JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 71, 590–607 (1979)

Homogenization in Open Sets with Holes

DOINA CIORANESCU AND JEANNINE SAINT JEAN PAULIN

Laboratoire Analyse Numérique, Université Pierre et Marie Curie, 4 place Jussieu, 75230 Paris Cedex 05, France

Submitted by J. L. Lions

Let Q^r be a cylindrical bar with r cylindrical cavities having generators parallel to those of Q^r. Let Ω be the cross-section of the bar, Ω^* the cross-section of the domain occupied by the material and $\Omega^i (i = 1,..., r)$ the cross- sections of the cavities:

$$\bar{\Omega}^i \subset \Omega; \ \bar{\Omega}^i \cap \bar{\Omega}^k = \phi, \ i \neq k.$$

The study of the elastic torsion of this bar leads to the following problem [see Lanchon (Thèse, Paris, 1972; *J. Mécanique* **13** (1974), 267–320)]:

$$\begin{aligned} &\Delta f_r + 2\mu\alpha = 0 \text{ in } \Omega^* \\ &f_{r|\partial\Omega} = 0 \\ &f_r = \text{constant on } \partial\Omega^i; \quad i = 1,..., r \end{aligned} \tag{1}$$

where μ is the shear modulus of the material, α is the angle of twist and f_r represents the stress function. In this paper the problem (1) with an increasing number of holes which are distributed periodically is considered. One would like to know if f_r has a limit f_∞ as $r \to +\infty$, and if so, the equation satisfied by this limit. This is an "homogenization" problem — the heterogeneous bar Q^r is replaced by a homogeneous one, the response of which under torsion approximates as closely as possible that of Q^r. A more general problem will be studied and the case of elastic torsion will be obtained as an application. The proof uses the energy method [see Lions (Collège de France, 1975–1977), Tartar (Collège de France, 1977)] and extension theorems. A related problem is the homogenization of a perforated plate [cf. Duvaut (to appear)].

1. NOTATIONS. VARIATIONAL FORMULATION

Let Y be the representative cell in $\mathbb{R}^2$

$$Y = [0, l_1[\times [0, l_2[.$$

Let τ_i ($i = 1,..., M$) be two-dimensional connected open sets whose boundaries are smooth, assumed to lie locally on one side of their boundary.

The τ_i are used to construct the holes.

0022-247X/79/100590–18$02.00/0

The part of Y occupied by the material is denoted by Y^*:

$$Y^* = Y - \bigcup_{i=1}^{M} (\bar{\tau}_i \cap Y); \theta = \frac{\text{meas } Y^*}{\text{meas } Y} = \frac{|Y^*|}{|Y|}$$

Let $\tilde{\chi}$ be the characteristic function of Y^* (this function is defined at every point of Y, and not merely almost everywhere in Y):

$$\begin{aligned} \tilde{\chi}(y) &= 1 \quad \text{if} \quad y \in Y^* \\ &= 0 \quad \text{if} \quad y \in \bar{\tau}_i \cap Y, \quad i = 1,..., M. \end{aligned}$$

The function $\tilde{\chi}$ is extended periodically in $\mathbb{R}^2$ and let χ be this extension. The "holes" T_ϵ^j ($j = 1, 2,...$) in $\mathbb{R}^2$ are defined as the (closed) connected components of the set

$$\left\{x \mid \chi\left(\frac{x}{\epsilon}\right) = 0\right\} \qquad (\epsilon > 0).$$

This means $\mathbb{R}^2$ is covered periodically by cells homothetic to the representative cell Y, the ratio being ϵ: 1.

Let Ω be a bounded connected two-dimensional open set whose boundary is not necessarily smooth.

Let Ω_ϵ^* denote the open subset of Ω representing the part of Ω occupied by the material.

We make the following assumptions:

(i) Ω_ϵ^* is a connected set.

(ii) the T_ϵ^j have a smooth boundary and they are locally on one side of their boundary.

We denote by Ω_ϵ^j an "interior hole", i.e. a T_ϵ^j which is included in Ω and does not intersect $\partial\Omega$. There is a finite number N_ϵ of such closed sets Ω_ϵ^j. Let:

$$\Omega_\epsilon^{**} = \bigcup_{j=1}^{N_\epsilon} \Omega_\epsilon^j = \bigcup_{j=1}^{N_\epsilon} \{T_\epsilon^j \mid T_\epsilon^j \subset \Omega;\ T_\epsilon^j \cap \partial\Omega = \phi\}.$$

Remark. One does not have

$$\Omega_\epsilon^* = \Omega - \Omega_\epsilon^{**}.$$

Let $\partial_{\text{ext}} \Omega_\epsilon^*$ be the exterior boundary of Ω_ϵ^*

$$\partial_{\text{ext}}\Omega_\epsilon^* = \partial\Omega_\epsilon^* - \partial\Omega_\epsilon^{**}.$$

This exterior boundary is not necessarily smooth: it may have angles and Ω_ϵ^* may not be locally on one side of $\partial_{\text{ext}} \Omega_\epsilon^*$. ∎

Consider the problem (Problem 1):

$$
\begin{aligned}
\mathbf{A}_\epsilon u_\epsilon &= -\operatorname{div}\left(A\left(\frac{x}{\epsilon}\right)\operatorname{grad} u_\epsilon\right) = f \quad \text{in} \quad \Omega_\epsilon^* \\
u_\epsilon|_{\partial_{\text{ext}}\Omega_\epsilon^*} &= 0 \\
u_\epsilon|_{\partial\Omega_\epsilon^i} &= \text{const.} \qquad i = 1,\ldots, N_\epsilon \\
\int_{\partial\Omega_\epsilon^i} \frac{\partial u_\epsilon}{\partial \nu_{A_\epsilon}}\, ds &= \int_{\Omega_\epsilon^i} f\, dx
\end{aligned} \tag{2}
$$

(the normal is directed towards the exterior of Ω_ϵ^*). Here $A(x/\epsilon)$ is the value of the matrix $(a_{ij}(x))_{i,j=1,2}$ calculated at the point x/ϵ.

We introduce the vector space

$$E_\epsilon = \{v \in H^1(\Omega_\epsilon^*),\ v = \text{const. on } \partial\Omega_\epsilon^i\ (i = 1,\ldots, N_\epsilon),\ v|_{\partial_{\text{ext}}\Omega_\epsilon^*} = 0\}$$

with the norm

$$\|v\|_{E_\epsilon} = \|\operatorname{grad} v\|_{[L^2(\Omega_\epsilon^*)]^2}.$$

The variational formulation of (2) is:

$$
\begin{aligned}
\int_{\Omega_\epsilon^*} A\left(\frac{x}{\epsilon}\right) \operatorname{grad} u_\epsilon \operatorname{grad} v_\epsilon\, dx &= \int_{\Omega_\epsilon} f v_\epsilon\, dx \\
&\forall v_\epsilon \in E_\epsilon .
\end{aligned} \tag{3}
$$

We make the following assumptions:

A.1. $f \in L^2(\Omega)$

A.2. The coefficients $a_{ij} \in L^\infty(\mathbb{R}^2)$, $i, j = 1, 2$.

