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Prime ideals in ultraproducts of commutative rings

Bruce Olberding [a,*], Jay Shapiro [b,1]

[a] *Department of Mathematical Sciences, New Mexico State University, Las Cruces, NM 88003-8001, USA*
[b] *Department of Mathematics, George Mason University, Fairfax, VA 22030, USA*

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Abstract

We describe classes of prime ideals in ultraproducts of commutative rings. We consider in particular prime ideals in ultraproducts of Noetherian rings, Krull domains, finite character rings and QR-domains.

Introduction

If $\{R_i\colon i \in I\}$ is a collection of commutative rings, then the ultraproduct $R^* = \prod_{\mathcal{U}} R_i$ preserves many properties of its component rings R_i. More precisely, by a theorem of Łos a first-order sentence in the language of commutative rings is satisfied by R^* if and only if it is satisfied by "almost all" of the R_i [1, Theorem 4.1.9, p. 216]. Thus ultraproducts are of importance in understanding the first-order theory of a given class of rings, and there are a number of applications of model-theoretic algebra (e.g., to the existence of various bounds in commutative algebra) which use the ultraproduct construction in a fundamental way. Some interesting instances of this technique can be found in [18–20] and [21].

* Corresponding author.
E-mail addresses: olberdin@nmsu.edu (B. Olberding), jshapiro@gmu.edu (J. Shapiro).
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Many important ideal-theoretic properties cannot be expressed as first-order sentences and fail to be preserved by ultraproducts. In particular, except in some very special cases, an ultraproduct of Noetherian rings is non-Noetherian, has infinite Krull dimension and a very complicated prime spectrum. Yet it is of interest in applications to understand ultraproducts of Noetherian rings. For example recent articles by H. Schoutens [19–21] rely on ultraproducts of Noetherian rings as a means of transferring characteristic $p > 0$ devices such as tight closure to affine $\mathbb{C}$-domains.

In the article [16] we consider the relationship between a Noetherian domain, its completion, valuations and ultrapowers. In the present article we focus on describing prime ideals in ultraproducts of several classes of rings, including those rings that are Noetherian. In Section 3 we describe all the maximal ideals in ultraproducts of finite character rings or QR-domains, and postpone till Section 5 a description of all the maximal ideals in an ultraproduct of d-dimensional Noetherian rings (see Corollary 5.6).

In Section 4 we describe the maximal prime divisors of any induced ideal (defined below) in an ultraproduct of a class of rings large enough to include Noetherian rings and QR-domains.

In general, nonzero ideals of an ultraproduct of rings have infinite height, but by introducing in Section 5 a notion of "ultra-height" for ideals of ultraproducts, we are able to partition and examine the set of prime ideals of finite ultra-height in an ultraproduct of rings from a class that includes the rings having a Noetherian prime spectrum. When $d \geqslant 0$ and R^* is an ultraproduct of rings all having dimension bounded above by d, then every prime ideal of R^* has finite ultra-height. Moreover, we show in Theorem 5.4 that every prime ideal of ultra-height n is contained in a prime ideal that is unique among ideals that are maximal with respect to having ultra-height n.

In Section 6 we consider chains of prime ideals in an ultraproduct and in Section 7 we use these results to describe in Corollary 7.5 all the prime ideals of ultra-height one in an ultraproduct of Krull domains. Hence we recover in more generality results from [11], in which prime ideals in an ultrapower of the ring of integers are described, and [2], which considers ultraproducts of Dedekind domains. We note also in Remark 7.4 how one may combine our results here with those of [16] to obtain a description of all the prime ideals in an ultrapower of a one-dimensional Noetherian domain having module-finite integral closure.

Our approach throughout this article turns on the basic problem: If R is a ring and S is a union of a collection $\mathcal{P}$ of prime ideals of R, what are the prime ideals of R that are maximal among ideals in S; equivalently, what are the maximal ideals of the ring R localized at $R \setminus S$? In Section 2 we examine a general setting in which this problem can be solved. This setting is large enough to encompass the ultraproducts we consider in later sections. We phrase this setting in terms of lattices of Zariski closed subsets of subspaces of prime ideals, and we show in Theorem 2.9 that maximal filters on this lattice correspond to prime ideals maximal in the union S of prime ideals. We deduce from this in Theorem 2.10 a characterization of when every maximal ideal of a ring can be described using filters on a certain lattice of closed subsets of a given collection of prime ideals.

1. Notation and preliminaries

In this section we discuss notation, lattices and the ultraproduct construction. All rings considered in this paper are commutative and have an identity. Some of our later results are phrased in terms of lattices. As usual, we write $\wedge$ and $\vee$ for the join and meet of a lattice. All the lattices we consider will be sublattices consisting of certain subsets of a given set. Hence all our lattices are distributive with meet $\cap$ and join $\cup$, and in all cases that we consider these lattices will contain a least element, namely the empty set ϕ. However the lattices we consider often will not have a top element.

1.1. Filters on lattices

If $\mathcal{L}$ is a lattice with least element ϕ, then a nonempty subset $\mathcal{F} \subseteq \mathcal{L}$ is a *filter* on $\mathcal{L}$ if for all $V, W \in \mathcal{L}$ we have:

(i) $V \wedge W \in \mathcal{F}$, if both $V \in \mathcal{F}$ and $W \in \mathcal{F}$;
(ii) if $V \in \mathcal{F}$ and $V \leqslant W$, then $W \in \mathcal{F}$.

A filter $\mathcal{F}$ on $\mathcal{L}$ is *proper* if it is properly contained in $\mathcal{L}$, and the proper filter $\mathcal{F}$ is a *maximal filter* if the only filter that contains it is $\mathcal{L}$. By Zorn's Lemma, every proper filter of a lattice $\mathcal{L}$ with least element extends to a maximal filter on $\mathcal{L}$.

1.2. Ultrafilters

Throughout the paper I will stand for a fixed index set. If $\mathcal{L}$ is the collection of all subsets of I, then an *ultrafilter on* I is a maximal filter on the lattice $\mathcal{L}$; equivalently, $\mathcal{U}$ is an ultrafilter on I if and only if for all $A \subseteq I$, either $A \in \mathcal{U}$ or $I \setminus A \in \mathcal{U}$. It is not hard to see that if an ultrafilter $\mathcal{U}$ contains a finite set, then it contains a singleton set, say $\{i\}$, and i is an element of every element of $\mathcal{U}$. In this case we say $\mathcal{U}$ is a *principal* ultrafilter. An ultrafilter that is not principal is called a *free* ultrafilter.

1.3. Ultraproducts and ultrapowers

Let $\{R_i\}_{i \in I}$ be a collection of rings indexed by a set I. If $\mathcal{U}$ is an ultrafilter on I, then we write $R^* = \prod_{\mathcal{U}} R_i$ for the *ultraproduct* of the R_i's with respect to the ultrafilter $\mathcal{U}$. An element of $\prod_{\mathcal{U}} R_i$ is an equivalence class of elements of $\prod_{i \in I} R_i$ defined by:

$$(a_i)_{i \in I} \sim (b_i)_{i \in I} \quad \Leftrightarrow \quad \{i \in I \colon a_i = b_i\} \in \mathcal{U}.$$

By an abuse of notation we denote by (a_i) the element of R^* determined by the equivalence class of (a_i). Since we only consider ultraproducts (with a brief exception in Definition 6.1), this should not cause any confusion. In the case that there is a ring R such that for all $i \in I$, $R_i = R$, then R^* is the *ultrapower* of R.

1.4. *Properties that hold for $\mathcal{U}$-many i*

Given a collection $\{X_i\}_{i\in I}$ of sets X_i indexed by I and an ultrafilter $\mathcal{U}$ on I, we say that a property $\mathcal{P}$ *holds for $\mathcal{U}$-many i* if the set of all i such that X_i satisfies $\mathcal{P}$ is an element of $\mathcal{U}$.

1.5. *Induced ideals of ultraproducts*

Let $\{R_i\}$ be a collection of rings indexed by a set I, and let $\mathcal{U}$ be an ultrafilter on I. Then R^*, as a homomorphic image of a product of commutative rings, is a commutative ring. If for each i, S_i is a subset of R_i, then we write (S_i) for the subset of R^* consisting of elements of the form (s_i), $s_i \in S_i$. An ideal A of R^* is *induced* if $A = (A_i)$ for some subsets A_i of R. Observe that $A = (A_i)$ is an induced ideal of R^* if and only if for $\mathcal{U}$-many i, A_i is an ideal of R_i. More information on induced ideals can be found in [15]. It is easily checked, using properties of ultrafilters, that an induced ideal $P = (P_i)$ of R^* is prime (respectively maximal) if and only if for $\mathcal{U}$-many i, P_i is a prime (respectively maximal) ideal of R_i. Thus R^* is a domain if and only if for $\mathcal{U}$-many i, R_i is a domain.

2. Prime ideals and maximal filters

In this section we introduce a general framework for describing prime ideals arising from certain lattices of closed sets of subspaces of $\mathrm{Spec}(R)$, where R is a commutative ring. In the following sections we apply this construction to the case where R is an ultraproduct of rings.

Definition 2.1. Let $\mathcal{P}$ be a set of prime ideals of a ring R. For A an ideal of R and V a nonempty subset of $\mathcal{P}$, we define

$$\mathbf{V}_{\mathcal{P}}(A) = \{P \in \mathcal{P}\colon\ A \subseteq P\},$$
$$\mathbf{J}(V) = \bigcap_{P\in V} P,$$
$$\mathbf{S}(V) = \bigcup_{P\in V} P.$$

If V is an empty set, we define $\mathbf{J}(V) = \mathbf{S}(V) = R$.

Let V be a nonempty collection of prime ideals of a ring R. Since $\mathbf{S}(V)$ is a union of prime ideals, it follows that $R \setminus \mathbf{S}(V)$ is a multiplicatively closed set. Thus there is a one-to-one correspondence between the prime ideals of R that are maximal in $\mathbf{S}(V)$ and the maximal ideals in the ring R localized at $R \setminus \mathbf{S}(V)$. Moreover by Zorn's Lemma the set of ideals of R contained in $\mathbf{S}(V)$ contains maximal members and these ideals are necessarily prime. In Theorem 2.9 we show that when the prime ideals in $\mathbf{S}(V)$ satisfy a prime avoidance property with respect to finitely generated ideals, then the prime ideals maximal in $\mathbf{S}(V)$ can be classified using ideals such as in (i) of the following definition.

Definition 2.2. Let $\mathcal{P} \subseteq \operatorname{Spec}(R)$, and let $\mathcal{L}$ be a sublattice of the lattice $\mathcal{L}(\mathcal{P})$ of all subsets of $\mathcal{P}$.

(i) If $\mathcal{F}$ a proper filter on $\mathcal{L}$, then

$$(\mathcal{F}) := \bigcup_{V \in \mathcal{F}} \mathbf{J}(V).$$

(ii) If Q is a prime ideal of R, then

$$\mathcal{F}(Q) := \{V \in \mathcal{L}\colon \mathbf{J}(V) \subseteq Q\}.$$

A priori $\mathcal{F}(Q)$ is merely a subset (possibly empty) of $\mathcal{L}$. We will show in Theorem 2.9 that under certain circumstances it is a maximal filter on $\mathcal{L}$. On the other hand $(\mathcal{F})$ is always at least a radical ideal of R. For let $a, b \in (F)$. Then $\mathbf{V}_{\mathcal{P}}(a) \cap \mathbf{V}_{\mathcal{P}}(b) \subseteq \mathbf{V}_{\mathcal{P}}(a+b)$. Clearly, since $\mathcal{F}$ is a filter, $\mathbf{V}_{\mathcal{P}}(a) \cap \mathbf{V}_{\mathcal{P}}(b)$ contains an element of $\mathcal{F}$. Thus $\mathcal{F}$ is closed under addition. Since $(\mathcal{F})$ is a union of radical ideals, this is sufficient to show that $(\mathcal{F})$ is a radical ideal. With an additional assumption on $\mathcal{L}$ we will show in Lemma 2.7 that $(\mathcal{F})$ is a prime ideal.

If $\mathcal{P}$ is a collection of prime ideals of a ring R, then $\mathcal{P}$ can be viewed as a topological subspace of $\operatorname{Spec}(R)$ (with the Zariski topology) under the subspace topology. A closed subset V of $\mathcal{P}$ is a set of the form $\mathbf{V}_{\mathcal{P}}(A)$, where A is an ideal of R. Thus V is closed in $\mathcal{P}$ if and only if $\mathbf{V}_{\mathcal{P}}(\mathbf{J}(V)) = V$.

We distinguish in the following definition three classes of sublattices of the lattice of closed subsets of a collection $\mathcal{P}$ of prime ideals.

