Journal of Combinatorial Theory, Series A **99**, 162–174 (2002)
doi:10.1006/jcta.2002.3267

On Pairs of Partitions with Steadily Decreasing Parts

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Communicated by George Andrews

Received December 6, 2002

A new generating function identity for special pairs of partitions with steadily decreasing parts is proved via a bijection. Viewing such pairs of partitions (or, more generally, special r-tuples of partitions) as coloured modular Young diagrams also allows to give bijective proofs for generating function identities due to Carlitz and Andrews. © 2002 Elsevier Science (USA)

Key Words: pairs of partitions; steadily decreasing parts; r-tuples of partitions; summatory maximum; generating functions.

1. INTRODUCTION

The starting point of this article was a problem arising in the work of Meinolf Geck on Hecke algebras of type B. The question was how to parameterize the simple modules for these algebras at $q = -1$ by suitable pairs of partitions; led by computations he conjectured a generating function for a special family of pairs of partitions with steadily decreasing parts. His conjecture is confirmed in this article as a consequence of Theorem 5.1. On the way towards proving this result some ideas were developed that provided easy constructive proofs of a classical theorem by Carlitz on pairs of partitions with steadily decreasing parts as well as its generalization to special r-tuples of partitions by Andrews.

In this article, we describe first a useful diagrammatic description for pairs of partitions with steadily decreasing parts. Then we use this description to provide a natural bijection proving Carlitz' Theorem. The diagrammatic description as well as the idea underlying the bijection generalize naturally also to the r-tuples of partitions considered by Andrews. In the final section, a more intricate map is constructed to transform pairs of partitions into distinct and steadily decreasing parts bijectively into pairs of partitions with odd resp. distinct and odd parts.

0097-3165/02 $35.00

2. PAIRS OF PARTITIONS WITH STEADILY DECREASING PARTS

Let $\alpha = (\alpha_1, \alpha_2, \ldots)$, $\beta = (\beta_1, \beta_2, \ldots)$ be partitions. If α is a partition of n and β a partition of m, we write $(\alpha, \beta) \vdash (n, m)$ and $|(\alpha, \beta)| = |\alpha| + |\beta| = n + m$.

We say that the pair (α, β) is a pair of partitions with steadily decreasing parts (see [1]) if the following condition is satisfied:

$$(*) \quad \min(\alpha_i, \beta_i) \geqslant \max(\alpha_{i+1}, \beta_{i+1}) \quad \text{for all } i.$$

For any $n, m \in \mathbb{N}_0$ we define

$$\mathscr{S}(n, m) = \{(\alpha, \beta) \vdash (n, m) \mid (\alpha, \beta) \text{ has steadily decreasing parts}\}$$

and we set $s(n, m) = |\mathscr{S}(n, m)|$; note $s(0, 0) = 1$.

We think of α, β in such a pair as partitions into different colours. For example, $\alpha = (6^2 421)$ and $\beta = (84^3 1^2)$ are depicted by

$$\alpha \quad \begin{array}{cccccc} | & | & | & | & | & | \\ | & | & | & | & | & | \\ | & | & | & | & & \\ | & | & & & & \\ | & & & & & \end{array} \qquad \beta \quad \begin{array}{cccccccc} - & - & - & - & - & - & - & - \\ - & - & - & - & & & & \\ - & - & - & - & & & & \\ - & - & - & - & & & & \\ - & & & & & & & \\ - & & & & & & & \end{array}$$

We then draw a diagram $Y(\alpha, \beta)$ for the pair by overlaying these diagrams:

$$Y(\alpha, \beta) \quad \begin{array}{cccccccc} + & + & + & + & + & + & - & - \\ + & + & + & + & | & | & & \\ + & + & + & + & & & & \\ + & + & - & - & & & & \\ + & & & & & & & \\ - & & & & & & & \end{array}$$

Condition $(*)$ is then equivalent to the condition that the length of each row in $Y(\alpha, \beta)$ is at most the length of the $+$ part of the previous row. In terms of the columns of the diagram it is also equivalent to the condition that each column in $Y(\alpha, \beta)$ contains at most one $|$ or one $-$ at its end (and that $Y(\alpha, \beta)$ has the shape of a Young diagram).