A.3. There is a positive number β such that

$$\sum_{i,j} a_{ij}(y)\, \zeta_i\zeta_j \geq \beta\zeta_i\zeta_i \qquad \text{for any } \zeta = (\zeta_k)_{k=1,2} \in \mathbb{R}^2.$$

Under these assumptions, classical theorems show that (3) has a unique solution $u_\epsilon \in E_\epsilon$. ∎

Now let $\epsilon \to 0$, hence $N_\epsilon \to +\infty$ (cf. the definitions of Ω_ϵ^*, Ω_ϵ^i and N_ϵ). The behavior of u_ϵ as $\epsilon \to 0$ will now be studied.

2. Extension Lemmas

Lemma 1. *There exists an extension operator*

$$P_\epsilon \in \mathscr{L}(E_\epsilon, H_0^1(\Omega))$$

such that

$$|\operatorname{grad} P_\epsilon v|_{[L^2(\Omega)]^2} \leqslant C\,|\operatorname{grad} v|_{[L^2(\Omega_\epsilon^*)]^2}, \qquad \forall v \in E_\epsilon .$$

where the constant C does not depend on ϵ.

Proof. Let $v \in E_\epsilon$, it is extended into each hole contained in Ω by its value on the boundary of the hole; if a hole ω_ϵ cuts the boundary $\partial\Omega$, v is extended by 0 in $\Omega \cap \omega_\epsilon$.

LEMMA 2. *Let $\Phi \in [L^2(Y^*)]^2$ be a solution of*

$$-\operatorname{div} \Phi = F \qquad \text{in} \qquad Y^*$$

with

$$\int_{\partial(\tau_i \cap Y)} \Phi \cdot n\, ds = \int_{\tau_i \cap Y} F\, dx \qquad i = 1,\ldots, M. \tag{4}$$

where $F \in L^2(Y)$ and n is the normal directed towards the exterior of Y^*.
Then there exists $\tilde{\Phi} \in [L^2(\bigcup_{i=1}^M (\tau_i \cap Y))]^2$ such that:

$$-\operatorname{div} \tilde{\Phi} = F \qquad \text{in} \qquad \bigcup_{i=1}^{M} (\tau_i \cap Y)$$

$$\tilde{\Phi} \cdot n\,|_{\partial(\tau_i \cap Y)} = \Phi \cdot n\,|_{\partial(\tau_i \cap Y)}$$

Moreover,

$$|\tilde{\Phi}|_{[L^2(\bigcup_{i=1}^M(\tau_i\cap Y))]^2} \leqslant C_1\,|F|_{L^2(Y)} + C_2\,|\Phi|_{[L^2(Y^*)]^2} \tag{5}$$

where C_1 and C_2 are constants.

Proof. We seek $\tilde{\Phi}$ under the form grad φ which leads to the solution of the following problem:

$$-\Delta\varphi = F \qquad \text{in} \qquad \bigcup_{i=1}^{M} (\tau_i \cap Y)$$

$$\left.\frac{\partial\varphi}{\partial n}\right|_{\partial(\tau_i \cap Y)} = \Phi \cdot n\,|_{\partial(\tau_i \cap Y)} \qquad (i = 1,\ldots, M)$$

where $\Phi \cdot n$ verifies (4).

This is a classical Neumann problem which has a solution φ in $H^1(\bigcup_{i=1}^M (\tau_i \cap Y))$, unique up to an additive constant. Moreover:

$$|\varphi|_{H^1(\bigcup_{i=1}^M(\tau_i\cap Y))} \leqslant C_1'\,|F|_{L^2(\bigcup_{i=1}^M(\tau_i\cap Y))} + C_2'\,|\Phi \cdot n|_{H^{-1/2}(\bigcup_{i=1}^M(\tau_i\cap Y))} \tag{6}$$

(C_1' and C_2' are constants).

Notice now that the application

$$v \mapsto v \cdot n$$

from $V = \{v \mid v \in [L^2(Y^*)]^2, \operatorname{div} v \in L^2(Y^*)\}$ to $H^{-1/2}(\bigcup_{i=1}^{M} \partial(\tau_i \cap Y))$ is continuous (see Lions–Magenes [1] for example).

It follows that

$$| \Phi \cdot n |_{H^{-1/2}(\bigcup_{i=1}^{n} \partial(\tau_i \cap Y))} \leqslant k_1 | \Phi |_{[L^2(Y^*)]^2} + k_2 | \operatorname{div} \Phi |_{L^2(Y^*)} . \tag{7}$$

Thus the inequality (5) is deduced from estimations (6) and (7) (recall that $-\operatorname{div} \Phi = F$ in Y^*).

3. The Elastic Torsion Problem

We add the following assumption:

A.4. The coefficients $a_{ij}(y)$ are Y-periodic.

Theorem 1. *Under the assumptions* A.1 *to* A.4 *there is an extension* $P_\epsilon u_\epsilon$ *of* u_ϵ *such that*

$$P_\epsilon u_\epsilon \rightharpoonup u^* \qquad in \qquad H_0^1(\Omega) \text{ weakly}$$

where u^* *is the solution of*

$$\mathscr{A} u^* = -\operatorname{div}(\mathscr{A} \operatorname{grad} u^*) = f \qquad in \qquad \Omega.$$

The constant matrix $\mathscr{A}$ *will be defined later.*

Proof. (i) *A priori estimates.* Using the assumptions and (2), it follows easily that

$$\| u_\epsilon \|_{H_0^1(\Omega_\epsilon^*)} \leqslant \text{constant} \qquad (\text{independently of } \epsilon). \tag{8}$$

Lemma 1 can be applied, u_ϵ is extended by $P_\epsilon u_\epsilon$; we get:

$$\| P_\epsilon u_\epsilon \|_{H_0^1(\Omega)} \leqslant \text{constant} \qquad (\text{independently of } \epsilon).$$

Hence we can extract a subsequence still denoted by $P_\epsilon u_\epsilon$ such that

$$P_\epsilon u_\epsilon \rightharpoonup u^* \qquad \text{in} \qquad H_0^1(\Omega) \text{ weakly}.$$

We now look for the equation satisfied by u_ϵ.