Definition 2.3. Let $\mathcal{P} \subseteq \operatorname{Spec}(R)$, and let $\mathcal{L}$ be a sublattice of the lattice of closed subsets of $\mathcal{P}$.

(i) $\mathcal{L}$ is *weakly saturated* if for each $V \in \mathcal{L}$ and subset W of V, $W \in \mathcal{L}$ if and only if $W = \mathbf{V}_V(A)$ for some finitely generated ideal A of R.
(ii) $\mathcal{L}$ is *saturated* if $\mathcal{L}$ is weakly saturated and for each $V \in \mathcal{L}$ and finitely generated ideal A of R with $\mathbf{V}_V(A) = \phi$, there exists $a \in A$ such that $\mathbf{V}_V(a) = \phi$.
(iii) $\mathcal{L}$ is *strongly saturated* if $\mathcal{L}$ is weakly saturated and for each $V \in \mathcal{L}$ and finitely generated ideal A of R, $\mathbf{V}_V(A) = \mathbf{V}_V(a)$ for some $a \in A$.

Remark 2.4. The saturated property (ii) is equivalent to a prime avoidance condition on the elements of a weakly saturated lattice $\mathcal{L}$. Namely $\mathcal{L}$ is saturated if and only if given any $V \in \mathcal{L}$ and finitely generated ideal A of R such that $A \subseteq \mathbf{S}(V)$, then $A \subseteq P$ for some $P \in V$. This is just the contrapositive of the statement in (ii). To see this note that the statement $\mathbf{V}_V(a) \neq \phi$ for all $a \in A$ is equivalent to $A \subseteq \mathbf{S}(V)$. Thus suppose that for some $V \in \mathcal{L}$ there is a finitely generated ideal $A \subseteq \mathbf{S}(V)$. Hence by (ii), $\mathbf{V}_V(A) \neq \emptyset$ and so $A \subseteq P$ for some $P \in V$.

One of our motivating examples of a weakly saturated lattice is that of the "basis lattice" for a collection of prime ideals:

Definition 2.5. If $\mathcal{P} \subseteq \operatorname{Spec}(R)$, then the *basis lattice* of $\mathcal{P}$ is the lattice $\mathcal{L}$ of subsets of $\mathcal{P}$ whose members are of the form $\mathbf{V}_{\mathcal{P}}(A)$, where A is a product of finitely many nonzero finitely generated ideals of R. In the lattice $\mathcal{L}$ meet is defined by $\cap$ and join is defined by $\cup$. Note that $\mathbf{V}_{\mathcal{P}}(A) \cap \mathbf{V}_{\mathcal{P}}(B) = \mathbf{V}_{\mathcal{P}}(A+B)$ and $\mathbf{V}_{\mathcal{P}}(A) \cup \mathbf{V}_{\mathcal{P}}(B) = \mathbf{V}_{\mathcal{P}}(AB)$.

That a basis lattice $\mathcal{L}$ for a collection $\mathcal{P}$ of prime ideals is weakly saturated is a consequence of the relevant definitions. It is important for the application of these notions in later sections to note that $\mathbf{V}_{\mathcal{P}}(0)$ $(= \mathcal{P})$ need not be in the basis lattice of $\mathcal{P}$. Indeed, $\mathcal{P}$ is a member of the basis lattice of $\mathcal{P}$ if and only if there exists a product $A = A_1 \cdots A_n$ of finitely many finitely generated *nonzero* ideals A_i of R such that A is contained in every prime ideal in $\mathcal{P}$. Thus it can only happen that $\mathcal{P}$ is not a member of the basis lattice of $\mathcal{P}$ if R is a domain. For if R contains zero-divisors, then necessarily 0 is a product of nonzero finitely generated ideals, so that $\mathcal{P} = \mathbf{V}_{\mathcal{P}}(0)$ is in the basis lattice of $\mathcal{P}$.

Example 2.6. We mention here for later reference three examples.

(i) Let $\mathcal{L}$ be a weakly saturated lattice on a set of incomparable prime ideals $\mathcal{P}$ such that for all $V \in \mathcal{L}$, V is a finite set. (Such examples are encountered later in Theorem 3.5, Remark 3.6(i) and Remark 4.4(i).) If A is any ideal of R and $V \in \mathcal{L}$, then since V is finite, the Prime Avoidance Theorem (see for example [4, Lemma 3.3]) implies that there is an element a of A contained in precisely the same prime ideals in V that contain A, that is, $\mathbf{V}_V(A) = \mathbf{V}_V(a)$. Hence $\mathcal{L}$ is strongly saturated.

(ii) If R is a ring containing an uncountable field, then any countable set $\mathcal{P}$ of prime ideals has the property that every finitely generated ideal contained in a union of prime ideals in $\mathcal{P}$ is contained in one of these prime ideals [23]. Thus if $\mathcal{L}$ is a weakly saturated lattice on $\mathcal{P} \subseteq \operatorname{Spec}(R)$ such that each element $V \in \mathcal{L}$ is countable, then $\mathcal{L}$ is saturated. Therefore, if R contains an uncountable field and $\mathcal{P}$ is a countable subset of $\operatorname{Spec}(R)$, then the basis lattice of $\mathcal{P}$ is saturated.

(iii) Let R be a ring having the property that the radical of any finitely generated ideal is the radical of a principal ideal. If A is a finitely generated ideal that is contained in a union $\bigcup_{P \in X} P$ of prime ideals of R, then $\sqrt{A} \subseteq \bigcup_{P \in X} P$ and by assumption $\sqrt{A} = \sqrt{aR}$ for some $a \in R$. By taking a suitable power of a, we may assume that $a \in A$. Thus any weakly saturated lattice on a collection $\mathcal{P}$ of prime ideals of R is necessarily strongly saturated since for each finitely generated ideal A of R, there exists $a \in A$ such that $\mathbf{V}_V(A) = \mathbf{V}_V(a)$ for any $V \in \mathcal{L}$.

The QR-domains provide a large and diverse class of rings having this property. An integral domain R is a *QR-domain* if every overring of R is a localization of R with respect to a multiplicatively closed subset. The QR-domains are precisely the Prüfer domains having the property that the radical of any finitely generated ideal of R is the radical of a principal ideal, where a domain is a *Prüfer domain* if every finitely generated ideal of R is invertible [6, Theorem 27.5]. Interesting examples of QR-domains include the ring of entire functions and holomorphy rings ([5, Proposition 8.1.1], [17]). More generally, it evident that any Prüfer domain with torsion Picard group is a QR-domain. Also, it is not hard to see that a QR-domain R is Noetherian if and only if R is a Dedekind domain with torsion class group. Thus the class of QR-domains is quite distinct from the class of Noetherian domains.

Lemma 2.7. *Let $\mathcal{P} \subseteq \mathrm{Spec}(R)$ and let $\mathcal{F}$ and $\mathcal{G}$ be maximal filters on a weakly saturated lattice $\mathcal{L}$ of closed subsets of $\mathcal{P}$.*

(i) *If $V, W \in \mathcal{L}$ with $V \cup W \in \mathcal{F}$, then $V \in \mathcal{F}$ or $W \in \mathcal{F}$.*
(ii) *$(\mathcal{F})$ is a prime ideal of R contained in $\mathbf{S}(V)$ for each $V \in \mathcal{F}$.*
(iii) *If $(\mathcal{F}) \subseteq (\mathcal{G})$, then $\mathcal{F} = \mathcal{G}$.*
(iv) *If A is a finitely generated ideal, $A \subseteq (\mathcal{F})$ and $V \in \mathcal{F}$, then $A \subseteq P$ for some $P \in V$.*

Moreover, the mapping $\mathcal{F} \mapsto (\mathcal{F})$ is an injection from the set of maximal filters on $\mathcal{L}$ into $\mathrm{Spec}(R)$.

Proof. (i) This follows from the fact that $\mathcal{L}$ is a distributive lattice and $\mathcal{F}$ is a maximal filter [3, Theorem 9.7, p. 186]. We note that this part of the lemma does not need the assumption that $\mathcal{L}$ is weakly saturated.

(ii) Suppose that $a, b \in R$ and $ab \in (\mathcal{F})$. Then there exists $W \in \mathcal{F}$ such that $ab \in \mathbf{J}(W)$. Hence $W \subseteq \mathbf{V}_{\mathcal{P}}(ab) = \mathbf{V}_{\mathcal{P}}(a) \cup \mathbf{V}_{\mathcal{P}}(b)$. Thus $\mathbf{V}_W(a) \cup \mathbf{V}_W(b) = W \in \mathcal{F}$ and since $\mathcal{L}$ is weakly saturated both $\mathbf{V}_W(a)$ and $\mathbf{V}_W(b)$ are in $\mathcal{L}$. Hence by (i) either $\mathbf{V}_W(a) \in \mathcal{F}$ or $\mathbf{V}_W(b) \in \mathcal{F}$. Thus $a \in (\mathcal{F})$ or $b \in (\mathcal{F})$, so $(\mathcal{F})$ is a prime ideal. Finally, if $V \in \mathcal{F}$, then since $(\mathcal{F}) = \bigcup_{\{W \in \mathcal{L}:\ W \subseteq V\}} \mathbf{J}(W)$, it follows that $(\mathcal{F}) \subseteq \mathbf{S}(V)$.

(iii) Let $V \in \mathcal{F}$. We will show that $V \in \mathcal{G}$. Let $U \in \mathcal{G}$. Since $\mathcal{G}$ is a filter and $V \cup U \in \mathcal{L}$, it is the case that $V \cup U \in \mathcal{G}$. Since $\mathcal{L}$ is weakly saturated, there exists a finitely generated ideal A of R such that $V = \mathbf{V}_{V \cup U}(A)$. Since $V \in \mathcal{F}$, it follows that $A \subseteq \mathbf{J}(V) \subseteq (\mathcal{F}) \subseteq (\mathcal{G})$. Therefore, since A is finitely generated, we see that $A \subseteq \mathbf{J}(W_1) + \cdots + \mathbf{J}(W_n)$ for some $W_1, W_2, \ldots, W_n \in \mathcal{G}$. Hence $W_1 \cap W_2 \cap \cdots \cap W_n \subseteq \mathbf{V}_{\mathcal{P}}(\mathbf{J}(W_1) + \cdots + \mathbf{J}(W_n)) \subseteq \mathbf{V}_{\mathcal{P}}(A)$. Thus $(V \cup U) \cap W_1 \cap \cdots \cap W_n \subseteq \mathbf{V}_{V \cup U}(A) = V$. Since $\mathcal{G}$ is a filter and $V \cup U$, $W_1, \ldots, W_n \in \mathcal{G}$, it follows that $V \in \mathcal{G}$. Hence $\mathcal{F} \subseteq \mathcal{G}$, and since $\mathcal{F}$ is a maximal filter, we have $\mathcal{F} = \mathcal{G}$.

(iv) Since A is finitely generated and contained in $(\mathcal{F})$, there exist $W_1, W_2, \ldots, W_n \in \mathcal{F}$ such that $A \subseteq \mathbf{J}(W_1) + \cdots + \mathbf{J}(W_n)$. Since $V, W_1, W_2, \ldots, W_n \in \mathcal{F}$, it follows that $V \cap W_1 \cap W_2 \cap \cdots \cap W_n \in \mathcal{F}$. In particular, there exists $P \in V \cap W_1 \cap W_2 \cap \cdots \cap W_n$, and necessarily $A \subseteq P$.

The final statement of the lemma is a consequence of (ii) and (iii). $\square$

Lemma 2.8. *Let $\mathcal{P} \subseteq \mathrm{Spec}(R)$ and let $\mathcal{L}$ be a weakly saturated lattice of closed subsets of $\mathcal{P}$. If $V \in \mathcal{L}$ and Q is maximal as an ideal contained in $\mathbf{S}(V)$, then $\mathbf{J}(V) \subseteq Q$ (equivalently, $V \in \mathcal{F}(Q)$).*

Proof. Since every element of $\mathbf{J}(V)$ is contained in every prime ideal in V, it follows that $\mathbf{J}(V) + Q \subseteq \mathbf{S}(V)$. Thus the maximality of Q in $\mathbf{S}(V)$ implies $\mathbf{J}(V) \subseteq Q$, so $V \in \mathcal{F}(Q)$. $\square$

Theorem 2.9. *Let $\mathcal{P} \subseteq \mathrm{Spec}(R)$ and let $\mathcal{L}$ be a weakly saturated lattice of closed subsets of $\mathcal{P}$. The following statements are equivalent.*

(i) *For each $V \in \mathcal{L}$, the mappings $\mathcal{F} \mapsto (\mathcal{F})$ and $Q \mapsto \mathcal{F}(Q)$ form a one-to-one correspondence between maximal filters $\mathcal{F}$ on $\mathcal{L}$ containing V and the prime ideals Q of R maximal among ideals contained in $\mathbf{S}(V)$.*

(ii) *$\mathcal{L}$ is saturated.*

Proof. To see that (i) implies (ii) it suffices, as noted in Remark 2.4, to show that if A is a finitely generated ideal of R with $A \subseteq \mathbf{S}(V)$ for some $V \in \mathcal{L}$, then $A \subseteq P$ for some $P \in V$. Since $A \subseteq \mathbf{S}(V)$, it is contained in an ideal Q that is maximal among ideals that are subsets of $\mathbf{S}(V)$. But (i) implies that $Q = (\mathcal{F})$ for some maximal filter $\mathcal{F}$ on $\mathcal{L}$ containing V. Hence by Lemma 2.7(iv) we are done.

