}coin * [j]e\text{-}coin \rightarrow coin * coin, \quad i \neq j \text{ and } i, j \in S.$$

Then, the situation of problem (P1) can be blocked by assuming the following condition:
$\vdash [i]e\text{-}coin * \cdots * [k]e\text{-}coin * [k]e\text{-}coin * \cdots * [j]e\text{-}coin \rightarrow prohibition.$
The situation of problem (P2) can also be blocked by assuming the following condition:
$\vdash [i]e\text{-}coin * \cdots * [k]e\text{-}coin * \cdots * [j]e\text{-}coin \rightarrow prohibition, \quad k \notin S.$

10. Concluding remarks

10.1. Decidability and complexity

Although the completeness theorems with respect to Kripke-type semantics were proved for the proposed logics, such as TSEILL, TSEILLs, BIAL and DIAL, other important logical properties, such as decidability, complexity and cut-eliminability, were not discussed in this paper, because of some difficulties in proving these properties. In the following, we review some related topics on the decidability, undecidability and complexity results of linear and affine logics, and explain the difficulties of showing decidability of some related logics.

Decidability: *overview.* It is known that the intuitionistic linear logic ILL (with !) is undecidable [36]. Thus, all the extended logics, which are extensions of ILL, may also be undecidable. On the other hand, some sublogics without ! are shown to be decidable. For example, WILL by Wansing, which is a sublogic of TSEILLs, the !-free fragment of ILL, and the !-free fragment of ITLL by Hirai can be decidable. It is also known that the first-order predicate intuitionistic linear logic without ! is decidable, but the second-order predicate intuitionistic linear logic without ! is undecidable. However, in the present works, the decidability of the underlying sublogics without !, such as TSEILL without !, is unknown. It is conjectured that such sublogics without ! are decidable. It is also known that the intuitionistic affine logic (with !) is decidable [33]. Then, are BIAL and DIAL decidable? This also remains unsolved. To solve this problem, it may be necessary to introduce the corresponding sequent calculi with the cut-elimination property for these logics, because, in the cases of linear and affine logics, there is no good purely semantic method that can derive decidability. Similar to the linear logic case, it is known that the first-order (second-order) intuitionistic affine logic without ! is decidable (undecidable, respectively). In both the linear logic and affine logic cases, the situations for the decidability and undecidability of the classical versions are almost the same as that of the intuitionistic versions.

Role of contraction. In general, it is difficult to show the decidability of substructural logics with the full *contraction rule* but without the full *weakening rule*, which structural rules are, respectively, of the forms (in a sequent calculus):

$$\frac{\alpha, \alpha, \Gamma \Rightarrow \gamma}{\alpha, \Gamma \Rightarrow \gamma}\ (\text{co}), \quad \frac{\Gamma \Rightarrow \gamma}{\alpha, \Gamma \Rightarrow \gamma}\ (\text{we}),$$

which, respectively, correspond to the Hilbert-style axiom schemes $(\alpha \rightarrow \alpha \rightarrow \beta) \rightarrow \alpha \rightarrow \beta$ and $\alpha \rightarrow \beta \rightarrow \alpha$. For example, in order to show the decidability of the implicational fragment of the logic R of relevant implication or equivalently the implicational fragment of ILL with (co), we must use a very complex way, called Kripke's method. It is known that R is undecidable, but R without the contraction axiom scheme is decidable. It is also known that the decision problem for the implicational fragment of the logic T of ticket entailment is a long standing open problem (for more information, see e.g. [2]). [16] By the explanation above, it can be seen that the contraction rule (or axiom scheme) has a crucial role to show the decidability of substructural logics.

Role of restricted contraction. Roughly speaking, a cause that derives the undecidability of the linear logics with ! is the presence of the restricted contraction rule of the form:

$$\frac{!\alpha, !\alpha, \Gamma \Rightarrow \gamma}{!\alpha, \Gamma \Rightarrow \gamma}\ (!\text{co}),$$

which corresponds to the Hilbert-style axiom scheme: $(!\alpha \rightarrow !\alpha \rightarrow \beta) \rightarrow !\alpha \rightarrow \beta$. A bottom-up proof search with respect to (!co) can derive an *infinite path* (in a proof search tree) such as

$$\begin{array}{c} \infty \\ \vdots\ (!\text{co}) \\ !\alpha, \ldots, !\alpha, \Gamma \Rightarrow \gamma \\ \vdots\ (!\text{co}) \\ !\alpha, \Gamma \Rightarrow \gamma. \end{array}$$