We also think of $Y(\alpha, \beta)$ as a generalized coloured 2-modular Young diagram, where 1 comes in two colours, denoted by the marked and the unmarked letter, and 2 carries both colours simultaneously:

$$\tilde{Y}(\alpha,\beta)\quad\begin{array}{cccccccc} 2 & 2 & 2 & 2 & 2 & 2 & 1' & 1' \\ 2 & 2 & 2 & 2 & 1 & 1 & & \\ 2 & 2 & 2 & 2 & & & & \\ 2 & 2 & 1' & 1' & & & & \\ 2 & & & & & & & \\ 1' & & & & & & & \end{array}$$

In fact, the conjugate diagram is a coloured 2-modular Young diagram, for the 2-coloured partition $(11', 8, 7', 7', 3, 3, 1', 1')$.

3. CARLITZ' THEOREM

Viewing bipartitions slightly differently, we define for $n, m \in \mathbb{N}$

$$\mathcal{T}(n,m) = \{\gamma \vdash (n,m) \mid \gamma \text{ has only parts of the form } (a, a-1), (a-1, a), (2a, 2a),\ a \in \mathbb{N}\}.$$

Then

THEOREM 3.1 (Carlitz [3]; see also Andrews [1, 12.4 and 12.5]). *For all $n, m \in \mathbb{N}_0$, $|\mathcal{S}(n,m)| = |\mathcal{T}(n,m)|$.*

Hence the generating function for pairs of partitions with steadily decreasing parts is

$$\sum_{n,m\in\mathbb{N}_0} s(n,m)x^n y^m = \prod_{a\in\mathbb{N}} (1 - x^a y^{a-1})^{-1}(1 - x^{a-1}y^a)^{-1}(1 - x^{2a}y^{2a})^{-1}.$$

Proof. We prove this by constructing a bijection $\varphi : \mathcal{S}(n,m) \to \mathcal{T}(n,m)$. Let $(\alpha,\beta) \in \mathcal{S}(n,m)$ and consider the corresponding diagram $\tilde{Y}(\alpha,\beta)$ defined in the previous section. For example, take $(\alpha,\beta) = (7^2 41, 9521^2)$, so

$$\tilde{Y}(\alpha,\beta)\quad\begin{array}{ccccccccc} 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1' & 1' \\ 2 & 2 & 2 & 2 & 2 & 1 & 1 & & \\ 2 & 2 & 1 & 1 & & & & & \\ 2 & & & & & & & & \\ 1' & & & & & & & & \end{array}$$

Now for obtaining the parts of $\varphi(\alpha, \beta) = \gamma \in \mathscr{T}(n, m)$, we transform the columns of $\tilde{Y}(\alpha, \beta)$ into parts of γ as follows:

$$\underbrace{2 \ldots 2}_{a-1} 1 \to (a, a-1),$$

$$\underbrace{2 \ldots 2}_{a-1} 1' \to (a-1, a),$$

$$\underbrace{2 \ldots 2}_{2a} \to (2a, 2a),$$

$$\underbrace{2 \ldots 2}_{2a-1} \to (a, a-1), (a-1, a).$$

Note that the weights of the colours are not changed in this process, so that indeed the resulting γ is a bipartition of (n, m). In the example above, after sorting the parts we obtain

$$\gamma = ((4,5), (3,2)^2, (2,2), (2,1)^3, (1,2), (0,1)^2).$$

Also, it is easy to see how to construct the inverse map. Given $\gamma \in \mathscr{T}(n, m)$, let m_a, m'_a be the multiplicity in γ of the parts $(a, a-1), (a-1, a)$, respectively, and let n_a be the multiplicity of the part $(2a, 2a)$, for $a \in \mathbb{N}$. Then the excess $|m_a - m'_a|$ gives the number of columns $\underbrace{2 \ldots 2}_{a-1} 1$ or $\underbrace{2 \ldots 2}_{a-1} 1'$ in the diagram $\tilde{Y}$ of a pair (α, β), depending on whether $m_a - m'_a$ is positive or negative. Furthermore, there are n_a columns of the form $\underbrace{2 \ldots 2}_{2a}$ and $\min(m_a, m'_a)$ columns of the form $\underbrace{2 \ldots 2}_{2a-1}$ in $\tilde{Y}$. Hence we have constructed $\tilde{Y}(\alpha, \beta)$ and thus we can read off (α, β). It is clear that these two maps are inverses to another, so φ gives a bijection as required. ∎