We now show that (ii) implies (i). Let $V \in \mathcal{L}$ and let Q be a prime ideal of R that is maximal among ideals contained in the set $\mathbf{S}(V)$. We claim that Q is equal to some $(\mathcal{F})$, where $\mathcal{F}$ is a maximal filter on $\mathcal{L}$ containing V. Define

$$\mathcal{E} = \left\{\mathbf{V}_V(A)\colon\ A \text{ is a finitely generated ideal with } A \subseteq Q\right\}.$$

Let A and B be finitely generated ideals that are contained in Q. Then $\mathbf{V}_V(A)$ and $\mathbf{V}_V(B)$ are in $\mathcal{E}$, and $\mathbf{V}_V(A) \cap \mathbf{V}_V(B) = \mathbf{V}_V(A+B) \in \mathcal{E}$. Hence $\mathcal{E}$ is closed under finite intersections. Also note that if $A \subseteq Q$, then since $Q \subseteq \mathbf{S}(V)$, we have that $A \subseteq \mathbf{S}(V)$. Since $\mathcal{L}$ is saturated and A is finitely generated, by the prime avoidance condition noted in Remark 2.4 it follows that A is contained in some element of V. In particular, $\mathbf{V}_V(A) \neq \phi$ and so $\phi \notin \mathcal{E}$. Therefore, since $\mathcal{E}$ does not contain the empty set and $\mathcal{E}$ is closed under finite intersections, it follows that $\mathcal{E}$ extends to a maximal filter $\mathcal{F}$ on $\mathcal{L}$.

We observe next that $Q \subseteq (\mathcal{F})$. Indeed if $a \in Q$, then $\mathbf{V}_V(a) \in \mathcal{E} \subseteq \mathcal{F}$, so $a \in \mathbf{J}(\mathbf{V}_V(a)) \subseteq (\mathcal{F})$; hence $Q \subseteq (\mathcal{F})$. Thus $Q = (\mathcal{F})$ since $(\mathcal{F})$ is an ideal of R contained in $\mathbf{S}(V)$ (by Lemma 2.7(ii)) and Q is maximal among ideals in $\mathbf{S}(V)$. It follows that each ideal that is maximal among ideals in $\mathbf{S}(V)$ is of the form $(\mathcal{F})$ for some maximal filter $\mathcal{F}$ on $\mathcal{L}$ containing V.

On the other hand, if $\mathcal{G}$ is a maximal filter on $\mathcal{L}$ containing V, then by Lemma 2.7(ii) $(\mathcal{G}) \subseteq \mathbf{S}(V)$. Moreover, if Q is maximal among ideals in $\mathbf{S}(V)$ containing $(\mathcal{G})$, then as we have established, $Q = (\mathcal{F})$ for some maximal filter $\mathcal{F}$ containing V. Thus $(\mathcal{G}) \subseteq (\mathcal{F})$, which by Lemma 2.7 implies $\mathcal{G} = \mathcal{F}$. Hence $(\mathcal{G})$ is maximal among ideals contained in $\mathbf{S}(V)$. Therefore the mapping $\mathcal{F} \mapsto (\mathcal{F})$ is a bijection between maximal filters $\mathcal{F}$ containing V and the ideals that are maximal among ideals contained in $\mathbf{S}(V)$.

It remains to show that if Q is maximal among ideals in $\mathbf{S}(V)$, then $\mathcal{F}(Q)$ is a maximal filter containing V. By Lemma 2.8 $V \in F(Q)$, so we have already established that $Q = (\mathcal{G})$ for some maximal filter $\mathcal{G}$ on $\mathcal{L}$. We must show that $\mathcal{G} = \mathcal{F}(Q)$. Observe that $\mathcal{G} \subseteq \mathcal{F}(Q)$, since if $W \in \mathcal{G}$, then $\mathbf{J}(W) \subseteq (\mathcal{G}) = Q$. It remains to show that $\mathcal{F}(Q) \subseteq \mathcal{G}$. Let $W \in \mathcal{F}(Q)$, and set $U = V \cup W$. Since $\mathcal{L}$ is weakly saturated, $W = \mathbf{V}_U(A)$ for some finitely generated ideal A of R. Thus $A \subseteq \mathbf{J}(W) \subseteq Q = (\mathcal{G})$. Since A is finitely generated there exist $W_1, W_2, \ldots, W_n \in \mathcal{G}$ such that $A \subseteq \mathbf{J}(W_1) + \cdots + \mathbf{J}(W_n)$. Thus

$U \cap W_1 \cap \cdots \cap W_n \subseteq U \cap \mathbf{V}_{\mathcal{P}}(A) = \mathbf{V}_U(A) = W$. Now $U \in \mathcal{G}$ since $V \in \mathcal{G}$ and $V \subseteq U$. Since $U, W_1, \ldots, W_n \in \mathcal{G}$ and $\mathcal{G}$ is a filter it follows that $W \in \mathcal{G}$. Hence $\mathcal{F}(Q) = \mathcal{G}$. $\square$

In the next section we will consider cases in which $\mathcal{P} \subseteq \mathrm{Max}(R)$. The next theorem characterizes when every member of $\mathrm{Max}(R)$ can be captured using maximal filters on the basis lattice of $\mathcal{P}$.

Theorem 2.10. *Let $\mathcal{P}$ be a nonempty subset of* $\mathrm{Max}(R)$, *and let $\mathcal{L}$ be the basis lattice of $\mathcal{P}$. Then the following statements are equivalent.*

(i) *The mappings $\mathcal{F} \mapsto (\mathcal{F})$ and $Q \mapsto \mathcal{F}(Q)$ form a one-to-one correspondence between the set of maximal filters $\mathcal{F}$ on $\mathcal{L}$ and the members Q of* $\mathrm{Max}(R)$.
(ii) *$\mathcal{L}$ is saturated and every finitely generated ideal of R is contained in some member of $\mathcal{P}$.*

Proof. (i) $\Rightarrow$ (ii). Since $\mathcal{L}$ is the basis lattice of $\mathcal{P}$, it is weakly saturated. We appeal to Theorem 2.9 to show that $\mathcal{L}$ is saturated. Indeed by the theorem it is enough to prove that for each $V \in \mathcal{L}$ the ideals that are maximal in $\mathbf{S}(V)$ are maximal ideals of R.

Let $V \in \mathcal{L}$, and let Q be an ideal which is maximal with respect to containment in $\mathbf{S}(V)$. We claim that Q is a maximal ideal of R. By Lemma 2.8, $\mathbf{J}(V)$ is contained in Q. Let M be a maximal ideal of R containing Q. By (i) $M = (\mathcal{F}(M))$. Since $\mathbf{J}(V) \subseteq Q \subseteq M$, we have $V \in \mathcal{F}(M)$, and since $M = (\mathcal{F}(M))$, it follows that M is a subset of $S(V)$ by Lemma 2.7(ii). Since Q is contained in M and Q is maximal in $\mathbf{S}(V)$ we have $Q = M$, which proves the claim.

It follows from the above argument that for any $V \in \mathcal{L}$ there is a bijection between the maximal filters on $\mathcal{L}$ that contain V and ideals that are maximal with respect to containment in $\mathbf{S}(V)$, given by $\mathcal{F} \mapsto (\mathcal{F})$. Hence by Theorem 2.9 $\mathcal{L}$ is saturated.

Finally, if B is any finitely generated ideal, then by (i) $B \subseteq (\mathcal{F})$ for some maximal filter $\mathcal{F}$ on $\mathcal{L}$. If W is any member of $\mathcal{F}$, then by Lemma 2.7(iv) B is contained in some member P of W.

(ii) $\Rightarrow$ (i). By Lemma 2.7 the mapping $\mathcal{F} \mapsto (\mathcal{F})$ is injective. By Theorem 2.9, for each maximal filter $\mathcal{F}$ on the basis lattice $\mathcal{L}$ of $\mathcal{P}$, $(\mathcal{F})$ is maximal among ideals in $\mathbf{S}(V)$ whenever $V \in \mathcal{F}$. Conversely, for each ideal Q of R maximal among ideals contained in $\mathbf{S}(V)$, $\mathcal{F}(Q)$ is a maximal filter on $\mathcal{L}$. Thus it remains to show that $\mathrm{Max}(R) = \{(\mathcal{F})\colon \mathcal{F}$ is a maximal filter on $\mathcal{L}\}$.

Let Q be a maximal ideal of R, and let a be a nonzero element of Q. Set $V = \mathbf{V}_{\mathcal{P}}(a) \in \mathcal{L}$. For every finitely generated ideal A of R containing a, $\mathbf{V}_{\mathcal{P}}(A) \subseteq V$. Since $\mathcal{L}$ is saturated, $\mathbf{V}_V(A) \neq \phi$ for all proper finitely generated ideals A of R. Thus since Q is the union of finitely generated ideals A of R containing a, it follows that $Q \subseteq \mathbf{S}(V)$. Hence there exists a maximal filter $\mathcal{F}$ on $\mathcal{L}$ such that $Q \subseteq (\mathcal{F})$. Since $Q \in \mathrm{Max}(R)$, we have $Q = (\mathcal{F})$.

Conversely, we claim that if $\mathcal{F}$ is a maximal filter on $\mathcal{L}$, then $(\mathcal{F}) \in \mathrm{Max}(R)$. Let $\mathcal{F}$ be a maximal filter on $\mathcal{L}$. Then $(\mathcal{F}) \subseteq Q$ for some maximal ideal Q of R. However we have established that $Q = (\mathcal{G})$ for some maximal filter $\mathcal{G}$ on $\mathcal{L}$. Thus by Lemma 2.7(iii) $\mathcal{F} = \mathcal{G}$, so $(\mathcal{F}) = Q \in \mathrm{Max}(R)$. $\square$

Lemma 2.11. *Let $\mathcal{P}$ be a collection of prime ideals of a ring R, and suppose $\mathcal{L}$ is a weakly saturated lattice of closed subsets of $\mathcal{P}$. Let $\mathcal{F}$ be a maximal filter on $\mathcal{L}$. Then $\mathcal{F}$ extends to an ultrafilter $\mathcal{U}$ on the set $\mathcal{P}$, and for any such ultrafilter $\mathcal{U}$ extending $\mathcal{F}$, the following statements hold.*

(i) *The diagonal mapping $R \to \prod_{\mathcal{U}} R/P$ has kernel $(\mathcal{F})$.*
(ii) *If R is a domain, F is the quotient field of R and $\delta : F \to \prod_{\mathcal{U}} F$ is the diagonal mapping, then δ is injective and $\delta(R_{(\mathcal{F})}) = \delta(F) \cap \prod_{\mathcal{U}} R_P$.*

Proof. When viewed as a collection of subsets of $\mathcal{P}$, $\mathcal{F}$ is closed under finite intersections and does not contain the empty set. Thus $\mathcal{F}$ extends to an ultrafilter on $\mathcal{P}$. Let $\mathcal{U}$ be any ultrafilter that extends $\mathcal{F}$, and observe that $\mathcal{U} \cap \mathcal{L}$ is a proper filter on $\mathcal{L}$ that contains $\mathcal{F}$. Since $\mathcal{F}$ is a maximal filter on $\mathcal{L}$, this forces $\mathcal{F} = \mathcal{U} \cap \mathcal{L}$.

(i) Let $\alpha : R \to \prod_{\mathcal{U}} R/P$ be the diagonal mapping, and let $W \in \mathcal{L}$. Let $a \in (\mathcal{F}) = \bigcup_{W \in \mathcal{F}} \mathbf{J}(W)$. Then $a \in \mathbf{J}(W)$ for some $W \in \mathcal{F}$. Thus $W = \mathbf{V}_{\mathcal{P}}(\mathbf{J}(W)) \subseteq \mathbf{V}_{\mathcal{P}}(a)$. Hence $\mathbf{V}_{\mathcal{P}}(a) \in \mathcal{U}$ and so $a \in \operatorname{Ker} \alpha$. Therefore $(\mathcal{F}) \subseteq \operatorname{Ker} \alpha$.