The decision problem for the multiplicative exponential fragment of ILL is known as an open problem. A notable restricted contraction rule used in Lafont's soft linear logic (SLL) [35] is the *multiplexing rule* of the form

$$\frac{\overbrace{\alpha, \ldots, \alpha}^{n}, \Gamma \Rightarrow \gamma}{!_s\alpha, \Gamma \Rightarrow \gamma}\ (\text{multi}),$$

where n can be any natural number, which corresponds to the Hilbert-style axiom schemes: $\alpha \rightarrow !_s\beta \rightarrow \alpha$, $!_s\alpha \rightarrow \alpha$ and $(\overbrace{\alpha \rightarrow \cdots \rightarrow \alpha}^{n} \rightarrow \beta) \rightarrow !_s\alpha \rightarrow \beta$ for $2 \leqslant n$. A bottom-up proof search with respect to (multi) may not derive an infinite path in a proof search tree. However, a bottom-up proof search with respect to (multi) derives an *infinite branch* (in a proof search tree) such as

$$\begin{array}{ccccccc} \vdots & \vdots & & \vdots & & \vdots \\ (\Gamma \Rightarrow \gamma) & (\alpha, \Gamma \Rightarrow \gamma) & \cdots & (\alpha, \alpha, \ldots, \alpha, \Gamma \Rightarrow \gamma) & \cdots & \infty \\ & & & \vdots & & \\ & & & !_s\alpha, \Gamma \Rightarrow \gamma, & & \end{array}$$

because n in (multi) can be infinite. As far as we know, it is unknown whether SLL is decidable or not. On the other hand, if n is bounded by a *fixed finite positive integer*, then the corresponding restricted soft linear logic can be decidable. Such a decidable logic obtained from SLL by restricting n in (multi) is discussed in [26].

Role of restricted weakening. For the case of affine logics, a cause that derives the decidability is the presence of the full weakening rule (we). This rule may restrict the lengths of all paths in an arbitrary proof search tree including the

[16] An attempt to solve the problem is in [19]. It was shown in [19] that if the contraction rule is restricted to the prime contraction rule, in which the principal formulas are restricted to propositional variables, then the corresponding system is decidable. However, this is not a solution to the problem.

applications of (!co). On the other hand, the restricted weakening rule (used in ILL) of the form

$$\frac{\Gamma \Rightarrow \gamma}{!\alpha, \Gamma \Rightarrow \gamma} \text{ (!we)},$$

which corresponds to the Hilbert-style axiom scheme: $\alpha \rightarrow !\beta \rightarrow \alpha$, may be not enough to restrict the lengths of all paths in an arbitrary proof search tree including the applications of (!co).

Directions. In the sequel, in order to show the decidability of some sublogics of the proposed logics, we can consider three cases: (1) ! is deleted, (2) ! is replaced by a restricted version of $!_s$, and (3) the full weakening rule (or axiom scheme) is assumed. In these cases, it may be very important to consider the general combination problem: Which combinations of modal operators preserve decidability?

Complexity issues. Even though the logics proposed in this paper are almost undecidable, some sublogics without ! are supposed to be decidable, and hence further algorithmic accounts, such as the complexity of decision procedures, will be needed. In the following, some typical examples for complexity issues on linear logics are briefly explained. It is known that the multiplicative linear logic MLL and the multiplicative additive linear logic MALL are NP-complete and PSPACE-complete, respectively (see e.g. [36]). The complexity (of decision procedures) for WILL and the !-free ITLL are unknown, and thus the complexity problems for some combined linear and affine logics are also unsolved. Though the results discussed above are that for the complexity of proof searches, recent trends in the complexity issues on linear logics are mainly intended to capture the polytime computation in a proofs-as-programs paradigm. Some typical examples for such issues are as follows. It was proved by Girard that every polytime function is representable by a proof of the light linear logic LLL, and conversely that every LLL-proof is normalizable in polytime. In order to develop such a motivation, various linear and affine logics are introduced. The intuitionistic light affine logic ILAL by Asperti and Roversi, and the soft linear logic SLL by Lafont are typical examples for such polytime logics (for more information, see e.g. [35]).

10.2. Related works

The existing combined modal logics (based on the classical logic) and the proposed logics are compared below.