Remark 3.2. The bijection φ provides further refinements of the Carlitz identity. We describe the relation between some nice parameters of the partition pairs in $\mathscr{S}(n, m)$ and $\mathscr{T}(n, m)$, respectively. For $(\alpha, \beta) \in \mathscr{S}(n, m)$ and $\gamma = \varphi(\alpha, \beta) \in \mathscr{T}(n, m)$ we use the same notation as above.

(i) Counting the multiplicity of entries 1 and 1′, respectively, in the diagram $\tilde{Y}(\alpha, \beta)$ we immediately obtain

$$\sum_{\alpha_i > \beta_i} (\alpha_i - \beta_i) = \sum_{m_a > m'_a} (m_a - m'_a),$$

$$\sum_{\alpha_i < \beta_i} (\beta_i - \alpha_i) = \sum_{m_a < m'_a} (m'_a - m_a).$$

Associating corresponding weights to (α, β) and γ, respectively, this gives an identity for the weighted generating function.

(ii) By collecting the first entries of all parts of γ we obtain a partition $\gamma^{(1)} = (\gamma_1^{(1)}, \ldots)$, and similarly, from the second entries we obtain a partition $\gamma^{(2)} = (\gamma_1^{(2)}, \ldots)$.

We then have

$$l(\alpha) = \max(\gamma_1^{(1)}, 2a-1 \mid a \in \mathbb{N} \text{ with } \min(m_a, m_a') > 0),$$
$$l(\beta) = \max(\gamma_1^{(2)}, 2a-1 \mid a \in \mathbb{N} \text{ with } \min(m_a, m_a') > 0).$$

Carlitz also proved a "finite version" of his generating function theorem. For any $n, m, k \in \mathbb{N}_0$, let

$$\mathcal{S}_k(n,m) = \{(\alpha, \beta) \in \mathcal{S}(n,m) \mid l(\alpha), l(\beta) \leqslant k\}$$

and set $s_k(n,m) = |\mathcal{S}_k(n,m)|$; note $s_0(0,0) = 1$.

Then Carlitz' result is

THEOREM 3.3 (Carlitz [3, 4]). *For all $k \in \mathbb{N}_0$, the generating function for pairs of partitions with steadily decreasing parts and of length at most k is*

$$\sum_{n,m \in \mathbb{N}_0} s_k(n,m) x^n y^m = \prod_{a=1}^{k} \frac{1 - x^{2a-1} y^{2a-1}}{(1 - x^a y^{a-1})(1 - x^{a-1} y^a)(1 - x^a y^a)}.$$

Proof. The equation above is equivalent to the equation

$$\prod_{\substack{b=k+1 \\ b \text{ odd}}}^{2k-1} (1 - x^b y^b)^{-1} \sum_{n,m \in \mathbb{N}_0} s_k(n,m) x^n y^m$$
$$= \prod_{a=1}^{k} (1 - x^a y^{a-1})^{-1} (1 - x^{a-1} y^a)^{-1} \prod_{\substack{b=2 \\ b \text{ even}}}^{k} (1 - x^b y^b)^{-1}.$$

The objects counted on the left-hand side can be thought of as special 2-modular Young diagrams, where a 2-modular Young diagram $\tilde{Y}(\alpha, \beta)$ with at most k rows (corresponding to $(\alpha, \beta) \in \mathcal{S}_k(n,m)$) is complemented to the left by columns of odd height larger than k and at most $2k-1$, consisting of 2's only; i.e., for $k = 6$ a typical example looks like

$$
\begin{array}{ccccccccccc}
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1' & 1' \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & & \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & & & & \\
2 & 2 & 2 & 2 & 2 & 1' & 1' & & & & \\
2 & 2 & 2 & 2 & & & & & & & \\
2 & 2 & 2 & 1' & & & & & & & \\
2 & 2 & 2 & & & & & & & & \\
2 & & & & & & & & & & \\
2 & & & & & & & & & &
\end{array}
$$