Conversely, if $a \in \operatorname{Ker} \alpha$, then $\mathbf{V}_{\mathcal{P}}(a) \in \mathcal{U}$. Let W be any member of $\mathcal{F}$. Then since $\mathcal{L}$ is weakly saturated, $\mathbf{V}_W(a) \in \mathcal{L}$. Since $\mathbf{V}_W(a) = W \cap \mathbf{V}_{\mathcal{P}}(a)$, the set $\mathbf{V}_W(a)$ is also an element of $\mathcal{U}$. But $\mathcal{U} \cap \mathcal{L} = \mathcal{F}$, so $\mathbf{V}_{\mathcal{P}}(a) \in \mathcal{F}$. Thus $a \in \mathbf{J}(\mathbf{V}_W(a)) \subseteq (\mathcal{F})$. This proves the reverse inequality, namely $\operatorname{Ker} \alpha \subseteq (\mathcal{F})$.

(ii) Clearly δ is injective. By (i), $(\mathcal{F}) = \{a \in R\colon \mathbf{V}_{\mathcal{P}}(a) \in \mathcal{U}\}$, so the equality $\delta(R_{(\mathcal{F})}) = \delta(F) \cap \prod_{\mathcal{U}} R_P$ follows. □

Let X be a subset of the ring R which is the union of a set of prime ideals. Throughout the rest of this paper we will use R_X to denote the ring R localized at the multiplicatively closed set $R \setminus X$.

Theorem 2.12. *Let $\mathcal{L}$ be a saturated lattice of subsets of a collection $\mathcal{P}$ of prime ideals of a ring R and let $V \in \mathcal{L}$. Suppose that for all $P \in V$ the ring R_P is a valuation domain. Then $R_{(\mathcal{F})}$ is a valuation domain whenever $\mathcal{F}$ is a maximal ultrafilter on $\mathcal{L}$ such that $V \in \mathcal{F}$. In particular $R_{\mathbf{S}(V)}$ is a Prüfer domain.*

Proof. To prove the first statement let $\mathcal{F}$ be a maximal filter on $\mathcal{L}$ containing V. Then by Lemma 2.11, $\mathcal{F}$ extends to an ultrafilter $\mathcal{U}$ on the set of all subsets of $\mathcal{P}$. Since R_P is a valuation domain for $\mathcal{U}$-many elements of the index set (namely for all $P \in V$), it follows that the ultraproduct $\prod_{\mathcal{U}} R_P$ is a valuation domain. Also by Lemma 2.11 $R_{(\mathcal{F})}$ is isomorphic to a domain that is an intersection of the image of its quotient field and a valuation ring containing it; hence $R_{(\mathcal{F})}$ is a valuation domain.

To see that the second statement follows from the first, recall that a domain is Prüfer if and only if each localization at a maximal ideal is a valuation domain. The maximal ideals of $R_{\mathbf{S}(V)}$ correspond to the ideals of R which are maximal with respect to containment in $\mathbf{S}(V)$. By Theorem 2.9 these ideals are all of the form $(\mathcal{F})$ for some maximal filter $\mathcal{F}$ on $\mathcal{L}$ containing V. Hence the second statement follows. □

Although it is not needed later, we record the following corollary as an application of the ideas in this section.

Corollary 2.13. *Let R be a domain containing an uncountable field, and let V be a countable collection of maximal ideals of R such that for all $M \in V$, R_M is a valuation domain. Then the ring $R_{\mathbf{S}(V)}$ is a Prüfer domain. In particular if $R = \bigcap_{M \in V} R_M$, then R is a Prüfer domain.*

Proof. Let $\mathcal{L}$ be the basis lattice of V. By Example 2.6(ii) $\mathcal{L}$ is a saturated lattice, so $R_{\mathbf{S}(V)}$ is a Prüfer domain by Theorem 2.12. If also $R = \bigcap_{M \in V} R_M$, then $R = R_{\mathbf{S}(V)}$ and the claim is clear. □

3. Saturated lattices in ultraproducts

Standing hypothesis for Section 3. We let R^* denote the ultraproduct of the commutative rings $\{R_i\colon i \in I\}$ with respect to an ultrafilter $\mathcal{U}$ on the set I.

As noted in 1.5 an induced ideal of R^* of the form (P_i), where each P_i is a prime ideal of R_i, is a prime ideal of R^*. However, finding other primes in the ultraproduct tends to be more problematic. In [11] the maximal ideals of R^*, where $R_i = \mathbb{Z}$ for all i, were shown to be in one-to-one correspondence with maximal filters on certain Boolean algebras. When each R_i is a domain [16] describes chains of primes in R^* using valuations on the quotient fields of the R_i's. In this section we will use the machinery developed thus far to generalize the work of [11] (albeit in a different notation) by describing some of the prime ideals of an arbitrary ultraproduct using maximal filters on lattices of certain subsets of the induced prime ideals. In some cases this technique can describe all the maximal ideals of R^* (see Remark 3.6 and Section 5).

Definition 3.1. We use the following variations on the notion of "induced" sets.

(i) A subset $W \subseteq \operatorname{Spec}(R^*)$ is said to be *induced by the family* $\{W_i\}_{i\in I}$ where $W_i \subseteq \operatorname{Spec}(R_i)$, if W consists of all induced primes of the form (P_i) with $P_i \in W_i$ (if W_i is empty, then $P_i = R_i$). If each W_i is a finite set, then we say that W is a *finitely induced set*. We note that in this case, even though each W_i is finite, in general W is an infinite set.

(ii) We write $\operatorname{Spec}^{\mathrm{ind}}(R^*)$ for the set of prime ideals induced by the family $\{\operatorname{Spec}(R_i)\}_{i\in I}$. Thus $\operatorname{Spec}^{\mathrm{ind}}(R^*)$ is precisely the set of induced prime ideals $P = (P_i)$ of R^*. Similarly we write $\operatorname{Max}^{\mathrm{ind}}(R^*)$ for the set of prime ideals induced by the family $\{\operatorname{Max}(R_i)\}$; these prime ideals are necessarily maximal ideals of R^* (see 1.5).

(iii) For each $i \in I$, let $\mathcal{P}_i \subseteq \operatorname{Spec}(R_i)$ and $\mathcal{L}_i$ be a weakly saturated lattice of subsets of $\mathcal{P}_i$. If $\mathcal{P}$ is induced by the family $\{\mathcal{P}_i\}$, then the *lattice $\mathcal{L}$ of subsets of $\mathcal{P}$ induced by the family* $\{\mathcal{L}_i\}$ is the set of subsets V of $\mathcal{P}$ that are induced by a family $\{V_i\}$, where for each $i \in I$, $V_i \in \mathcal{L}_i$. That $\mathcal{L}$ is indeed a lattice with respect to $\cup$ and $\cap$ follows from the next lemma.

Lemma 3.2. *Let $\mathcal{P}$ be a subset of* $\mathrm{Spec}^{\mathrm{ind}}(R^*)$ *induced by a family* $\{\mathcal{P}_i\}$. *If* $A = (A_i)$ *is an induced ideal of* R^* *and* V *and* W *are subsets of* $\mathcal{P}$ *induced by families* $\{V_i\}$ *and* $\{W_i\}$ *respectively, then*:

(i) $V \setminus W$ *is the set induced by the family* $\{V_i \setminus W_i\}$;
(ii) $V \cup W$ *is the set induced by the family* $\{V_i \cup W_i\}$;
(iii) $V \cap W$ *is the set induced by the family* $\{V_i \cap W_i\}$;
(iv) $V \subseteq W$ *if and only if* $V_i \subseteq W_i$ *for* $\mathcal{U}$*-many* i;
(v) $\mathbf{J}(V)$ *is the induced ideal* $(\mathbf{J}(V_i))$;
(vi) $\mathbf{S}(V) = \{(a_i) \in R^*\colon a_i \in \mathbf{S}(V_i)\}$;
(vii) $R^*_{\mathbf{S}(V)} \cong \prod_{\mathcal{U}} (R_i)_{\mathbf{S}(V_i)}$;
(viii) $\mathbf{V}_{\mathcal{P}}(A)$ *is the subset of* $\mathcal{P}$ *induced by the family* $\{\mathbf{V}_{\mathcal{P}_i}(A_i)\}$.

Proof. (i) Let $P = (P_i) \in V \setminus W$. Then $P_i \in V_i$ for $\mathcal{U}$-many i and $P_i \notin W_i$ for $\mathcal{U}$-many i. Since $\mathcal{U}$ is a filter, $P_i \in V_i \setminus W_i$ for $\mathcal{U}$-many i; hence P is in the set induced by the family $\{V_i \setminus W_i\}$. Conversely, if $P = (P_i)$ is in the set induced by $\{V_i \setminus W_i\}$, then P_i is in $V_i \setminus W_i$ for $\mathcal{U}$-many i. Since P_i is in V_i for $\mathcal{U}$-many i, $P \in V$. Also, since $\mathcal{U}$ is an ultrafilter and P_i is not in W_i for $\mathcal{U}$-many i, $P \notin W$.

(ii) If $P = (P_i) \in \mathcal{P}$, then $P \in V \cup W$ if and only if for $\mathcal{U}$-many i, $P_i \in V_i \cup W_i$. Statement (ii) follows.

(iii) The proof of (iii) is similar to (ii).

(iv) Suppose $V \subseteq W$ and $X := \{i \in I\colon V_i \not\subseteq W_i\} \in \mathcal{F}$. For each $i \in X$, let $P_i \in V_i \setminus W_i$, and for each $i \in I \setminus X$, set $P_i = R_i$. Then $(P_i) \in V \setminus W$, contrary to assumption. Conversely, suppose $V_i \subseteq W_i$ for $\mathcal{U}$-many i. Let $P = (P_i) \in V$. Then for $\mathcal{U}$-many i, $P_i \in V_i \subseteq W_i$, so that $P \in W$.

(v) If $a \in R^* \setminus \mathbf{J}(V)$, then $a \notin P$ for some $P = (P_i) \in V$. Since V is induced by $\{V_i\}$, we have that for $\mathcal{U}$-many i, $P_i \in V_i$. Thus for $\mathcal{U}$-many i, $\mathbf{J}(V_i) \subseteq P_i$. Hence $(\mathbf{J}(V_i)) \subseteq P$. Consequently $a \notin (\mathbf{J}(V_i))$. On the other hand if $a = (a_i) \in R^* \setminus (\mathbf{J}(V_i))$, then for $\mathcal{U}$-many i, there exists $P_i \in V_i$ such that $a_i \notin P_i$. Thus $(P_i) \in V$ and $a \notin (P_i)$. This proves (v).

(vi) If $a = (a_i) \in \mathbf{S}(V)$, then $a \in P$ for some $P = (P_i) \in V$. Since V is induced by the family $\{V_i\}$, we have that $P_i \in V_i$ for $\mathcal{U}$-many i. Hence $a_i \in \mathbf{S}(V_i)$ for $\mathcal{U}$-many i, and so a is in the right side of the equality of (vi). Conversely, suppose that $a = (a_i)$ where $a_i \in \mathbf{S}(V_i)$ for all $i \in I$. Then for all i, $a_i \in P_i$ for some $P_i \in V_i$. Hence $a \in (P_i) \subseteq \mathbf{S}(V)$ which proves the desired set equality.

(vii) Define a mapping $f : \prod_{\mathcal{U}} (R_i)_{\mathbf{S}(V_i)} \to R^*_{\mathbf{S}(V)}$ by $f((\frac{a_i}{b_i})) = \frac{(a_i)}{(b_i)}$, where for each $i \in I$, $a_i \in R_i$ and $b_i \notin \mathbf{S}(V_i)$. By (vi) this mapping is well-defined and onto. Also, it is injective since if $\frac{(a_i)}{(b_i)} = \frac{(c_i)}{(d_i)}$ in $R^*_{\mathbf{S}(V)}$, then there exists (e_i) of $R^* \setminus \mathbf{S}(V)$ such that $(e_i)(a_i d_i - b_i c_i) = (0)$. Hence by part (vi) $\frac{a_i}{b_i} = \frac{c_i}{d_i}$ in $(R_i)_{\mathbf{S}(V_i)}$ for $\mathcal{U}$-many i, and it follows that $(\frac{a_i}{b_i}) = (\frac{c_i}{d_i})$.