Let

$$\xi_\epsilon = A\left(\frac{x}{\epsilon}\right) \operatorname{grad} u_\epsilon \qquad \text{in} \qquad \Omega_\epsilon^*.$$

Using the assumptions and (8) we get:

$$|\xi_\epsilon|_{[L^2(\Omega_\epsilon^*)]^2} \leqslant \text{constant} \qquad (\text{independently of } \epsilon). \tag{9}$$

Moreover ξ_ϵ verifies:

$$-\operatorname{div} \xi_\epsilon = f \qquad \text{in} \qquad \Omega_\epsilon^* \tag{10}$$

and

$$\int_{\partial\Omega_\epsilon{}^i} \xi_\epsilon \cdot n \, ds = \int_{\Omega_\epsilon{}^i} f \, dx.$$

In order to pass to the limit, it is necessary to obtain equations and estimates in Ω, or at least in any relatively compact open subset of Ω.

Let Ω' be such a subset. We seek an extension $Q_\epsilon\xi_\epsilon$ of ξ_ϵ preserving the equation (10) in Ω' and such that

$$|Q_\epsilon\xi_\epsilon|_{[L^2(\Omega')]^2} \leqslant \text{constant} \qquad (\text{independently of } \epsilon).$$

Let $y = x/\epsilon$ and $\Phi(y) = \xi_\epsilon(\epsilon y)$. It will be noticed that:

$$-\operatorname{div} \Phi = F \qquad \text{in} \qquad Y^*$$

$$\int_{\partial(\tau_i \cap Y)} \Phi \cdot n \, ds = \int_{\tau_i \cap Y} F \, dy \qquad i = 1, \ldots, M$$

with $F \in L^2(Y^*)$ and hence Lemma 2 can be applied. Let Q denote the extension operator given by this lemma ($Q\Phi = \Phi$ in Y^*, $Q\Phi = \tilde{\Phi}$ in $\bigcup_{i=1}^M (\tau_i \cap Y)$ and define now:

$$(Q_\epsilon\xi_\epsilon)(\epsilon y) = (Q\Phi)(y).$$

It follows that

$$-\operatorname{div} Q_\epsilon\xi_\epsilon = f \qquad \text{in} \qquad \epsilon Y$$

$$|Q_\epsilon\xi_\epsilon|_{[L^2(\epsilon Y)]^2} \leqslant C_1' \, |f|_{L^2(\epsilon Y)} + C_2' \, |\xi_\epsilon|_{[L^2(\epsilon Y^*)]^2}$$

If ϵY is extended periodically to $\mathbb{R}^2$ we obtain

$$-\operatorname{div} Q_\epsilon\xi_\epsilon = f \qquad \text{in} \qquad \Omega_\epsilon = \Omega_\epsilon^* \cup \left(\bigcup_{i=1}^{N_\epsilon} \Omega_\epsilon{}^i\right)$$

$$|Q_\epsilon\xi_\epsilon|_{[L^2(\Omega_\epsilon)]^2} \leqslant C_1 \, |f|_{L^2(\Omega)} + C_2 \, |\xi_\epsilon|_{[L^2(\Omega_\epsilon^*)]^2} \tag{11}$$

where C_1 and C_2 are constants independent of ϵ.

Since Ω' is a relatively compact subset of Ω, if ϵ is small enough, $\partial\Omega'$ does not meet the holes cutting the boundary $\partial\Omega$ (the distance between Ω' and $\partial\Omega$ is positive).

Recalling (11) and the a priori estimate (9) we conclude that

$$-\operatorname{div} Q_\epsilon \xi_\epsilon = f \quad \text{in} \quad \Omega' \tag{12}$$

and

$$| Q_\epsilon \xi_\epsilon |_{[L^2(\Omega')]^2} \leqslant \text{constant} \qquad (\text{independently of } \epsilon).$$

Consequently we can extract a subsequence, still denoted by $Q_\epsilon \xi_\epsilon$, such that

$$Q_\epsilon \xi_\epsilon \underset{\epsilon \to 0}{\rightharpoonup} \xi^* \quad \text{in} \quad [L^2(\Omega')]^2 \text{ weakly}$$

with ξ^* verifying the limit equation

$$-\operatorname{div} \xi^* = f \quad \text{in} \quad \Omega' \tag{13}$$

obtained from (12).

(ii) *Definition of the homogenized operator.* The purpose is to establish a relation between ξ^* and u^*. The argument uses energy method.

For each $\lambda \in \mathbb{R}^2$ define $w_\lambda(y)$ by

$$\begin{aligned}
&-\operatorname{div}(A^*(y) \operatorname{grad} w_\lambda(y)) = 0 \quad \text{in} \quad Y^* \\
&(w_\lambda - \lambda \cdot y) \text{ periodic} \quad \text{in} \quad Y^* \\
&\int_{\partial(\tau_i \cap Y)} \frac{\partial w_\lambda}{\partial \nu_{A^*}}\, ds = 0, \quad i = 1,\dots, M \\
&w_\lambda = \text{constant} \quad \text{on} \quad \partial(\tau_i \cap Y).
\end{aligned}$$

Let $\tilde{P}w_\lambda$ be the extension of w_λ inside the hole τ_i by its value on the boundary of τ_i $(i = 1,\dots, M)$.

Set

$$\eta_\lambda = A^* \operatorname{grad} w_\lambda$$

and notice that $\Phi = \eta_\lambda$ verifies the assumptions of Lemma 2 with $F = 0$. Let $Q\eta_\lambda$ be the extension of η_λ to Y given by this lemma. We have:

$$(\tilde{P}w_\lambda - \lambda \cdot y) \text{ periodic} \quad \text{in} \quad Y$$

and

$$-\operatorname{div}(Q\eta_\lambda) = 0 \quad \text{in} \quad Y.$$

Moreover

$$\mathfrak{M}(\operatorname{grad} \tilde{P}w_\lambda) = \lambda$$

($\mathfrak{M}$ is the average in Y: $\mathfrak{M}g = (1/|Y|) \int_Y g\, dx$).

Observing that w_λ is linear in λ and that Q is a linear operator we can define a matrix $\mathscr{A}$ by

$$\forall \lambda \in \mathbb{R}^2 \qquad \mathscr{A}\lambda = \mathfrak{M}(QA^*(y) \operatorname{grad} w_\lambda(y)). \tag{14}$$

DEFINITION. The matrix $\mathscr{A}$ given by (14) defines an operator $\mathscr{A}$ called homogenized operator associated with problem 1.