We now apply our previously defined map φ to this diagram, reading the parts of the image under φ off the columns exactly as before. One sees immediately that the parts are of the form $(a, a-1)$, $(a-1, a)$ for $a \leqslant k$, and $(2a, 2a)$ for $2a \leqslant k$. The corresponding bipartitions are exactly the ones counted by the right-hand side of the equation above. ∎

4. ANDREWS' GENERALIZATION OF CARLITZ' THEOREM

Let $r \in \mathbb{N}$. For an r-tuple $(s_1, \ldots, s_r)$ of natural numbers the *summatory maximum* is defined as

$$
\begin{aligned}
\operatorname{smax}(s_1, \ldots, s_r) &= \left(\sum_{i=1}^{r} s_i \right) - (r-1) \min(s_1, \ldots, s_r) \\
&= \min(s_1, \ldots, s_r) + \sum_{i=1}^{r} (s_i - \min(s_1, \ldots, s_r)).
\end{aligned}
$$

The generalization of condition $(*)$ in Section 2 on a pair of partitions is now the following condition on an r-tuple of partitions $(\alpha^{(1)}, \ldots, \alpha^{(r)})$:

$$
(*)_r \quad \min(\alpha_i^{(1)}, \ldots, \alpha_i^{(r)}) \geqslant \operatorname{smax}(\alpha_{i+1}^{(1)}, \ldots, \alpha_{i+1}^{(r)}) \quad \text{for all } i.
$$

Let $\mathscr{S}(n_1, \ldots, n_r)$ denote the set of r-tuples of partitions $(\alpha^{(1)}, \ldots, \alpha^{(r)}) \vdash (n_1, \ldots, n_r)$ satisfying $(*)_r$, and set $s(n_1, \ldots, n_r) = |\mathscr{S}(n_1, \ldots, n_r)|$.

The analogue of the set $\mathscr{T}(n, m)$ is given as follows. Let $\mathscr{T}(n_1, \ldots, n_r)$ be the set of $\gamma \vdash (n_1, \ldots, n_r)$ such that γ has only parts of the form

$$
v_i(a) = (a, \ldots, a, a+1, a, \ldots a) \text{ with } a+1 \text{ at position } i \in \{1, \ldots, r\}
$$

or of the form

$$w_j(a) = (ra + j, \ldots, ra + j), \quad j \in \{2, \ldots, r\},$$

where $a \in \mathbb{N}_0$.

For indeterminates $x_1, \ldots, x_r$ and an r-tuple $v = (v_1, \ldots, v_r)$ we use the notation $x^v := x_1^{v_1} \cdots x_r^{v_r}$.

THEOREM 4.1 (Andrews [1, Section 12.4; 2]). *For all $n_1, \ldots, n_r \in \mathbb{N}_0$ we have*

$$|\mathcal{S}(n_1, \ldots, n_r)| = |\mathcal{T}(n_1, \ldots, n_r)|.$$

Hence the generating function for the r-tuples of partitions of type $\mathcal{S}$ is

$$\sum_{\mathbf{n}=(n_1,\ldots,n_r)\in\mathbb{N}_0^r} s(n_1, \ldots, n_r)x^{\mathbf{n}} = \prod_{a\in\mathbb{N}_0} \prod_{i=1}^{r} (1 - x^{v_i(a)})^{-1} \prod_{j=2}^{r} (1 - x^{w_j(a)})^{-1}.$$

Proof. For proving the assertion, we construct a bijection $\psi : \mathcal{S}(n_1, \ldots, n_r) \to \mathcal{T}(n_1, \ldots, n_r)$ generalizing the previous bijection φ.