(viii) Suppose $P = (P_i) \in \mathbf{V}_{\mathcal{P}}(A)$. Then $A_i \subseteq P_i$ for $\mathcal{U}$-many i. Since $P \in \mathcal{P}$, we have also that $P_i \in \mathcal{P}_i$ for $\mathcal{U}$-many i. Hence (since $\mathcal{U}$ is a filter) we have that $P_i \in \mathbf{V}_{\mathcal{P}_i}(A_i)$ for $\mathcal{U}$-many i. Therefore P is in the set induced by $\{\mathbf{V}_{\mathcal{P}_i}(A_i)\}$. To prove the reverse inclusion, suppose that $P = (P_i)$ is in the set induced by $\{\mathbf{V}_{\mathcal{P}_i}(A_i)\}$. Then for $\mathcal{U}$-many i, $A_i \subseteq P_i$ and $P_i \in \mathcal{P}_i$. Hence $P \supseteq A$ and $P \in \mathcal{P}$, so that $P \in \mathbf{V}_{\mathcal{P}}(A)$. $\square$

Lemma 3.3. *Let $\mathcal{P}$ be a nonempty set of prime ideals of R^* induced by a family $\{\mathcal{P}_i\}$, where each $\mathcal{P}_i$ is a collection of prime ideals of R_i. If for each $i \in I$, $\mathcal{L}_i$ is a strongly saturated lattice of closed subsets of $\mathcal{P}_i$, then the lattice $\mathcal{L}$ induced by the family $\{\mathcal{L}_i\}$ is a strongly saturated lattice of closed subsets of $\mathcal{P}$.*

Proof. If $V \in \mathcal{L}$, then V is induced by a family $\{V_i\}$, where for each $i \in I$, V_i is a closed subset of $\mathcal{P}_i$. By Lemma 3.2(v), $\mathbf{J}(V) = (\mathbf{J}(V_i))$ and so by Lemma 3.2(viii), $\mathbf{V}_{\mathcal{P}}(\mathbf{J}(V))$ is induced by the family $\{\mathbf{V}_{\mathcal{P}_i}(\mathbf{J}(V_i))\}$. Since each V_i is closed in $\mathcal{P}_i$, it follows that $V_i = \mathbf{V}_{\mathcal{P}_i}(\mathbf{J}(V_i))$. Hence we have that $\mathbf{V}_{\mathcal{P}}(\mathbf{J}(V))$ is induced by the family $\{V_i\}$ and so $\mathbf{V}_{\mathcal{P}}(\mathbf{J}(V)) = V$. Thus we have shown that every member of $\mathcal{L}$ is a closed subset of $\mathcal{P}$.

Next we show that if $V \in \mathcal{L}$ and A is a finitely generated ideal of R^*, then $\mathbf{V}_V(A) \in \mathcal{L}$. Since A is finitely generated, it must be an induced ideal. Hence we can write $A = (A_i)$ for finitely generated ideals $A_i \subseteq R_i$. By Lemma 3.2(viii) $\mathbf{V}_{\mathcal{P}}(A)$ is induced by the family $\{\mathbf{V}_{\mathcal{P}_i}(A_i)\}$, and by assumption V is induced by a family $\{V_i\}$, where for each $i \in I$, $V_i \in \mathcal{L}_i$. Thus by Lemma 3.2(viii) $\mathbf{V}_V(A)$ is induced by the family $\{\mathbf{V}_{V_i}(A_i)\}$. Since for each $i \in I$, $\mathcal{L}_i$ is a weakly saturated lattice, it follows that $\mathbf{V}_{V_i}(A_i) \in \mathcal{L}_i$. Hence $\mathbf{V}_V(A) \in \mathcal{L}$, which is what we wanted.

Now suppose $W \in \mathcal{L}$ and that $W \subseteq V$. We must show that there exists a finitely generated ideal A of R^* such that $W = \mathbf{V}_V(A)$. Since $W \in \mathcal{L}$, it is induced by a family $\{W_i\}$, where each $W_i \in \mathcal{L}_i$. Furthermore by Lemma 3.2(iv) the set $X = \{i \in I\colon\ W_i \subseteq V_i$ and $W_i \in \mathcal{L}_i\}$ is in $\mathcal{U}$. If $i \in X$, then since $\mathcal{L}_i$ is weakly saturated, there exists an ideal A_i of R_i such that $W_i = \mathbf{V}_{V_i}(A_i)$. For each $i \in I \setminus X$, set $A_i = 0$. For each i, since $\mathcal{L}_i$ is strongly saturated there exists $a_i \in A_i$ such that $\mathbf{V}_{V_i}(A_i) = \mathbf{V}_{V_i}(a_i)$. Then by Lemma 3.2(viii) $W = \mathbf{V}_V(a)$, where $a = (a_i)$. Hence $\mathcal{L}$ is a weakly saturated lattice. Moreover this argument shows that if A is any finitely generated ideal of R^*, then there exists $a \in A$ such that $\mathbf{V}_V(A) = \mathbf{V}_V(a)$. Thus $\mathcal{L}$ is strongly saturated. $\square$

Remark 3.4. Although we have not pursued this approach here, it is possible to view the induced sets in Definition 3.1 as ultraproducts of sets. By doing so one may obtain Lemmas 3.2 and 3.3 as an application of Łos's theorem.

Using Lemma 3.3 we record now our main theorem of this section.

Theorem 3.5. *Let $\mathcal{P}$ be a nonempty set of prime ideals of R^* induced by a family $\{\mathcal{P}_i\}$, where each $\mathcal{P}_i$ is a collection of incomparable prime ideals of R_i. Then the collection of all finitely induced subsets of $\mathcal{P}$ is a strongly saturated lattice.*

Proof. For each $i \in I$, let $\mathcal{L}_i$ be the collection of finite subsets of $\mathcal{P}_i$. By Example 2.6(i) each $\mathcal{L}_i$ is a strongly saturated lattice, so by Lemma 3.3 the result is proved. $\square$

Remark 3.6. We note that if a lattice $\mathcal{L}$ of subsets of a collection of prime ideals is saturated, then Theorem 2.9 can be applied to describe the prime ideals that are maximal among ideals contained in $\mathbf{S}(V)$ for any $V \in \mathcal{L}$. Along these lines, the maximal ideals in an ultraproduct of d-dimensional Noetherian rings R_i, where the maximal ideals in each R_i all have height d, are described in Section 5.

Also of interest is the case where the basis lattice of the set of induced maximal ideals of R^* is saturated. In this case we can describe all the maximal ideals of R^* using our methods. In particular we have by Theorem 2.10 that the mappings $\mathcal{F} \mapsto (\mathcal{F})$ and $Q \mapsto \mathcal{F}(Q)$ form a one-to-one correspondence between maximal filters $\mathcal{F}$ on the basis lattice of the set of induced maximal ideals of R^* and the members Q of $\mathrm{Max}(R^*)$. We note here two such interesting cases in which the basis lattice $\mathcal{L}$ of the set $\mathcal{P}$ of induced maximal ideals is strongly saturated.

(i) A ring R has *finite character* if each nonzero ideal of R is contained in only finitely many maximal ideals of R. Suppose that R^* is an ultraproduct of finite character rings R_i. For each $i \in I$, the basis lattice $\mathcal{L}_i$ of $\mathrm{Max}(R_i)$ is the collection of finite subsets of $\mathrm{Max}(R_i)$, so by Example 2.6(i) $\mathcal{L}_i$ is strongly saturated. Thus by Theorem 3.5 the lattice $\mathcal{L}'$ of closed subsets of the collection $\mathcal{P}$ of induced maximal ideals of R^* induced by the family $\{\mathcal{L}_i\}$ is strongly saturated. The basis lattice $\mathcal{L}$ of $\mathcal{P}$ is a sublattice of $\mathcal{L}'$. For if $\mathbf{V}_{\mathcal{P}}(A) \in \mathcal{L}$, where A is a product of nonzero finitely generated ideals of R, then A is induced, so by Lemma 3.2(viii) $\mathbf{V}_{\mathcal{P}}(A)$ is induced by a family of finite subsets of $\mathrm{Max}(R_i)$; hence $\mathbf{V}_{\mathcal{P}}(A) \in \mathcal{L}'$. Now $\mathcal{L}$ (since it is the basis lattice of $\mathcal{P}$) is weakly saturated, and since $\mathcal{L}$ is a sublattice of a strongly saturated lattice, $\mathcal{L}$ is strongly saturated.

(ii) If for each $i \in I$, R_i is a QR-domain, then by Example 2.6(iii) the basis lattice $\mathcal{L}_i$ for $\mathrm{Max}(R_i)$ is strongly saturated. The basis lattice $\mathcal{L}$ of the set of induced maximal ideals of R^* is contained in the lattice induced by the family $\{\mathcal{L}_i\}$. By Lemma 3.3 the latter lattice is strongly saturated, so the sublattice $\mathcal{L}$ is also strongly saturated.

4. Maximal prime divisors in ultraproducts

Standing hypothesis for Section 4. As before R^* is an ultraproduct of commutative rings $\{R_i\colon i \in I\}$ with respect to an ultrafilter $\mathcal{U}$ on I.

If A is an ideal of a ring R and $Z(A) = \{r \in R\colon \exists s \in R \setminus A \text{ such that } rs \in A\}$, then $R \setminus Z(A)$ is a multiplicatively closed set and it follows that if P is an ideal of R maximal among ideals in $Z(A)$, then P is a prime ideal. The prime ideals maximal among ideals in $Z(A)$ are the *maximal prime divisors* of A. We denote this set by $\mathrm{Max}(A)$. If R is a Noetherian ring, then every proper ideal has only finitely many maximal prime divisors (these of course are the primes maximal among the associated primes of A). The prime ideals minimal over A are contained in $Z(A)$ and are the *minimal prime divisors* of A (see [13, 7.4]). The set of all minimal prime divisors of A is denoted $\mathrm{Min}(A)$.

We describe in Theorem 4.3 the maximal prime divisors of an induced ideal $A = (A_i)$ of R^* in the case where each $\mathrm{Max}(A_i)$ has a strongly saturated basis lattice. This occurs for example when each R_i is a Noetherian ring or a QR-domain (see Remark 4.4).

The following lemma is proved in Theorem 6.6 of [15] under the assumption that R is an integral domain.

Lemma 4.1. *The following statements are equivalent for a commutative ring R.*

(i) *Each nonzero prime ideal of R is contained in a unique maximal ideal of R.*

(ii) *For all nonzero nonunits x, y, z of R such that $xR + yR = R$, there exist $x', y' \in R$ such that $xR + x'R = R$, $yR + y'R = R$ and $x'y' \in Rz$.*

Proof. (i) $\Rightarrow$ (ii). Let x, y, z be nonunits in R such that $Rx + Ry = R$. Define $S_1 = \mathbf{S}(\mathbf{V}_{\mathrm{Max}(R)}(x))$, $S_2 = \mathbf{S}(\mathbf{V}_{\mathrm{Max}(R)}(y))$ and $S = \{ab \in R\colon a \in R \setminus S_1$ and $b \in R \setminus S_2\}$. We claim first that $zR \cap S \neq \phi$. By way of contradiction, suppose that $zR \cap S = \phi$. Since S is a multiplicatively closed subset of R, there exists a nonzero prime ideal P of R such that $z \in P$ and $P \cap S = \phi$. Therefore we deduce that $P \subseteq S_1 \cap S_2$. (For if not, then say $p \in P$, with $p \notin S_1$. Then $p = p1 \in S$ a contradiction.) Thus there exist prime ideals Q_1 and Q_2 such that $P \subseteq Q_1 \cap Q_2$ and Q_1 and Q_2 are maximal among ideals in S_1 and S_2, respectively. Now $xR + Q_1 \subseteq S_1$ since x is in every prime ideal in $\mathbf{V}_{\mathrm{Max}(R)}(x)$, so the maximality of Q_1 in S_1 implies $x \in Q_1$. Similarly, $y \in Q_2$. If M_1 is a maximal ideal containing Q_1, then $x \in M_1$, so $M_1 \in \mathbf{V}_{\mathrm{Max}(R)}(x)$ and it must be that $M_1 = Q_1$ since $M_1 \subseteq S_1$. Hence Q_1 is a maximal ideal of R containing P. Similarly, Q_2 is a maximal ideal of R containing P, so (i) forces $Q_1 = Q_2$. However since $xR + yR = R$ and $x, y \in Q_1 = Q_2$, this is impossible. Thus $zR \cap S \neq \phi$ and there exist $x' \in R \setminus S_1$ and $y' \in R \setminus S_2$ such that $x'y' \in zR$. Moreover $xR + x'R = R$ and $yR + y'R = R$.