(iii) *The homogenized equation.* From the results obtained in the first two steps, we get:

$$\begin{aligned} P_\epsilon u_\epsilon &\underset{\epsilon\to 0}{\rightharpoonup} u^* \quad \text{in} \quad H_0^1(\Omega) \text{ weakly} \\ \tilde{Q}_\epsilon \xi_\epsilon &\underset{\epsilon\to 0}{\rightharpoonup} \xi^* \quad \text{in} \quad [L^2(\Omega')]^2 \text{ weakly} \end{aligned} \tag{15}$$

and

$$-\operatorname{div} \xi^* = f \quad \text{in} \quad \Omega'.$$

Next let

$$w_\epsilon(x) = \epsilon \tilde{P} w_\lambda \left(\frac{x}{\epsilon}\right)$$

$$\eta_{\lambda\epsilon}(x) = \eta_\lambda \left(\frac{x}{\epsilon}\right).$$

The gradient of w_ϵ is periodic by construction. To extend $\eta_{\lambda\epsilon}$ we use the same technique as the one used to extend ξ_ϵ and we define

$$(\tilde{Q}_\epsilon \eta_{\lambda\epsilon})(x) = (\tilde{Q}\eta_\lambda)\left(\frac{x}{\epsilon}\right).$$

From the step (ii) and the preceeding remarks, it follows that

$$-\operatorname{div} \tilde{Q}_\epsilon \eta_{\lambda\epsilon} = 0 \tag{16}$$

$$\begin{aligned} w_\epsilon &\underset{\epsilon\to 0}{\rightharpoonup} w^* \quad \text{in} \quad H^1(\Omega) \text{ weakly} \\ \operatorname{grad} w_\epsilon &\underset{\epsilon\to 0}{\rightharpoonup} \lambda \quad \text{in} \quad [L^2(\Omega)]^2 \text{ weakly} \end{aligned} \tag{17}$$

and

$$\tilde{Q}_\epsilon \eta_{\lambda\epsilon} \underset{\epsilon\to 0}{\rightharpoonup} \mathfrak{M}(\tilde{Q}_\epsilon \eta_{\lambda\epsilon}) = \mathscr{A}\lambda \quad \text{in} \quad [L^2(\Omega)]^2 \text{ weakly}. \tag{18}$$

Moreover

$$\operatorname{grad} w^* = \lambda.$$

Fix $\varphi \in \mathfrak{D}(\Omega)$ and choose a relatively compact open subset Ω' of Ω such that

$$\operatorname{supp} \varphi \subset \Omega' \Subset \Omega.$$

Multiplying (12) by $\varphi \cdot w_\epsilon$ and (16) by $\varphi \cdot P_\epsilon u_\epsilon$, subtracting one from the other it follows that:

$$\begin{aligned} &\int_{\Omega'} \tilde{Q}_\epsilon \xi_\epsilon \cdot \nabla\varphi \cdot w_\epsilon \, dx + \int_{\Omega'} \tilde{Q}_\epsilon \xi_\epsilon \cdot \varphi \cdot \nabla w_\epsilon \, dx \\ &\qquad - \int_{\Omega'} \tilde{Q}_\epsilon \eta_{\lambda\epsilon} \cdot \nabla\varphi \cdot P_\epsilon u_\epsilon \, dx - \int_{\Omega'} \tilde{Q}_\epsilon \eta_\epsilon \cdot \varphi \cdot \nabla(P_\epsilon u_\epsilon) \, dx \\ &= \int_{\Omega'} f \cdot \varphi \cdot w_\epsilon \, dx. \end{aligned} \tag{19}$$

We use the definitions of the extension operators to compute the following expression in (19):

$$\int_{\Omega'} Q_\epsilon \xi_\epsilon \cdot \varphi \cdot \nabla w_\epsilon \, dx - \int_{\Omega'} Q_\epsilon \eta_{\lambda\epsilon} \cdot \varphi \cdot \nabla(P_\epsilon u_\epsilon) \, dx$$

$$= \int_{\Omega_\epsilon^* \cap \Omega'} \left(A\left(\frac{x}{\epsilon}\right) \nabla u_\epsilon \cdot \nabla w_\epsilon - A^*\left(\frac{x}{\epsilon}\right) \cdot \nabla w_\epsilon \cdot \nabla u_\epsilon \right) \varphi \, dx \qquad (20)$$

$$+ \int_{(\bigcup_{i=1}^{N_\epsilon} \Omega_\epsilon^i) \cap \Omega'} [Q_\epsilon \xi_\epsilon \cdot \nabla w_\epsilon - Q_\epsilon \eta_{\lambda\epsilon} \cdot \nabla(P_\epsilon u_\epsilon)] \, \varphi \, dx.$$

This expression is equal to zero. Indeed the first term in the right hand side of (20) is zero since A^* is the adjoint of A, and the second term is also zero by the definitions of $P_\epsilon u_\epsilon$ and of w_ϵ.

Using this remark in (18), it follows that

$$\int_{\Omega'} Q_\epsilon \xi_\epsilon \cdot \nabla\varphi \cdot w_\epsilon \, dx - \int_{\Omega'} Q_\epsilon \eta_{\lambda\epsilon} \cdot \nabla\varphi \cdot P_\epsilon u_\epsilon \, dx = \int_{\Omega'} f \cdot \varphi \cdot w_\epsilon \, dx$$

and we can pass to the limit in this expression when $\epsilon \to 0$ because of the convergences (15), (17) and (18).

We thus deduce:

$$\int_{\Omega'} \xi^* \cdot \nabla\varphi \cdot w^* \, dx - \int_{\Omega'} \mathscr{A}\lambda \cdot \nabla\varphi \cdot u^* \, dx = \int_{\Omega'} f \cdot \varphi \cdot w^* \, dx.$$

Recalling (13) and the fact that supp $\varphi \subset \Omega'$, we get:

$$-\int_{\Omega} \xi^* \cdot \lambda \cdot \varphi \, dx + \int_{\Omega} \mathscr{A}\lambda \cdot \varphi \cdot \nabla u^* \, dx = 0.$$

This is true for any $\lambda \in \mathbb{R}^2$ and any $\varphi \in \mathfrak{D}(\Omega)$. Hence

$$\xi^* = \mathscr{A}\nabla u^*$$

which implies

$$-\operatorname{div}(\mathscr{A}\nabla u^*) = f \quad \text{in} \quad \Omega. \qquad (21)$$

We call (21) the homogenized equation associated with problem 1.

Remarks. 1. We have constructed independent extensions for u_ϵ and $A(x/\epsilon)$ grad u_ϵ. Indeed

$$A\left(\frac{x}{\epsilon}\right) \operatorname{grad} P_\epsilon u_\epsilon \neq Q_\epsilon\left(A\left(\frac{x}{\epsilon}\right) \operatorname{grad} u_\epsilon\right).$$

In Ω_ϵ^i, ξ_ϵ cannot be extended by 0 (which is the value assumed by $A(x/\epsilon)$ grad $P_\epsilon u_\epsilon$ there) because we want to preserve the equation $-\operatorname{div} \xi_\epsilon = f$, while $\xi_\epsilon \cdot n \not\equiv 0$ on $\partial\Omega_\epsilon^i$.