Combined modal logic: *fusion*, *product and fibring.* There are many combined multi-modal logics based on the standard classical logic. In the following, some examples of such multi-modal logics are briefly explained based on the review in [45]. Typical examples of the combined logics may be classified as (1) the fusion of modal logics, (2) the product of modal logics, (3) the fibring of modal logics, and (4) the combination or modification of (1)–(3). Approach (1) refers to the combination of some different modal logics using intermediate axiom schemes, and is the approach adopted in the present paper based on linear and affine logics instead of the classical logic. By approach (2), the multi-modal logics are validated in products of different Kripke frames. Approach (3), which was originally proposed by Gabbay, is a more general combination mechanism. Indeed, fibring captures fusion as a special case. If the fibring technique can be applied to the proposed logics, it may be shown that the technique would not only make the logics more comprehensive, but would allow the interpretation of the various aspects to be considered.

Bunched logic. A type of non-modal combined logic, named by O'Hearn and Pym, the logic of bunched implications (BI), has been studied by many researchers [39,42]. This logic is a natural combination of the intuitionistic multiplicative non-modal linear logic (MILL) and the intuitionistic logic (IL), and has a number of applications in computer science, such as in Petri net specifications, memory allocation models and logic programming languages (for more information about BI, see [42] and references therein). The connection between BI and fibring is also discussed in [42]. It is remarked that the part of the linear multiplicative connectives for the Kripke semantics of BI has a common structure to that for the logics proposed in the present paper. In addition, it can be observed that BI is an extension of the (negation-less) logic RW_+ of contraction-less relevant implication, by adding the additive intuitionistic implication and some additive constants [30]. Thus, BI is a natural extension of not only MILL and IL, but also RW_+.

Hybrid logic. There are a number of ways to introduce various useful modal operators, but one of the best candidates is a "hybrid logic" approach. Roughly speaking, this approach involves extending an ordinary language of modal logics by adding other types of atomic formulas, as "nominals" [6]. By this approach, each nominal must be true at exactly one point in any Kripke model. Various types of new modal operators, such as temporal operators and binding operators, can be expressed, and many applications have been proposed. It is remarked that an operator $@_i$ on a hybrid logic has similar interpretations to the proposed logics in terms of the spatial operator $[l_i]$, e.g. a formula $@_i \alpha$ is interpreted as "α is true at time i".

Spatial logic. A number of spatial logics have been proposed by many logicians and computer scientists. In the following, some recent topics on spatial logics are briefly reviewed from the point of view of both computer science and pure mathematical logic.

From the point of view of computer science, some examples of recent promising topics are as follows. Cardelli and Gordon [10] introduced a space-time modal logic for mobile ambients, Caires and Cardelli [8] introduced a spatial logic for concurrency, Merz et al. [37] proposed a spatio-temporal logic for mobile systems, and Calcagno et al. [9] proposed a spatial logic for finite trees. [17]

From the point of view of mathematical logic, some examples of more recent topics are as follows. Aiello et al. [1] presented a topological space interpretation for S4 by using a notion of bisimulation, Bennet et al. [4] introduced a multi-dimensional spatial modal logic, Kutz et al. [34] introduced some modal logics of metric spaces, Andréka et al. [3] presented some logical axiomatizations of space-time.

In the paper [24] addressed by the presented discussion, a new logic, spatio-temporal soft linear logic (SSLL), was introduced based on a sequent calculus. The setting of the spatial operators of this spatial logic is essentially the same as that of the proposed logics. SSLL lacks the disjunction connective $\vee$ and some modal operators, and hence the semantics for SSLL can be simplified. On the other hand, the semantics for SSLL cannot deal with a wide range of substructural logics uniformly. Thus, the present paper's semantics can be viewed as a generalization of that of SSLL. In addition, the framework of the spatial operator $[l]$ used in this paper can be adapted for the standard modal logic S4 based on the classical logic [28].

10.3. Conclusions

In this paper, a number of combined multi-modal logics based on the resource-sensitive logics were studied from a purely logical point of view. The traditional S5-type, S4-type and K-type modal operators were used to represent "knowledge", "any-time" and "next-time", respectively, and the linear exponential, spatial and strong negation operators were also combined in order to represent "reusable resource", "locations" and "refutability", respectively.

By the proposed logics, a resource-sensitive account of various useful modal operators was clarified from both syntactical and semantical points of view. The completeness theorems for the proposed logics were the main results of this paper, including not only the completeness results for the existing linear logics, such as ITLL by Hirai, (propositional) MLL by Kobayashi et al. and WILL by Wansing, but also for further extensions with the knowledge and dynamic operators. TSEILLs, for example, is an extension of the three logics ITLL, MLL and WILL. A more general completeness framework for the class of normal modal logics over BIAL was also derived. Some intuitive interpretations, such as informational and Petri net interpretations, and some illustrative examples, such as medical and distributed systems, were also addressed from the resource-sensitive and computational points of view.

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