(ii) $\Rightarrow$ (i). Suppose P is a nonzero prime ideal of R contained in two distinct maximal ideals M and N of R. Let $0 \neq z \in P$, and let $x \in M \setminus N$ and $y \in N \setminus M$ such that $xR + yR = R$. Then by (ii) there exist $x', y' \in R$ such that $xR + x'R = R$, $yR + y'R = R$ and $x'y' \in zR$. Since $x \in M$, $x' \notin M$. Similarly, $y' \notin N$. However $x'y' \in zR \subseteq P$, so $x' \in P$ or $y' \in P$, and since $P \subseteq M \cap N$, this is a contradiction. □

Lemma 4.2. *If every member of a collection of commutative rings has the property that each nonzero prime ideal is contained in a unique maximal ideal, then every ultraproduct of these rings also has this property.*

Proof. As in [15] the lemma is an immediate consequence of Lemma 4.1 and Łos's theorem. Alternatively, the lemma can be verified directly using Lemma 4.1. □

Theorem 4.3. *Let $A = (A_i)$ be an induced ideal of R^* such that for $\mathcal{U}$-many $i \in I$, the basis lattice $\mathcal{L}_i$ of $\mathrm{Max}(A_i)$ is strongly saturated. Let V be the subset of $\mathrm{Spec}^{\mathrm{ind}}(R^*)$ induced by the family $\{\mathrm{Max}(A_i)\}$. The following statements hold for A.*

(i) *The mappings $\mathcal{F} \mapsto (\mathcal{F})$ and $Q \mapsto \mathcal{F}(Q)$ form a one-to-one correspondence between maximal filters $\mathcal{F}$ on the basis lattice of V and the members Q of the set $\mathrm{Max}(A)$.*
(ii) *If for $\mathcal{U}$-many i, $\mathrm{Max}(A_i) = \mathrm{Min}(A_i)$, then each minimal prime divisor of A is contained in a unique maximal prime divisor of A.*

Proof. (i) For each $i \in I$, let $V_i = \mathrm{Max}(A_i)$, so that V is induced by the family $\{V_i\}$, and let $\mathcal{L}_i$ be the basis lattice of V_i. Note that $\mathbf{S}(V_i) = Z(A_i)$. By Theorem 3.3 the lattice induced by the $\{\mathcal{L}_i\}$ is strongly saturated. Since the basis lattice of V (which, as a basis lattice, is necessarily weakly saturated) is a sublattice of this lattice, it is strongly saturated. Thus by Theorem 2.9 it suffices to show that $\mathbf{S}(V) = Z(A)$. By Lemma 3.2 $\mathbf{S}(V) = \{(a_i) \in R^*\colon a_i \in \mathbf{S}(V_i)\}$. Hence for $a = (a_i) \in R^*$, $a \in \mathbf{S}(V)$ if and only if for $\mathcal{U}$-many i, $a_i \in \mathbf{S}(V_i) =$

$Z(A_i)$; if and only if for $\mathcal{U}$-many i there exists $b_i \in R_i \setminus A_i$ such that $a_i b_i \in A_i$. Therefore, $a \in \mathbf{S}(V)$ if and only if there exists $b \in R^* \setminus A$ such that $ab \in A$. Hence $S(A) = Z(A)$.

(ii) By passing to the ring $R^*/A \cong \prod_{\mathcal{U}} R_i/A_i$ we may assume without loss of generality that $A = 0$. For each $i \in I$, let V_i be the set of minimal prime ideals of R_i. By assumption for each $i \in I$, V_i coincides with the set of maximal prime divisors of R_i. Thus if for each $i \in I$, T_i is the ring R_i localized at $R_i \setminus \mathbf{S}(V_i)$, then T_i has Krull dimension zero. By Lemma 3.2 (vii) $R^*_{\mathbf{S}(V)} \cong \prod_{\mathcal{U}} T_i$, so by Lemma 4.2 $R^*_{\mathbf{S}(V)}$ has the property that each nonzero prime ideal of R^* is contained in a unique maximal ideal of $R^*_{\mathbf{S}(V)}$. From (i) it follows that the prime ideals maximal in $\mathbf{S}(V)$ are the maximal prime divisors of A. Hence each minimal prime divisor of A is contained in a unique maximal prime divisor of A. $\square$

Remark 4.4. In both of the following cases if A is a proper induced ideal of R^*, then the maximal prime divisors of A can be described via Theorem 4.3 by maximal filters on the basis lattice of a collection of induced prime ideals of R^*.

(i) If for each $i \in I$, R_i is a Noetherian ring, then for every proper ideal A_i of R_i, $\mathrm{Max}(A_i)$ is finite. Hence by the Prime Avoidance Theorem (see Example 2.6(i)) the basis lattice of $\mathrm{Max}(A_i)$ is strongly saturated, and Theorem 4.3 applies.

(ii) Similarly, if for each $i \in I$, R_i is a QR-domain, then for every proper ideal A_i of R_i, the basis lattice of $\mathrm{Max}(A_i)$ is strongly saturated (see Example 2.6(iii)), and Theorem 4.3 applies.

Remark 4.5. In the setting of Theorem 4.3(ii) it need not be the case that every maximal prime divisor of A is a minimal prime divisor of A. For example, let K be a field and for each $i \in \mathbb{N}$ set $R_i = K[[x]]/(x^i)$. If $\mathcal{U}$ is a free ultrafilter on $\mathbb{N}$, then $\prod_{\mathcal{U}} R_i$ has infinite Krull dimension (see for example [7]). Thus in each R_i, the zero ideal does not have an embedded prime, yet the induced maximal ideal (xR_i) of R^* is a maximal prime divisor of 0 that is not a minimal prime divisor of 0.

5. Ultra-height in ultraproducts

Standing hypotheses for Section 5. As usual R^* is an ultraproduct of rings $\{R_i\colon i \in I\}$ with respect to an ultrafilter $\mathcal{U}$ on I. In addition, n denotes a nonnegative integer such that for all $i \in I$, the Krull dimension $\dim(R_i)$ of R_i is at least n. Finally we assume that each ideal of R_i of height n is contained in only finitely many height n prime ideals.

Two of our motivating examples for the rings considered in this section are Krull domains and the rings R for which $\mathrm{Spec}(R)$ is a Noetherian topological space. In a Krull domain every ideal of height 1 is contained in only finitely many prime ideals of height 1, so our standing hypotheses are satisfied in the case $n = 1$ when each R_i is a Krull domain. On the other hand, if R is a ring with Noetherian prime spectrum, then every ideal of R has at most finitely many minimal prime ideals [14]. Thus if for each $i \in I$, R_i is a ring with Noetherian prime spectrum, then the standing hypotheses are satisfied for all choices of n.

We introduce now a height function for ideals in ultraproducts that resembles the usual height function $\mathrm{ht}(A)$ for ideals A, and we use this new height function to partition the set of prime ideals of finite "ultra-height." We describe also the maximal elements in these sets.

Definition 5.1. To each ideal of R^* we associate an *ultra-height* in the following way.

(i) If $A = (A_i)$ is an induced ideal of R^* and there exists an integer $m \geqslant 0$ such that for $\mathcal{U}$-many i, $\operatorname{ht}(A_i) = m$, then we define the *ultra-height* of A to be m and write $\operatorname{uht}(A) = m$. If no such integer m exists, then we define $\operatorname{uht}(A) = \infty$.
(ii) If B is an arbitrary (not necessarily induced) ideal of R^*, then we define $\operatorname{uht}(B) = \sup\{\operatorname{uht}(A)\colon A \text{ is an induced ideal of } R^* \text{ with } A \subseteq B\}$, assuming this supremum exists; otherwise we set $\operatorname{uht}(B) = \infty$.

If there exists $d \geqslant 0$ such that for $\mathcal{U}$-many i, $\dim(R_i) \leqslant d$, then every ideal of R has ultra-height $\leqslant d$.

Remark 5.2. It is easy to find examples of ideals of infinite ultra-height in ultraproducts of Noetherian rings. Even in an ultrapower of Noetherian rings it is possible that there exist induced ideals of infinite ultra-height. For example, let R be a Noetherian domain of infinite Krull dimension, say with maximal ideals $\{M_i\colon i \in \mathbb{N}\}$, such that for each $i \in \mathbb{N}$, $\operatorname{ht}(M_i) = i$. (See [13, Example 1, p. 203].) If $\mathcal{U}$ is a free ultrafilter on $\mathbb{N}$, then for any $n \in \mathbb{N}$, the set $\{i \in \mathbb{N}\colon \operatorname{ht}(M_i) = n\}$ is finite and hence not in $\mathcal{F}$. Thus $\operatorname{uht}((M_i)) = \infty$.

Definition 5.3. Associated to R^* and the nonnegative integer n, we have the following sets:

$$\operatorname{Spec}_n(R^*) = \{P \in \operatorname{Spec}(R^*)\colon \operatorname{uht}(P) = n\},$$
$$\operatorname{Max}_n(R^*) = \text{maximal elements of } \operatorname{Spec}_n(R^*),$$
$$\operatorname{Spec}_n^{\text{ind}}(R^*) = \{P \in \operatorname{Spec}_n(R^*)\colon P \text{ is an induced ideal of } R^*\},$$
$$\mathcal{L}_n(R^*) = \text{finitely induced subsets of } \operatorname{Spec}_n^{\text{ind}}(R^*).$$

A priori it is not clear that $\operatorname{Max}_n(R^*)$ is a nonempty set. However in Theorem 5.4 we show that (under our standing assumptions on the R_i) the set $\operatorname{Max}_n(R^*)$ is nonempty and we describe the elements of $\operatorname{Max}_n(R^*)$ using maximal filters on $\mathcal{L}_n(R^*)$.

Recall the standing assumption of this section that each finitely generated ideal A_i of R_i of height n is contained in only finitely many primes of height n. This will be important in the next result.

Theorem 5.4. *Each prime ideal in* $\operatorname{Spec}_n(R^*)$ *is contained in a unique member of* $\operatorname{Max}_n(R^*)$. *Furthermore, the mappings* $\mathcal{F} \mapsto (\mathcal{F})$ *and* $Q \mapsto \mathcal{F}(Q)$ *form a one-to-correspondence between maximal filters* $\mathcal{F}$ *on the lattice* $\mathcal{L}_n(R^*)$ *and the members* Q *of* $\operatorname{Max}_n(R^*)$.

Proof. In the proof we abbreviate $\mathbf{V}_{\operatorname{Spec}_n^{\text{ind}}(R^*)}(A)$ as $\mathbf{V}(A)$ for all ideals A of R^*. By Theorem 3.5 $\mathcal{L}_n(R^*)$ is a strongly saturated lattice, so by Theorem 2.9 the mapping $\mathcal{F} \mapsto$

$(\mathcal{F})$ is an injection from the set of maximal filters $\mathcal{F}$ on $\mathcal{L}_n(R^*)$ to the set of prime ideals of R^*. Moreover, for each $V \in \mathcal{L}_n(R^*)$ the mappings $\mathcal{F} \mapsto (\mathcal{F})$ and $Q \mapsto \mathcal{F}(Q)$ form a one-to-one correspondence between the maximal filters $\mathcal{F}$ on $\mathcal{L}_n(R^*)$ that contain V and the prime ideals Q that are maximal among ideals in $\mathbf{S}(V)$.

We show first that if $\mathcal{F}$ is a maximal filter on $\mathcal{L}_n(R^*)$, then the prime ideal $(\mathcal{F})$ has ultra-height n. Let $V \in \mathcal{F}$. Then by Lemma 3.2(v) $\mathbf{J}(V) = (\mathbf{J}(V_i))$, where V is induced by a family $\{V_i\}$ of finite sets. Since each $\mathbf{J}(V_i)$ is an intersection of finitely many height n prime ideals, it follows that $\mathrm{uht}((\mathcal{F})) \geqslant n$. If $\mathrm{uht}((\mathcal{F})) > n$, then there exists an induced ideal $A = (A_i) \subseteq (\mathcal{F})$ of R^* such that $\mathrm{uht}(A) > n$. Then $A \subseteq (\mathcal{F}) \subseteq \mathbf{S}(V)$, so that by Lemma 3.2(vi) for $\mathcal{U}$-many i, $A_i \subseteq \mathbf{S}(V_i)$. For all i, V_i is a finite set of height n prime ideals of R_i. Therefore by prime avoidance we have that A_i is contained in a height n prime ideal of R_i for $\mathcal{U}$-many i. But then $\mathrm{uht}(A) \leqslant n$, a contradiction. It follows that $\mathrm{uht}((\mathcal{F})) = n$.