2. The "local" character of the proof in step (iii) should be noticed. In order to obtain equation (21), we multiply the equations verified by $Q_\epsilon\xi_\epsilon$ and $Q_\epsilon\eta_{\lambda\epsilon}$ by functions φ with compact support. This is the reason why an extension of ξ_ϵ is needed only in Ω' and not in Ω, though an extension of u_ϵ in Ω was used.

THEOREM 2. *The homogenized operator $\mathscr{A}$ and the limit function u^* do not depend on the extension operators P_ϵ, Q_ϵ and $\tilde{P}$.*

Proof. Notations

$$a_{Y^*}^*(\varphi, \psi) = \int_{Y^*} A^*(y) \operatorname{grad} \varphi \operatorname{grad} \psi \, dy$$

$$= \int_{Y^*} a_{ij}(y) \frac{\partial\varphi}{\partial y_i} \frac{\partial\psi}{\partial y_j} \, dy$$

$-\chi^i = w_{\lambda_i} - y_i$ where $\lambda_1 = (1, 0)$ and $\lambda_2 = (0, 1)$

$$\mathscr{A} = (q_{ij})_{i,j=1,2}$$

$$Q\eta_\lambda = ((Q\eta_\lambda)_1, (Q\eta_\lambda)_2).$$

From the definition of the homogenized operator, it follows

$$\mathscr{A}\lambda = \frac{1}{|Y|}\left(\int_{Y^*} A^*(y) \operatorname{grad} w_\lambda(y)\, dy + \int_{\bigcup_{i=1}^M(\tau_i \cap Y)} Q\eta_\lambda \, dy\right).$$

Using the definition of $Q\eta_\lambda$ and integrating by parts, we get:

$$q_{ij} = \frac{1}{|Y|}\left[\int_{Y^*} a_{lk}(y) \frac{\partial w_{\lambda_i}}{\partial y_k} \frac{\partial y_j}{\partial y_l} \, dy + \int_{\bigcup_{i=1}^M(\tau_i \cap Y)} (Q\eta_{\lambda_i})_l \frac{\partial y_j}{\partial y_l} \, dy\right]$$

$$= \frac{1}{|Y|}\left[a_{Y^*}^*(\chi^i - y_i, -y_j) + \int_{\bigcup_{i=1}^M(\tau_i \cap Y)} (-\operatorname{div} Q\eta_{\lambda_i})\, y_j \, dy\right.$$

$$\left. + \int_{\bigcup_{i=1}^M \partial(\tau_i \cap Y)} (Q\eta_{\lambda_i} \cdot n_1)\, y_j \, ds\right]$$

(n_1 is the normal directed towards the exterior of τ_i).

Since

$$-\operatorname{div} Q\eta_{\lambda_i} = 0 \quad \text{in} \quad \bigcup_{i=1}^M (\tau_i \cap Y)$$

by construction, and

$$\int_{\bigcup_{i=1}^{M}\partial(\tau_i\cap Y)} (\underset{\sim}{Q}\eta_{\lambda_i}\cdot n_1)\, y_j\, ds = -\int_{\bigcup_{i=1}^{M}\partial(\tau_i\cap Y)} (\eta_{\lambda_i}\cdot n)\, y_j\, ds$$

$$-\int_{\bigcup_{i=1}^{M}\partial(\tau_i\cap Y)} \frac{\partial w_{\lambda_i}}{\partial \nu_{A^*}}\, y_j\, ds$$

it follows:

$$q_{ij} = \frac{1}{|Y|}\left[a_{Y^*}^{*}(\chi^i - y_i, -y_j) + \int_{\bigcup_{i=1}^{M}\partial(\tau_i\cap Y)} \frac{\partial(\chi^i - y_i)}{\partial \nu_{A^*}}\, y_j\, ds\right]. \qquad (22)$$

The functions χ^j are periodic in Y^* (i.e. they take equal values on opposite sides of Y). Multiplying the equation

$$-\operatorname{div}(A^*(y)\operatorname{grad} w_{\lambda_i}) = 0 \qquad \text{in} \qquad Y^*$$

by χ^j and integrating by parts we get

$$a_{Y^*}^{*}(w_{\lambda_i}, \chi^j) = \int_{\bigcup_{i=1}^{M}\partial(\tau_i\cap Y)} \frac{\partial w_{\lambda_i}}{\partial \nu_{A^*}}\, \chi^j\, ds.$$

Using this result in (22) it follows:

$$q_{ij} = \frac{1}{|Y|}\, a_{Y^*}^{*}(\chi^i - y_i, \chi^j - y_j)$$

since

$$\int_{\bigcup_{i=1}^{M}\partial(\tau_i\cap Y)} \frac{\partial(\chi^i - y_i)}{\partial \nu_{A^*}}\, (\chi^j - y_j)\, ds = 0$$

by the definition of w_{λ_i}.

This formula gives q_{ij} independently of any extension used in the proof of Theorem 1. The assertion that u^* is also independent of these extensions is a trivial consequence of the unicity of the solution of

$$-\operatorname{div}(\mathscr{A}\operatorname{grad} u^*) = f$$
$$u^* \in H_0^1(\Omega).$$

4. The Dirichlet and Neumann Problems. Homogenization Theorems

(i) Make the assumptions A.1, A.2, A.3 and consider the Dirichlet problem (Problem 2):

$$\begin{aligned} \mathbf{A}_\epsilon u_\epsilon &= -\operatorname{div}\left(A\left(\frac{x}{\epsilon}\right)\operatorname{grad} u_\epsilon\right) = f \qquad \text{in} \qquad \Omega_\epsilon^* \\ u_\epsilon|_{\partial_{\mathrm{ext}}\Omega_\epsilon^*} &= 0 \\ u_\epsilon|_{\partial\Omega_\epsilon^i} &= 0 \qquad i = 1,\ldots, N_\epsilon . \end{aligned} \qquad (23)$$

THEOREM 3. *There exists an extension $P_\epsilon u_\epsilon$ of u_ϵ such that $P_\epsilon u_\epsilon \in H_0^1(\Omega)$, and*

$$P_\epsilon u_\epsilon \underset{\epsilon \to 0}{\rightharpoonup} 0 \qquad in \qquad H_0^1(\Omega) \textit{ weakly}.$$

Proof. By assumptions A.1 to A.3, the system (23) has a unique solution $u_\epsilon \in H_0^1(\Omega_\epsilon^*)$. Moreover

$$\| u_\epsilon \|_{H_0^1(\Omega_\epsilon^*)} \leqslant \text{constant} \qquad (\text{independently of } \epsilon). \tag{24}$$

Let $P_\epsilon u_\epsilon$ be the extension of u_ϵ by 0 in $\Omega \backslash \Omega_\epsilon^*$. From (24) it follows

$$\| P_\epsilon u_\epsilon \|_{H_0^1(\Omega)} \leqslant \text{constant} \qquad (\text{independently of } \epsilon)$$

consequently, there exists a weakly convergent subsequence $P_\epsilon u_\epsilon$ with limit, say u^*, i.e.

$$P_\epsilon u_\epsilon \underset{\epsilon \to 0}{\rightharpoonup} u^* \qquad \text{in} \qquad H_0^1(\Omega) \text{ weakly}$$

and hence in $L^2(\Omega)$ strongly.