For $\alpha = (\alpha^{(1)}, \ldots, \alpha^{(r)})$ we draw a diagram $Y(\alpha)$ by overlaying in the ith row the r contributions $m := \min(\alpha_i^{(1)}, \ldots, \alpha_i^{(r)})$ from the r partitions in α, and then ending on contributions $\alpha_i^{(j)} - m$ in r different colours in some order (note that in any given row only contributions in $r-1$ colours appear). So for $r = 3$ the diagram for $\alpha = (951^2, 861, 763)$ in the colours $|$, $-$ and $\tilde{}$ looks like

$$Y(\alpha) \quad \begin{array}{cccccccccc} \tilde{+} & \tilde{+} & \tilde{+} & \tilde{+} & \tilde{+} & \tilde{+} & \tilde{+} & | & | & - \\ \tilde{+} & \tilde{+} & \tilde{+} & \tilde{+} & \tilde{+} & - & \sim & & & \\ \tilde{+} & \sim & \sim & & & & & & & \\ | & & & & & & & & & \end{array}$$

The corresponding generalized coloured r-modular diagram is then

$$\tilde{Y}(\alpha) \quad \begin{array}{cccccccccc} 3 & 3 & 3 & 3 & 3 & 3 & 3 & 1 & 1 & 1' \\ 3 & 3 & 3 & 3 & 3 & 1' & 1'' & & & \\ 3 & 1'' & 1'' & & & & & & & \\ 1 & & & & & & & & & \end{array}$$

where we have used here the 3 coloured versions of 1: $1, 1', 1''$. In the general case, we will denote the r versions of 1 by 1_j, $j = 1, \ldots, r$. The smax-condition is exactly tailored to provide the analogous condition to the one we have used before, namely that each column in $\tilde{Y}$ ends on r or on at most one 1_j. So as before we turn the columns of $\tilde{Y}$ into parts of $\gamma = \psi(\alpha)$ as follows:

$$\underbrace{r \dots r}_{a} 1_j \to v_j(a),$$
$$\underbrace{r \dots r}_{ra+j} \to w_j(a) \quad \text{for } j = 2, \dots, r,$$
$$\underbrace{r \dots r}_{ra+1} \to v_1(a), \dots, v_r(a),$$

where $a \in \mathbb{N}_0$. As before, one easily constructs the inverse map to ψ to see that ψ is bijective. ∎

Similar as in the case of Carlitz' Theorem, the explicit bijection provides further refinements and allows to relate some natural parameters of the r-tuples of partitions in $\mathcal{S}(n_1, \dots, n_r)$ and $\mathcal{T}(n_1, \dots, n_r)$ in a nice way. We refrain here from spelling out the analogue of Remark 3.2 in detail.

Also, similarly as before, we can deduce a "finite version" of Andrews' Theorem. For stating this, define for $k \in \mathbb{N}_0$

$$\mathcal{S}_k(n_1, \dots, n_r) = \{(\alpha^{(1)}, \dots, \alpha^{(r)}) \in \mathcal{S}(n_1, \dots, n_r) \mid l(\alpha^{(i)}) \leqslant k \text{ for } i = 1, \dots r\}$$

and set $s_k(n_1, \dots, n_r) = |\mathcal{S}_k(n_1, \dots, n_r)|$.

Furthermore, for $a \in \mathbb{N}_0$ set $w(a) = (a, \dots, a) \in \mathbb{N}_0^r$. Then an argument similar to the one used for Theorem 3.3 shows:

THEOREM 4.2. *For all $k \in \mathbb{N}_0$, the generating function for r-tuples of partitions of type $\mathcal{S}$ and of length at most k is*

$$\sum_{\mathbf{n}=(n_1,\dots,n_r)\in\mathbb{N}_0^r} s_k(n_1, \dots, n_r) x^{\mathbf{n}} = \prod_{a=1}^{k} \frac{1 - x^{w_1(a)}}{(1 - x^{w(a)}) \prod_{i=1}^{r} (1 - x^{v_i(a)})}.$$

5. PAIRS OF PARTITIONS INTO DISTINCT PARTS

In this section, we now consider the special set of partition pairs which turned up in the work of Meinolf Geck on Hecke algebras, and which had originally motivated this article.

For $n \in \mathbb{N}$, let $D(n)$ be the set of partitions of n into distinct parts, and set $D = \bigcup_n D(n)$. We consider pairs of partitions with steadily decreasing parts and some further conditions:

$$\mathcal{R}(n, m) = \{(\alpha, \beta) \in \mathcal{S}(n, m) \mid \alpha = (\alpha_1, \alpha_2, \dots),\ \beta = (\beta_1, \beta_2, \dots) \in D,$$
$$\alpha_i \neq \beta_i \text{ for } i = 1, \dots, \min(l(\alpha), l(\beta))\}.$$

We then set

$$\mathscr{R}(k) = \bigcup_{\substack{n,m \in \mathbb{N}_0 \\ n+m=k}} \mathscr{R}(n,m)$$

and $\mathscr{R} = \bigcup_k \mathscr{R}(k)$. In the context of Geck's work, the set $\mathscr{R}(k)$ was suspected to be a suitable labelling set for simple modules of the Hecke algebra under consideration.