We show next that if $P \in \mathrm{Spec}_n(R^*)$, then there is a maximal filter $\mathcal{F}$ on $\mathcal{L}_n(R^*)$ such that $P \subseteq (\mathcal{F})$. By our comments at the beginning of the proof it is enough to show that $P \subseteq \mathbf{S}(V)$ for some $V \in \mathcal{L}_n(R^*)$. Let $A = (A_i)$ be an induced ideal of R^* contained in P such that $\mathrm{uht}(A) = n$, and let $V = \mathbf{V}(A)$. Then $V \in \mathcal{L}_n(R^*)$, since for $\mathcal{U}$-many i, A_i has at most finitely prime ideals of height n containing it. If $B \subseteq P$ is an induced ideal of R containing A, then since B is induced by height n ideals, $\phi \neq \mathbf{V}(B) \subseteq V$. Thus since P is the union of all induced ideals $B \subseteq P$ containing A, we have that $P \subseteq \mathbf{S}(V)$. Hence by our above comments $P \subseteq (\mathcal{F})$ for some maximal filter $\mathcal{F}$ on $\mathcal{L}_n(R^*)$.

Finally, if $\mathcal{F}$ is a maximal filter on $\mathcal{L}_n(R^*)$, then $(\mathcal{F}) \in \mathrm{Max}_n(R^*)$. For if $(\mathcal{F})$ is not a maximal member of $\mathrm{Spec}_n(R^*)$, then we have established that there is a maximal filter $\mathcal{G}$ on $\mathcal{L}_n(R^*)$ such that $(\mathcal{F}) \subsetneq (\mathcal{G})$. But by Lemma 2.7 this implies that $\mathcal{F} = \mathcal{G}$, a contradiction. Hence $(\mathcal{F}) \in \mathrm{Max}_n(R^*)$. We conclude that every prime ideal in $\mathrm{Spec}_n(R^*)$ is contained in a prime ideal of the form $(\mathcal{F})$, where $\mathcal{F}$ is an maximal filter on $\mathcal{L}_n(R^*)$, and that $(\mathcal{F}) \in \mathrm{Max}_n(R^*)$.

It remains to show that if $P \in \mathrm{Spec}_n(R^*)$, then P is contained in a *unique* member of $\mathrm{Max}_n(R^*)$. Let $P \in \mathrm{Spec}_n(R^*)$ and let $A = (A_i) \subseteq P$ be an induced ideal of R^* with $\mathrm{uht}(A) = n$. Suppose that P is contained in two members of $\mathrm{Max}_n(R^*)$. Then $P \subseteq (\mathcal{F}) \cap (\mathcal{G})$ for some maximal filters $\mathcal{F}$ and $\mathcal{G}$ on $\mathcal{L}_n(R^*)$. Let $U \in \mathcal{F}$ and $W \in \mathcal{G}$. Set $V = U \cup W$ and observe that $V \in \mathcal{F} \cap \mathcal{G}$. Thus by Lemma 2.7 $(\mathcal{F}), (\mathcal{G}) \subseteq \mathbf{S}(V)$. Since $V \in \mathcal{L}_n(R^*)$, V is induced by a family $\{V_i\}$ of finite sets. For each $i \in I$, let T_i be the localization of R_i/A_i at $R \setminus \mathbf{S}(V_i)$. Then by Lemma 3.2(vi) $(R^*/A)_{\mathbf{S}(V)} \cong \prod_{\mathcal{U}} T_i$. For $\mathcal{U}$-many i, T_i is a zero-dimensional ring, so by Lemma 4.2 every prime ideal of R^*/A that survives in $(R^*/A)_{\mathbf{S}(V)}$ is contained in a unique maximal ideal of this ring. Thus since $(\mathcal{F})$ and $(\mathcal{G})$ are maximal among ideals contained in $\mathbf{S}(V)$ and $A \subseteq P \subseteq (\mathcal{F}) \cap (\mathcal{G})$, then $(\mathcal{F}) = (\mathcal{G})$, and this proves the theorem. $\square$

We next show how to obtain a class of maximal ideals of R^* determined by the maximal ideals of R_i. Then with certain additional assumptions, we will describe all the maximal ideals of R^*. Note that our next result (and only our next result) does not use the standing assumption of this section regarding finite height ideals of R_i.

Proposition 5.5. *For each $i \in I$, let $\mathcal{P}_i = \operatorname{Max}(R_i)$ and let $\mathcal{P}$ be the set induced by the family $\{\mathcal{P}_i\}$. Let $\mathcal{L}$ be the lattice of all finitely induced subsets of $\mathcal{P}$. Then $(\mathcal{F})$ is a maximal ideal of R^* for each maximal filter $\mathcal{F}$ on $\mathcal{L}$.*

Proof. It suffices to show that if $a = (a_i) \in R^* \setminus (\mathcal{F})$, then there exists $s \in (\mathcal{F})$ and $t \in R^*$ such that $s + at = 1$. To say that $a \notin (\mathcal{F})$ means that no subset of $\mathbf{V}_{\mathcal{P}}(a)$ is in $\mathcal{F}$. This we claim, implies that $\mathcal{P} \setminus \mathbf{V}_{\mathcal{P}}(a)$ must contain a subset W that is in $\mathcal{F}$. To see this let $V \in \mathcal{F}$ be arbitrary and suppose that it is induced by the family $\{V_i\}$. Let $X_i = \{P \in V_i\colon a_i \in P\}$ and $W_i = V_i \setminus X_i$. Then let X and W be the sets of prime ideals of R^* induced by the families $\{X_i\}$ and $\{W_i\}$ respectively. Thus by Lemma 3.2 $V = X \cup W$. Since $X, W \in \mathcal{L}$, it follows from Lemma 2.7 that one of these sets is in $\mathcal{F}$. However, $X \subseteq \mathbf{V}_{\mathcal{P}}(a)$, and therefore $W \subseteq \mathcal{P} \setminus \mathbf{V}_{\mathcal{P}}(a)$ must be in $\mathcal{F}$ as claimed.

Since $\mathbf{V}_W(a) = \phi$ it follows that for $\mathcal{U}$-many i, $\mathbf{V}_{W_i}(a_i) = \phi$. As $\mathcal{P}_i$ consists of all the maximal ideals of R_i and W_i is a finite set, there must exist $s_i \in \mathbf{J}(W_i)$ and $t_i \in R_i$ such that $s_i + a_i t_i = 1$. Now let $s = (s_i)$ and $t = (t_i)$. By Lemma 3.2, $s \in \mathbf{J}(W)$. Furthermore, $\mathbf{J}(W) \subseteq (\mathcal{F})$ and $s + at = 1$, so we are done. □

The next two corollaries show that with additional assumptions on the coordinate rings R_i, we can use the above results to describe all the maximal ideals of R^*.

Corollary 5.6. *Suppose that for $\mathcal{U}$-many $i \in I$, $\dim(R_i) = n$ and $\operatorname{Spec}(R_i)$ is a Noetherian space. Then $\operatorname{Max}_n(R^*) \subseteq \operatorname{Max}(R^*)$. Furthermore, if $M \in \operatorname{Max}(R^*)$, then $M \in \operatorname{Max}_r(R^*)$ for some $0 \leqslant r \leqslant n$. In particular, $M = (\mathcal{F})$ for some maximal filter $\mathcal{F}$ on $\mathcal{L}_r(R^*)$.*

Proof. It is safe to assume that all the R_i have dimension n. Then it follows from our hypothesis and Proposition 5.5 that $\operatorname{Max}_n(R^*) \subseteq \operatorname{Max}(R^*)$.

Now let $M \in \operatorname{Max}(R^*)$ and suppose that $\operatorname{uht}(M) = r$ (which could be less than n). Then $M \in \operatorname{Spec}_r(R^*)$ and clearly M is a maximal element of this set. Furthermore, since $\operatorname{Spec}(R)$ is Noetherian, each ideal of height r is contained in only finitely many prime ideals of height r (since they all would be minimal primes over the ideal) [14]. The last statement now follows from Theorem 5.4. □

Corollary 5.7. *Suppose that for $\mathcal{U}$-many $i \in I$, $\operatorname{Spec}(R_i)$ is a Noetherian space and each maximal ideal of R_i has height n. Then $\operatorname{Max}_n(R^*) = \operatorname{Max}(R^*)$.*

Proof. By Corollary 5.6, we know $\operatorname{Max}_n(R^*) \subseteq \operatorname{Max}(R^*)$. To complete the proof we have to show that if $M \in \operatorname{Max}(R^*)$, then $\operatorname{uht}(M) = n$. Suppose that $\operatorname{uht}(M) = r < n$. From Theorem 5.4 we know $M = (\mathcal{F})$ for some maximal filter $\mathcal{F}$ on $\mathcal{L}_r(R^*)$. Pick $V \in \mathcal{F}$. Then V is induced by a family $\{V_i\}$ where each V_i is a finite set of prime ideals of R_i of height r.

By assumption, no element of V_i is maximal. Thus for each $P \in V_i$ we can pick a maximal ideal M of R_i such that $P \subsetneq M$. Denote this finite set of maximal ideals by W_i and let W be the set induced by the family $\{W_i\}$. Hence $W \in \mathcal{L}_n(R^*)$. Each element of W corresponds in a natural fashion to an element of V. Thus each subset of W corresponds in a natural way to a subset of V. Hence we can define a maximal filter $\mathcal{G}$ on the lattice of subsets of W by declaring $W' \subseteq W$ is in $\mathcal{G}$ if and only if the corresponding set $V' \subseteq V$ is in $\mathcal{F}$. That $\mathcal{G}$ is a maximal filter follows from the definition.

Next we claim that $M = (\mathcal{F}) \subseteq (\mathcal{G})$. To see this let $a \in (\mathcal{F})$. Then $a \in \mathbf{J}(Y)$ for some $Y \in \mathcal{F}$. Clearly $Y \cap V \in \mathcal{F}$ since $\mathcal{F}$ is a filter. Let W' be the subset of W corresponding to $Y \cap V$. Thus $W' \in \mathcal{G}$ and

$$\mathbf{J}(Y) \subseteq \mathbf{J}(Y \cap V) \subseteq \mathbf{J}\bigl(W'\bigr) \subseteq (\mathcal{G}).$$

Therefore the claim is proved. Finally, note that $\mathrm{uht}((\mathcal{G})) = n$. Hence $(\mathcal{F}) \subsetneq (\mathcal{G})$. But this is a contradiction, since $M = (\mathcal{F})$ was assumed to be maximal. $\square$

Remark 5.8. The following two problems remain open.

(i) Describe the prime ideals of finite ultra-height in an ultraproduct of Noetherian rings.
(ii) Describe the prime ideals in an ultraproduct of Artinian rings.

These two problems are in fact equivalent. For if Q is a prime ideal in an ultraproduct of Artinian rings, then $\mathrm{uht}(Q) = 0$. Conversely, suppose Q is a prime ideal of ultra-height n is an ultraproduct R^* of Noetherian rings $\{R_i\colon\ i \in I\}$. Let $A = (A_i) \subseteq Q$ be an induced ideal of R^* with $\mathrm{uht}(A) = n$. We may assume each A_i has height n. For each $i \in I$, let V_i be the finite set of height n prime ideals containing A_i. Let $V \in \mathcal{L}_n(R^*)$ be induced by the family $\{V_i\}$. For each $i \in I$, let T_i be the ring R_i/A_i. By Lemma 3.2(vii) $(R^*/A)_{\mathbf{S}(V)} \cong \prod_{\mathcal{U}} (T_i)_{S(V_i)}$. Thus Q/A corresponds to a prime ideal in the ultraproduct $\prod_{\mathcal{U}} (T_i)_{S(V_i)}$ of Artinian rings.

In Section 7 we describe all the prime ideals of ultra-height one in an ultraproduct of Krull domains.

6. Chains of primes ideals in $\mathrm{Spec}_n(R^*)$

Standing hypotheses for Section 6. In this section we use the same assumptions as in Section 5. Namely, $\dim(R_i) \geqslant n$ for some fixed $n \geqslant 0$, and every ideal of height n is contained in only finitely many height n prime ideals.

Definition 6.1. For each $i \in I$ let E_i denote the set of all functions e_i from the set of height n prime ideals of R_i to $\mathbb{Z}_{\geqslant 0}$ with finite support (i.e., $e_i(L) = 0$ for all but finitely many $L \in \mathrm{Spec}(R_i)$ with $\mathrm{ht}(L) = n$). Set $E = \prod_{i \in I} E_i$. Let $P = (P_i) \in \mathrm{Spec}_n(R^*)$ and for each $e = (e_i)_{i \in I} \in E$, define $P^e = (P_i^{e_i(P_i)})$. (Since P is an induced ideal in R^*, this definition is independent of the representation of P as (P_i).)