Next

$$\chi_{\bigcup_{i=1}^{N_\epsilon} \Omega_\epsilon^i} \cdot P_\epsilon u_\epsilon = 0 \qquad \forall \epsilon \tag{25}$$

(χ_A is the characteristic function of the set A).

Since

$$\chi_{\bigcup_{i=1}^{N_\epsilon} \Omega_\epsilon^i} \underset{\epsilon \to 0}{\rightharpoonup} 1 - \theta \qquad \text{in} \qquad L^2(\Omega) \text{ weakly}$$

passing to the limit in (25), it follows

$$(1 - \theta)\, u^* = 0$$

hence

$$u^* = 0.$$

COROLLARY (Problem 3). *Suppose that the representative cell Y has M holes ($M > 1$) and that the boundary conditions are: a Dirichlet condition on at least one hole and a Neumann condition on all the other holes. Then*

$$u^* = 0.$$

The proof is similar to that of Theorem 3.

(ii) We now prove a homogenization result for the Neumann problem with an extension technique similar to the one used in the proof of Theorem 1 (see Tartar [5]).

The following assumption is added:

A.5. The holes do not meet the boundary $\partial\Omega$.

This assumption restricts the geometry of the open set Ω. (Example: Ω is a finite union of rectangles homothetic to the representative cell).

Consider the Neumann problem (Problem 4)

$$\mathbf{A}_\epsilon u_\epsilon \equiv -\operatorname{div}\left(A\left(\frac{x}{\epsilon}\right)\operatorname{grad} u_\epsilon\right) = f \qquad \text{in} \qquad \Omega_\epsilon^*$$

$$u_\epsilon\,|_{\partial\Omega} = 0$$

$$\left.\frac{\partial u_\epsilon}{\partial \nu_{A_\epsilon}}\right|_{\partial\Omega_\epsilon^i} = 0 \qquad i = 1,\ldots, N_\epsilon .$$

THEOREM 4. *Under the assumptions* A.1 *to* A.5, *there exists an extension* $\tilde{P}_\epsilon u_\epsilon \in H_0^1(\Omega)$ *such that:*

$$\tilde{P}_\epsilon u_\epsilon \underset{\epsilon\to 0}{\rightharpoonup} u^* \qquad in \qquad H_0^1(\Omega) \text{ weakly}$$

where u^* is the solution of the equation

$$\tilde{\mathscr{A}} u = -\operatorname{div}(\tilde{\mathscr{A}} \operatorname{grad} u^*) = \theta f \qquad in \qquad \Omega.$$

The matrix $\tilde{\mathscr{A}}$ has constant coefficients and will be defined later.

Proof. The idea is the same as in Theorem 1.

In Ω_ϵ^* we have the following estimates:

$$\left.\begin{aligned} \| u_\epsilon \|_{H_0^1(\Omega_\epsilon^*)} &\leqslant \text{constant} \\ \| \xi_\epsilon \|_{[L^2(\Omega_\epsilon^*)]^2} &\leqslant \text{constant} \end{aligned}\right\} \text{independently of } \epsilon$$

and the equation:

$$-\operatorname{div} \xi_\epsilon = f_\epsilon \equiv f\,|_{\Omega_\epsilon^*} \qquad \text{in} \qquad \Omega_\epsilon^*$$

with

$$\xi_\epsilon \cdot n = 0 \qquad \text{on} \qquad \partial\Omega_\epsilon^i; \qquad i = 1,\ldots, N_\epsilon . \tag{26}$$

We want to construct extensions $\tilde{Q}_\epsilon \xi_\epsilon \in [L^2(\Omega)]^2$ and $R_\epsilon f_\epsilon \in L^2(\Omega)$ such that

$$| \tilde{Q}_\epsilon \xi_\epsilon |_{[L^2(\Omega)]^2} \leqslant C_1(| \xi_\epsilon |_{[L^2(\Omega_\epsilon^*)]^2} + | f |_{L^2(\Omega)}). \tag{27}$$

$$| R_\epsilon f_\epsilon |_{L^2(\Omega)} \leqslant C_2 \, | f_\epsilon |_{L^2(\Omega_\epsilon^*)} \tag{28}$$

and

$$-\operatorname{div} \tilde{Q}_\epsilon \xi_\epsilon = R_\epsilon f_\epsilon \qquad \text{in} \qquad \Omega. \tag{29}$$

with constants C_1 and C_2 independent of ϵ.

By the boundary condition (26), we extend ξ_ϵ and f_ϵ by 0 in Ω_ϵ^i. Let $\tilde{Q}_\epsilon \xi_\epsilon$ and $R_\epsilon f_\epsilon$ denote these extensions. Notice that

$$R_\epsilon f_\epsilon = \chi_{\Omega_\epsilon^*} f.$$

Then the estimates (27), (8) and the equation (29) follow easily.

Hence, we can extract subsequences, still denoted by $\{\tilde{Q}_\epsilon \xi_\epsilon\}$ and $\{R_\epsilon f_\epsilon\}$, such that

$$\tilde{Q}_\epsilon \xi_\epsilon \underset{\epsilon \to 0}{\rightharpoonup} \xi^* \quad \text{in} \quad [L^2(\Omega)]^2 \text{ weakly}$$

$$R_\epsilon f_\epsilon \underset{\epsilon \to 0}{\rightharpoonup} \theta f \quad \text{in} \quad L^2(\Omega) \text{ weakly}$$

and

$$-\operatorname{div} \xi^* = \theta f.$$

We now seek an extension $\tilde{P}_\epsilon u_\epsilon \in H_0^1(\Omega)$ such that:

$$| \operatorname{grad} \tilde{P}_\epsilon u_\epsilon |_{[L^2(\Omega)]^2} \leqslant C_3 \, | \operatorname{grad} u_\epsilon |_{[L^2(\Omega_\epsilon^*)]^2} . \tag{30}$$

It is possible to use the Lemma of Bramble–Hilbert which gives the existence of such an extension, but is not constructive. Another possibility is to construct actually an extension verifying the inequality (30; see Tartar [5]).