We will use again the diagrams introduced in Section 2 for a better understanding of the partition pairs in $\mathscr{R}$. So for example, the diagram of $(\alpha, \beta) = ((14, 11, 7, 6, 3, 2, 1), (11, 10, 9, 7, 4, 3)) \in \mathscr{R}$ looks like this:

$$\begin{array}{llllllllllllllll}
 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 \\
 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & & & \\
 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1' & 1' & & & & & \\
\tilde{Y}(\alpha, \beta) & 2 & 2 & 2 & 2 & 2 & 2 & 1' & & & & & & & \\
 & 2 & 2 & 2 & 1' & & & & & & & & & & \\
 & 2 & 2 & 1' & & & & & & & & & & & \\
 & 1 & & & & & & & & & & & & &
\end{array}$$

The conditions defining $\mathscr{R}$ can be translated into the following conditions on the diagram $\tilde{Y}$:

(i) Each row ends on a (non-empty) sequence of letters 1 or on a (non-empty) sequence of letters $1'$.

(ii) The length of each row is at most the length of the 2-part of the previous row, and if these lengths are equal then the rows end on the same colour.

Furthermore, let $O(n)$ be the set of partitions into odd parts only, and set $O = \bigcup_n O(n)$. Then we define

$$\mathscr{Q}(k) = \{(\lambda, \mu) \mid \lambda \in D \cap O,\ \ \mu \in O,\ \ |\lambda| + |\mu| = k\}.$$

THEOREM 5.1. *For all $k \in \mathbb{N}$ we have $|\mathscr{R}(k)| = |\mathscr{Q}(k)|$.*

Proof. We prove the assertion by constructing a bijection $\Phi : \mathscr{R}(k) \to \mathscr{Q}(k)$.

We start with a pair $(\alpha, \beta) \in \mathscr{R}(k)$ and its associated diagram $\tilde{Y}$ and show how to obtain the parts of $\Phi(\alpha, \beta) = (\lambda, \mu) \in \mathscr{Q}(k)$.

In the first step we take out all repeated columns from $\tilde{Y}$ ending on 1 or $1'$ (i.e., leaving only one column of each such type), remove the marks, and turn them directly into parts of μ. Let $\lambda^{(0)}$ be the partition corresponding to the left-over diagram (reading this as a 2-modular Young diagram), and let $\mu^{(0)}$ be the partition where we have collected the parts corresponding to the columns taken out. So $\lambda^{(0)}$ is a partition in $D \cap O$ with parts in two colours, and $\mu^{(0)} \in O$. In our example above we obtain in this first step:

$$
\begin{array}{cccccccccccc}
 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 \\
 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & \\
 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1' & & & \\
\lambda^{(0)} & 2 & 2 & 2 & 2 & 2 & 2 & 1' & & & & \\
 & 2 & 2 & 2 & 1' & & & & & & & \\
 & 2 & 2 & 1' & & & & & & & & \\
 & 1 & & & & & & & & & &
\end{array}
\qquad
\begin{array}{cccc}
 & 2 & 2 & 1 \\
\mu^{(0)} & 1 & & \\
 & 1 & &
\end{array}
$$

i.e., $\lambda^{(0)} = (21, 19, 15', 13', 7', 5', 1)$, $\mu^{(0)} = (5, 1^2)$. We observe that no information is lost at this point, since the colours of the columns moved as parts into $\mu^{(0)}$ can be recovered from the one copy of the column of each type that remained in $\lambda^{(0)}$. Furthermore, note that in $\lambda^{(0)}$ consecutive odd numbers have to be of the same colour. Note also that $\max \mu^{(0)} \leqslant 2l(\lambda^{(0)}) - 1$ since $\max \mu^{(0)}$ corresponds to a column in $\lambda^{(0)}$.