We first construct extensions on the representative cell Y and then we derive extensions on Ω by the same method as in the proof of Theorem 1.

LEMMA 3. *There exists on extension operator*

$$\tilde{P} \in \mathscr{L}(H^1(Y^*), H^1(Y))$$

such that

$$| \operatorname{grad} \tilde{P}\varphi |_{[L^2(Y)]^2} \leqslant C_4 \, | \operatorname{grad} \varphi |_{[L^2(Y^*)]^2} , \qquad \forall \varphi \in H^1(Y^*).$$

Proof. Let $\varphi \in H^1(Y^*)$. We may write φ in the form:

$$\varphi = \mathfrak{M}_{Y^*}(\varphi) + \psi \qquad \text{where} \qquad \mathfrak{M}_{Y^*}(\psi) = 0.$$

Let $S \in \mathscr{L}(H^1(Y^*), H^1(Y))$ be any extension operator (such an operator exists since the boundaries of the holes are smooth enough). Then:

$$\| S\psi \|_{H^1(Y)} \leqslant C \, \| \psi \|_{H^1(Y^*)} .$$

Since the average of ψ in Y^* is zero, we have

$$\| \psi \|_{H^1(Y^*)} \leqslant C' \, | \operatorname{grad} \psi |_{[L^2(Y^*)]^2} = C' \, | \operatorname{grad} \varphi |_{[L^2(Y^*)]^2} .$$

Hence

$$\| S\psi \|_{H^1(Y)} \leq C' \, | \operatorname{grad} \varphi |_{[L^2(Y^*)]^2} . \tag{31}$$

Set

$$\tilde{P}\varphi = \mathfrak{M}_{Y^*}(\varphi) + S\psi.$$

By (31) this extension has the required properties. ∎

The extension given by Lemma 3 can now be used to extend u_ϵ.

Let $y = x/\epsilon$ and define the function $\tilde{u}_\epsilon$ by

$$\tilde{u}_\epsilon(y) = \frac{1}{\epsilon} u_\epsilon(\epsilon y). \tag{32}$$

This function is defined on Y since u_ϵ is defined in $\Omega_\epsilon^* = \epsilon Y^*$. Notice that

$$\tilde{u}_\epsilon \in H^1(Y^*).$$

By Lemma 3 we have:

$$\tilde{P}\tilde{u}_\epsilon = \mathfrak{M}_{Y^*}(\tilde{u}_\epsilon) + Sv_\epsilon$$

where

$$v_\epsilon = \tilde{u}_\epsilon - \mathfrak{M}_{Y^*}(\tilde{u}_\epsilon).$$

The function $\tilde{P}\tilde{u}_\epsilon$ is defined on Y; define $\tilde{P}_\epsilon u_\epsilon$ on $\Omega = \epsilon Y$ by:

$$(\tilde{P}_\epsilon u_\epsilon)(x) = \epsilon(\tilde{P}\tilde{u}_\epsilon)\left(\frac{x}{\epsilon}\right) \qquad x \in \epsilon Y.$$

It remains to show that this extension satisfies inequality (30).

Since

$$(\nabla(\tilde{P}_\epsilon u_\epsilon))(x) = \frac{1}{\epsilon} (\nabla(\tilde{P}_\epsilon u)) \left(\frac{x}{\epsilon}\right)$$

it follows that:

$$\int_\Omega | \nabla(\tilde{P}_\epsilon u_\epsilon)|^2 \, dx = \int_\Omega \left| \frac{1}{\epsilon} (\nabla(\tilde{P}_\epsilon u_\epsilon)) \left(\frac{x}{\epsilon}\right) \right|^2 dx$$

$$= \epsilon^2 \int_{\Omega/\epsilon} |(\nabla(\tilde{P}\tilde{u}_\epsilon)(y)|^2 \, dy.$$

The domain Ω/ϵ is covered by cells Y (with sides l_1 and l_2) and the number of such cells is of order of $(1/\epsilon^2)$ (meas Ω/meas Y);

$$\epsilon^2 \int_{\Omega/\epsilon} |(\nabla(\tilde{P}\tilde{u}_\epsilon))(y)|^2 \, dy$$

is of the same order as

$$\epsilon^2 \sum_{p,q} \int_{pl_1}^{(p+1)l_1} \int_{ql_2}^{(q+1)l_2} |(\nabla(\tilde{P}\tilde{u}_\epsilon)(y)|^2 \, dy \tag{33}$$

(the number of terms in the above sum is of the order of $(1/\epsilon^2)$ (meas Ω/meas Y)). We shall now estimate this sum; each term has the form

$$\int_{Y_k} |(\nabla(\tilde{P}\tilde{u}_\epsilon))(y)|^2 \, dy$$

(Y_k is a translate of the cell Y).

By Lemma 3, it follows that:

$$\int_{Y_k} |(\nabla(\tilde{P}\tilde{u}_\epsilon))(y)|^2 \, dy \leqslant C \int_{Y_k^*} |(\nabla\tilde{u}_\epsilon)(y)|^2 \, dy.$$

By definition (32), we have

$$(\nabla\tilde{u}_\epsilon)(y) = (\nabla u_\epsilon)(\epsilon y) \qquad y \in Y^*$$

and hence

$$\int_{Y_k} |(\nabla(\tilde{P}\tilde{u}_\epsilon))(y)|^2 \, dy \leqslant C \int_{\epsilon Y_k^*} \frac{1}{\epsilon^2} |(\nabla u_\epsilon)(x)|^2 \, dx.$$

Therefore, the sum (33) is bounded by

$$\epsilon^2 \sum_{k=1}^{(1/\epsilon^2)(\text{meas}\,\Omega/\text{meas}\,Y)} \int_{\epsilon Y_k^*} |(\nabla u_\epsilon)(x)|^2 \, dx$$

which is of the same order as $\int_{\Omega_\epsilon^*} |\nabla u_\epsilon|^2 \, dx$. This completes the proof of (30) (the cells ϵY_k^* cover Ω_ϵ^*). ∎

By inequality (30), we can extract a subsequence (denoted by $\tilde{P}_\epsilon u_\epsilon$) such that

$$\tilde{P}_\epsilon u_\epsilon \underset{\epsilon\to 0}{\rightharpoonup} u^* \qquad \text{in} \qquad H_0^1(\Omega) \text{ weakly}.$$

In order to find the equation satisfied by u^*, we proceed as in the proof of Theorem 1.