Now to obtain the final pair (λ, μ), we use the following algorithm: for the start of any consecutive sequence of colour $1'$ in $\lambda^{(0)}$ we iteratively take out a corresponding co-hook. More precisely, we proceed as follows. Consider the lowest sequence of consecutive odd numbers of colour $1'$ in $\lambda^{(0)}$ (possibly of length 1). Take out the co-hook corresponding to the $1'$ of the largest part of this sequence, or equivalently, remove the 'L'-shaped hook consisting of this part as well as one 2 from all the parts above it. Below, the entries in the corresponding L-shaped hook are marked in bold:

$$
\begin{array}{cccccccccccc}
 & \mathbf{2} & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 \\
 & \mathbf{2} & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & \\
 & \mathbf{2} & 2 & 2 & 2 & 2 & 2 & 2 & 1' & & & \\
\lambda^{(0)} & \mathbf{2} & 2 & 2 & 2 & 2 & 2 & 1' & & & & \\
 & \mathbf{2} & \mathbf{2} & \mathbf{2} & \mathbf{1'} & & & & & & & \\
 & 2 & 2 & 1' & & & & & & & & \\
 & 1 & & & & & & & & & &
\end{array}
$$

We stretch this L-hook into a part, remove the mark on $1'$ and put this part into $\mu^{(0)}$ to obtain $\mu^{(1)}$; note that this is automatically the largest part of this new partition. After taking this L-hook out of $\lambda^{(0)}$, we also remove all the marks on the letters $1'$ in the consecutive sequence we have just considered, and we thus obtain $\lambda^{(1)}$. In the example above, we have

$$\lambda^{(1)} \quad \begin{array}{llllllllll} \mathbf{2} & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 \\ \mathbf{2} & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & \\ \mathbf{2} & \mathbf{2} & \mathbf{2} & \mathbf{2} & \mathbf{2} & \mathbf{2} & \mathbf{1'} & & & \\ 2 & 2 & 2 & 2 & 2 & 1' & & & & \\ 2 & 2 & 1 & & & & & & & \\ 1 & & & & & & & & & \end{array} \qquad \mu^{(1)} \quad \begin{array}{llllllll} 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 \\ 2 & 2 & 1 & & & & & \\ 1 & & & & & & & \\ 1 & & & & & & & \end{array}$$

Note that we can reverse this step: by construction, the largest part of $\mu^{(1)}$ is now larger than $2l(\lambda^{(1)}) - 1$. We then insert this part as an L-hook into $\lambda^{(1)}$ at the highest possible place such that the resulting partition is in $D \cap O$; this is uniquely defined and gives a new part in the resulting partition which starts a consecutive sequence of odd parts. We then put marks on all the letters 1 in the parts of this sequence. More precisely, if $\lambda^{(1)} = (l_1, \ldots, l_k)$, $\mu^{(1)} = (m_1, \ldots)$ and $m_1 > 2k - 1$, then let $j \in \mathbb{N}_0$ be minimal such that $(l_1 + 2, l_2 + 2, \ldots, l_j + 2, m_1 - 2j, l_{j+1}, \ldots, l_k)$ is a partition into distinct parts; this recovers $\lambda^{(0)}$.

We now take the next step in the process of computing $\Phi(\alpha, \beta)$ by iterating the previous step; i.e., we consider again the highest part in the lowest consecutive sequence of colour $1'$ in $\lambda^{(1)}$, remove the corresponding L-hook (marked above in boldface), turn it into a part for the next partition $\mu^{(2)}$ and delete all the marks on the letters $1'$ on the parts of this sequence. Note that as our partition $\lambda^{(0)}$ is in $D \cap O$ (up to the colouring), the size of the L-hooks removed in this procedure is weakly increasing so that the new part on the μ-side is always the largest part at this step. In our example, we obtain in the next step

$$\lambda^{(2)} \quad \begin{array}{lllllllll} 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & \\ 2 & 2 & 2 & 2 & 2 & 1 & & & \\ 2 & 2 & 1 & & & & & & \\ 1 & & & & & & & & \end{array} \qquad \mu^{(2)} \quad \begin{array}{lllllllll} 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & \\ 2 & 2 & 1 & & & & & & \\ 1 & & & & & & & & \\ 1 & & & & & & & & \end{array}$$

The procedure ends when there are no more marked letters $1'$, say the final pair is $\lambda^{(r)}, \mu^{(r)}$. We then set $\Phi(\alpha, \beta) = (\lambda, \mu) = (\lambda^{(r)}, \mu^{(r)})$. So in our example, $\lambda = \lambda^{(2)} = (17, 15, 11, 5, 1)$, $\mu = \mu^{(2)} = (17, 15, 5, 1^2)$.

Note that one can also write down a (somewhat clumsy) formula for (λ, μ) directly from the parts of $(\lambda^{(0)}, \mu^{(0)})$, corresponding to taking out all the L-hooks at once.

By the remarks made above it is clear how to construct the inverse map. The main point is that as long as the maximal part on the μ-side is larger than the longest column on the λ-side it is always possible to insert this maximal part from the μ-side as an L-hook into the λ-side such that it is the starting part of a consecutive sequence of odd parts. Once the largest part is at most as large as the largest column on the λ-side we are at the easy part of the procedure where we just put back all these small parts as repeating columns into the λ-side. ∎

As it is easy to write down the generating function for partition pairs of type $\mathcal{Q}$, we now immediately obtain the generating function for partition pairs of type $\mathcal{R}$, confirming the conjecture by Geck mentioned in the Introduction. Let $r(k) = |\mathcal{R}(k)|$ for $k \in \mathbb{N}$, $r(0) = 1$.

Corollary 5.2. *The generating function for the pairs of partitions of type $\mathcal{R}$ is*

$$\sum_{k \geqslant 0} r(k)x^k = \prod_{k \geqslant 1} \frac{1 + x^{2k-1}}{1 - x^{2k-1}} = \prod_{k \geqslant 1} (1 + x^{2k-1})(1 + x^k).$$

Remark 5.3. Let $(\alpha, \beta) \in \mathcal{R}(k)$ and $\Phi(\alpha, \beta) = (\lambda, \mu) \in \mathcal{Q}(k)$, $\alpha = (\alpha_1, \alpha_2, \ldots)$, $\beta = (\beta_1, \beta_2, \ldots)$, $\lambda = (\lambda_1, \ldots)$, $\mu = (\mu_1, \ldots) = (1^{m_1} 2^{m_2} \ldots)$.

There are a number of nice relations between the parameters of (α, β) and (λ, μ) that can easily be derived from the description of the bijection above. These can be viewed as giving refinements of Theorem 5.1 and Corollary 5.2, respectively.

(i) $\max(\alpha_1, \beta_1) = \alpha_1$ if and only if $\max(\lambda_1, \mu_1) = \lambda_1$.

More precisely, we have the following:

If $\lambda_1 \geqslant \mu_1$, then $\alpha_1 = \frac{1}{2}(\lambda_1 + 1) + l(\mu)$ and $\beta_1 = \frac{1}{2}(\lambda_1 - 1) + l(\mu) - m_1$.

If $\lambda_1 < \mu_1$, then $\alpha_1 = \frac{1}{2}(\mu_1 - 1) + l(\mu) - 1 - m_1$ and $\beta_1 = \frac{1}{2}(\mu_1 + 1) + l(\mu) - 1$.

(ii) $\sum_i |\alpha_i - \beta_i| = l(\lambda) + l(\mu)$.

(iii) Let $b \in \mathbb{N}_0$ be minimal such that $\mu_{b+1} \leqslant 2(l(\lambda) + b) - 1$. Then

(a) b is the number of connected $1'$-components in $\tilde{Y}(\alpha, \beta)$, where we consider two entries $1'$ in the diagram as connected if the corresponding boxes of the diagram intersect in an edge or in a vertex.

(b) $l(\lambda) + b = \max(l(\alpha), l(\beta))$.

ACKNOWLEDGMENTS

I am grateful to Meinolf Geck for communicating his generating function conjecture to me. Thanks go also to the Isaac Newton Institute for Mathematical Sciences of the University of Cambridge for their hospitality during my stay there in the frame of the programme *Symmetric functions and Macdonald polynomials*, where part of the work for this article was done.

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