Now w_λ depends on the new boundary conditions. For any $\lambda \in \mathbb{R}^2$ define $\tilde{w}_\lambda$ by

$$-\text{div}(A^*(y) \,\text{grad}\, \tilde{w}_\lambda(y)) = 0 \qquad \text{in} \qquad Y^*$$
$$(\tilde{w}_\lambda - \lambda \cdot y) \text{ periodic} \qquad \text{in} \qquad Y^*$$
$$\frac{\partial \tilde{w}_\lambda}{\partial \nu_{A^*}} = 0 \qquad \text{on} \qquad \partial(\tau_i \cap Y), \qquad i = 1,\dots, M.$$

The function $\tilde{\eta}_\lambda = A^* \operatorname{grad} \tilde{w}_\lambda$ is extended by 0 inside τ_i $(i = 1,..., M)$. Let $\tilde{Q}\tilde{\eta}_\lambda$ denote this extension.

The matrix $\tilde{\mathscr{A}}$ is defined by

$$\tilde{\mathscr{A}}\lambda = \mathfrak{M}(\tilde{Q}\tilde{\eta}_\lambda) \qquad \text{for any } \lambda \in \mathbb{R}^2$$

and we introduce the functions:

$$\tilde{w}_\epsilon(x) = \epsilon(\tilde{P}\tilde{w}_\lambda)\left(\frac{x}{\epsilon}\right)$$

$$\tilde{\eta}_{\lambda\epsilon} = \tilde{\eta}_\lambda\left(\frac{x}{\epsilon}\right)$$

and

$$(\tilde{Q}_\epsilon\tilde{\eta}_{\lambda\epsilon})(x) = (\tilde{Q}\tilde{\eta}_\lambda)\left(\frac{x}{\epsilon}\right).$$

We have

$$-\operatorname{div} \tilde{Q}_\epsilon\tilde{\eta}_{\lambda\epsilon} = 0 \qquad \text{in} \qquad \Omega_\epsilon^*. \tag{34}$$

By the definitions of $\tilde{w}_\lambda$ and $\tilde{\eta}_\lambda$ we can now extract subsequences $\{w_\epsilon\}$ and $\{\tilde{Q}_\epsilon\tilde{\eta}_{\lambda\epsilon}\}$ such that

$$\tilde{w}_\epsilon \underset{\epsilon\to 0}{\rightharpoonup} \tilde{w}^* \qquad \text{in} \qquad H^1(\Omega) \text{ weakly}$$

$$\operatorname{grad} \tilde{w}_\epsilon \underset{\epsilon\to 0}{\rightharpoonup} \lambda \qquad \text{in} \qquad [L^2(\Omega)]^2 \text{ weakly}$$

$$\tilde{Q}_\epsilon\tilde{\eta}_{\lambda\epsilon} \underset{\epsilon\to 0}{\rightharpoonup} \tilde{\mathscr{A}}\lambda \qquad \text{in} \qquad [L^2(\Omega)]^2 \text{ weakly}$$

and

$$\operatorname{grad} \tilde{w}^* = \lambda.$$

Let $\varphi \in \mathfrak{D}(\Omega)$. Multiplying (29) by $\varphi\tilde{w}_\epsilon$ and (34) by $\varphi \cdot \tilde{P}_\epsilon u_\epsilon$ we get:

$$\begin{aligned}\int_\Omega \tilde{Q}_\epsilon\xi_\epsilon \cdot \nabla\varphi \cdot \tilde{w}_\epsilon\, dx &+ \int_\Omega \tilde{Q}_\epsilon\xi_\epsilon \cdot \varphi \cdot \nabla\tilde{w}_\epsilon\, dx \\ &- \int_\Omega \tilde{Q}_\epsilon\tilde{\eta}_{\lambda\epsilon} \cdot \nabla\varphi \cdot \tilde{P}_\epsilon u_\epsilon\, dx - \int_\Omega \tilde{Q}_\epsilon\tilde{\eta}_{\lambda\epsilon} \cdot \varphi \cdot \nabla(\tilde{P}_\epsilon u_\epsilon)\, dx \\ = \int_\Omega R_\epsilon f_\epsilon \cdot \varphi \cdot \tilde{w}_\epsilon\, dx.\end{aligned}$$

Therefore

$$\int_\Omega \xi^* \cdot \nabla\varphi \cdot \tilde{w}^*\, dx - \int_\Omega \tilde{\mathscr{A}}\lambda \cdot \nabla\varphi \cdot u^*\, dx = \int_\Omega \theta f \cdot \varphi \cdot \tilde{w}^*\, dx$$

which completes the proof.

Remarks. 1. A computation similar to the one used in the proof of Theorem 2 gives the coefficients $\tilde{q}_{ij}$ of the matrix $\tilde{\mathscr{A}}$:

$$\tilde{q}_{ij} = \frac{1}{|Y|} a^*_{Y^*}(\tilde{\chi}^i - y_i, \tilde{\chi}^j - y_j)$$

where

$$\tilde{\chi}^i = -(\tilde{w}_{\lambda_i} - y_i); \qquad \lambda_1 = (1, 0) \qquad \text{and} \qquad \lambda_2 = (0, 1).$$

Consequently, the homogenized matrix $\tilde{\mathscr{A}}$ and the limit function u do not depend on the extensions used in the proof.

2. Assumption A.5 is necessary to overcome the difficulties of extending u_ϵ in the holes intersecting the boundary $\partial\Omega$. However, we can always extend u_ϵ in any relatively compact open subset Ω' of Ω. In Ω' we extend u_ϵ by $P'_\epsilon u_\epsilon$ and we get

$$P'_\epsilon u_\epsilon \underset{\epsilon \to 0}{\rightharpoonup} u^* \qquad \text{in} \qquad H^1(\Omega') \text{ weakly}$$

where u^* is a solution of

$$\tilde{\mathscr{A}} u = -\operatorname{div}(\tilde{\mathscr{A}} \nabla u^*) = \theta f$$

but we know nothing about the value of u^* on $\partial\Omega$. The homogenization of the Neumann problem without the assumption A.5 is still an open problem.

3. In the case of Problems 2, 3 and 4, the method of asymptotic expansions (cf. Lions [4]) gives precise results regarding the order of convergence.

References

1. J. L. Lions and E. Magenes, "Non-Homogeneous Boundary Value Problems and Applications," Vols. 1 and 2, Springer-Verlag, New York/Heidelberg/Berlin, 1972.
2. H. Lanchon, Thèse, Paris, 1972.
3. H. Lanchon, Torsion élastoplastique d'un arbre cylindrique de section simplement ou multiplement connexe, *J. Mécanique* 13 (1974), 267–320.
4. J. L. Lions, Cours d'homogénéisation, Collège de France, 1975, 1976, 1977.
5. L. Tartar, Problèmes d'homogénéisation dans les équations aux dérivées partielles, Cours Peccot, Collège de France, 1977.
6. G. Duvaut, Comportement microscopique d'une plaque perforée périodiquement, to appear.