Recall that if $Q \in \mathrm{Max}_n(R^*)$, then by Theorem 5.4 $Q = (\mathcal{F}(Q))$ for a unique maximal filter $\mathcal{F}(Q)$ on $\mathcal{L}_n(R^*)$, the lattice of finitely induced subsets of $\mathrm{Spec}_n(R^*)$. Using these constructs we define a family (in fact a chain) of ideals contained in Q which under certain circumstances turn out to be prime ideals.

Definition 6.2. Let $Q \in \mathrm{Max}_n(R^*)$ and $e \in E$. We define

$$Q_e = \sqrt{\bigcup_{V \in \mathcal{F}(Q)} \Bigl(\bigcap_{P \in V} P^e\Bigr)}.$$

We note that $\bigcup_{V \in \mathcal{F}(Q)} (\bigcap_{P \in V} P^e)$ is an ideal. To see this let a and b be in the set. Then $a \in \bigcap_{P \in V_1} P^e$ and $b \in \bigcap_{P \in V_2} P^e$, where $V_1, V_2 \in \mathcal{F}(Q)$. Thus $a + b \in \bigcap_{P \in V_1 \cap V_2} P^e$. Since $\mathcal{F}(Q)$ is a filter, $V_1 \cap V_2 \in \mathcal{F}(Q)$ and it follows that the set is closed under addition. As it is the union of ideals it is clearly closed under multiplication. Hence it is an ideal.

Applying the relevant definitions, it is not hard to see that if $Q \in \mathrm{Max}_n(R^*)$ is an induced ideal, say $Q = (Q_i)$, then

$$Q_e = \sqrt{Q^e} = \sqrt{\bigl(Q_i^{e_i(Q_i)}\bigr)}.$$

Proposition 6.3. *The set $\{Q_e\colon e \in E\}$ is a chain under set-inclusion $\subseteq$.*

Proof. Let V be an arbitrary element of $\mathcal{F}(Q)$ induced by say the family $\{V_i\}$ and let $e, f \in E$. Let X and Y be the subsets of V induced by the families $\{X_i\}$ and $\{Y_i\}$ respectively, where $X_i = \{P_i \in V_i\colon e_i(P_i) \geqslant f_i(P_i)\}$ and $Y_i = \{P_i \in V_i\colon f_i(P_i) > e_i(P_i)\}$. Then $V = X \cup Y$ by Lemma 3.2(ii). Therefore by Lemma 2.7(i) one of X or Y is in $\mathcal{F}(Q)$. We will first assume that $X \in \mathcal{F}(Q)$.

Let V' be any another element of $\mathcal{F}(Q)$ and partition V' into X' and Y' as above. If $Y' \in \mathcal{F}(Q)$, we would have by Lemma 3.2(iii) $\phi = X \cap Y' \in \mathcal{F}(Q)$, which is a contradiction. Hence $X' \in \mathcal{F}(Q)$ for any other choice of $V' \in \mathcal{F}(Q)$.

Now let $a \in Q_e$. Then for some $V \in \mathcal{F}(Q)$ and some $n > 0$, we have $a^n \in \cap_{P \in V} P^e$. Let X and Y be the partition of V as above. Then we have $\bigcap_{P \in V} P^e \subseteq \bigcap_{P \in X} P^e \subseteq \bigcap_{P \in X} P^f \subseteq Q_f$. Since Q_f is a radical ideal we can conclude that $a \in Q_f$; hence $Q_e \subseteq Q_f$. If we started with the assumption that $Y \in \mathcal{F}(Q)$, then by the same argument we would have $Q_f \subseteq Q_e$. $\square$

Let P be a prime ideal of a domain R such that $\bigcap_{k=1}^{\infty} P^k = 0$. We define the "order filtration" $\mathrm{ord}_P : R \to \mathbb{Z} \cup \{\infty\}$ via $\mathrm{ord}_P(0) = \infty$ and, for $0 \neq a = (a_i) \in R^*$, $\mathrm{ord}_P(a) = \max\{k\colon a \in P^k\}$, where P^0 is defined to be R. Let $P = (P_i)$ be an induced prime ideal of R^*. Then we define the induced function $\mathrm{ord}^*_P : R^* \to \mathbb{Z}^* \cup \{\infty\}$ via $\mathrm{ord}^*_P(0) = \infty$ and $\mathrm{ord}^*_P(a) = (\mathrm{ord}_{P_i}(a_i))$ for all $0 \neq a = (a_i) \in R^*$. For more on this construction see [16].

We say that the order filtration ord_P is *k-additive* for a positive integer k if for all $a, b \in R$, $\mathrm{ord}_P(ab) \leqslant k(\mathrm{ord}_P(a) + \mathrm{ord}_P(b))$. Similarly, ord^*_P is k-additive if for all $a, b \in R^*$, $\mathrm{ord}^*_P(ab) \leqslant k(\mathrm{ord}^*_P(a) + \mathrm{ord}^*_P(b))$.

Theorem 6.4. *Let $Q \in \mathrm{Max}_n(R^*)$ and suppose that there exists $V \in \mathcal{F}(Q)$ and $k > 0$ such that for all $P \in V$, ord^*_P is a k-additive function. Then $Q_e \in \mathrm{Spec}_n(R^*)$ for each $e \in E$.*

Proof. Let V be induced by the family $\{V_i\}$. We first claim that for $\mathcal{U}$-many i, V_i satisfies the property that for all $P_i \in V_i$, the function ord_{P_i} is k-additive. If not, then for $\mathcal{U}$-many i

there exists $P_i \in V_i$ such that ord_{P_i} is not k-additive. Hence the function ord_P^* for $P = (P_i)$ cannot be k-additive, contrary to assumption.

Next we show that Q_e is a prime ideal when $e = (e_i) \in E$. Let $ab \in Q_e$, where $a, b \in R^*$. Thus for some $n > 0$ and some $W \in \mathcal{F}(Q)$, $a^n b^n \in \bigcap_{P \in W} P^e$. Write $a = (a_i)$ and $b = (b_i)$. Let $X_i = \{P_i \in W_i\colon \mathrm{ord}_{P_i}(a_i) \geqslant \mathrm{ord}_{P_i}(b_i)\}$ and $Y_i = \{P_i \in W_i\colon \mathrm{ord}_{P_i}(b_i) > \mathrm{ord}_{P_i}(a_i)\}$. Let X and Y be the subsets of W induced by the families $\{X_i\}$ and $\{Y_i\}$ respectively. By Lemma 3.2(ii), $W = X \cup Y$. Thus by Lemma 2.7(i) either X or Y is in $\mathcal{F}(Q)$.

First assume that $X \in \mathcal{F}(Q)$. We know that for $\mathcal{U}$-many i, ord_{P_i} is k-additive for all $P_i \in V_i$. Therefore if $P = (P_i) \in X$ we have, since $a^n b^n \in P^e$, that $e_i(P_i) \leqslant \mathrm{ord}_{P_i}(a_i^n b_i^n) \leqslant nk^n(\mathrm{ord}_{P_i}(a_i) + \mathrm{ord}_{P_i}(b_i)) \leqslant 2nk^n \cdot \mathrm{ord}_{P_i}(a_i) \leqslant \mathrm{ord}_{P_i}(a_i^{2nk^n})$. Thus $a_i^{2nk^n} \in P_i^{e_i(P_i)}$. Hence $a^{2nk^n} \in \bigcap_{P \in X} P^e \subseteq Q_e$. Since Q_e is a radical ideal, we have $a \in Q_e$. If $Y \in \mathcal{F}(Q)$, then a symmetric argument shows that $b \in Q_e$. Hence Q_e is prime.

The only thing left is to show that $\mathrm{uht}(Q_e) = n$. However, Q_e clearly contains induced ideals A with $\mathrm{uht}(A) = n$, and on the other hand $Q_e \subseteq Q$. Thus we are done. $\square$

Corollary 6.5. *If for each $i \in I$, R_i is a Krull domain, then for each $Q \in \mathrm{Max}_1(R^*)$ and $e \in E$, Q_e is a prime ideal of R^*.*

Proof. For each height one prime ideal P_i of R_i, ord_{P_i} is a 1-additive function; indeed, ord_{P_i} extends to a valuation on the quotient field of R_i [12, Corollary, p. 88]. It follows that for each $P = (P_i) \in \mathrm{Spec}_1^{\mathrm{ind}}(R^*)$, ord_P^* is a 1-additive function. Now apply Theorem 6.4. $\square$

We note in the next corollary another significant case in which the Q_e are prime ideals. In order to appeal to Theorem 6.4 we apply a recent theorem of Hochster and Huneke: If P is a prime ideal of height k in a regular local ring R that contains a field, then $P^{(km)} \subseteq P^m$ for all $m > 0$ [10] (see also [19] for a "nonstandard" proof). Here $P^{(m)}$ denotes the mth symbolic power $P^m R_P \cap R$. It is an open question whether the theorem of Hochster and Huneke holds for all regular local rings of mixed characteristic. (Swanson has shown that given a prime ideal P of any regular local ring there exists $k > 0$ such that $P^{(km)} \subseteq P^m$ for all $m > 0$, but it is not known whether k can be chosen in such a way that it depends only on the height of P [24].)

If R is a regular local ring containing a field and P is a prime ideal of R of height k, then ord_{PR_P} is well known to be (in our terminology) a 1-additive function on R_P. Thus, applying the result of Hochster and Huneke we have that $\mathrm{ord}_{PR_P}(x) \leqslant k(\mathrm{ord}_P(x) + 1) - 1 \leqslant 2k \cdot \mathrm{ord}_P(x)$, since $\mathrm{ord}_P(x) = 0$ implies $\mathrm{ord}_{PR_P}(x) = 0$. Thus $\mathrm{ord}_P(xy) \leqslant \mathrm{ord}_{PR_P}(xy) \leqslant \mathrm{ord}_{PR_P}(x) + \mathrm{ord}_{PR_P}(y) \leqslant 2k(\mathrm{ord}_P(x) + \mathrm{ord}_P(y))$. Hence ord_P is a $2k$-additive function.

Corollary 6.6. *Suppose that for each $i \in I$, R_i is a regular local ring containing a field. Then for each $Q \in \mathrm{Max}_n(R^*)$ and $e \in E$, Q_e is a prime ideal of R^*.*

Proof. If $P = (P_i)$ is a prime ideal of R^*, where each P_i is a height n prime ideal of R_i, then it follows from the preceding discussion that ord_P^* is $2n$-additive, so we may apply Theorem 6.4. $\square$

When it is assumed that the R_i are Noetherian rings, then given any prime ideal P in $\mathrm{Spec}_n(R^*)$ it is possible to find an ideal of the form Q_e, $Q \in \mathrm{Max}_n(R^*)$, contained in P. This will be the content of the next theorem. First, however, we need a lemma.

Lemma 6.7. *Let $Q \in \mathrm{Max}_n(R^*)$. If $A = (A_i) \subseteq Q$ is an induced ideal of R^*, then $(\sqrt{A_i}) \subseteq Q$.*

Proof. Let $a = (a_i) \in (\sqrt{A_i})$. Then for each $i \in I$, there exists $f_i > 0$ such that $a_i^{f_i} \in A_i$. Thus $b := (a_i^{f_i}) \in A \subseteq (\mathcal{F}(Q)) = Q$, where $\mathcal{F}(Q)$ is a maximal filter on $\mathcal{L}_n(R^*)$. Therefore $b \in \mathbf{J}(V) = (\mathbf{J}(V_i))$, for some $V \in \mathcal{F}(Q)$ which is induced by a family $\{V_i\}$. Hence for $\mathcal{U}$-many i, $a_i^{f_i} \in \mathbf{J}(V_i)$. Therefore $a_i \in \mathbf{J}(V_i)$, since $\mathbf{J}(V_i)$ is a radical ideal. Thus $a \in \mathbf{J}(V) \subseteq (\mathcal{F}(Q))$. $\square$

Theorem 6.8. *For each $i \in I$, let R_i be a Noetherian ring. If $L \in \mathrm{Spec}_n(R^*)$, then there exists a unique member Q of $\mathrm{Max}_n(R^*)$ such that $Q_e \subseteq L \subseteq Q$ for some $e \in \mathbb{N}$.*

Proof. In the following we abbreviate $\mathbf{V}_{\mathrm{Spec}_n^{\mathrm{ind}}(R^*)}(A)$ as $\mathbf{V}(A)$ for every ideal A of R^*. Let $A = (A_i) \subseteq L$ be an induced ideal of R^* with $\mathrm{uht}(A) = n$. We may assume that $\mathrm{ht}(A_i) = n$ for all $i \in I$. We first reduce to the case that for each $i \in I$, A_i is an intersection of powers of height n prime ideals. For each $i \in I$, write $A_i = B_i \cap C_i$, where B_i is the intersection of the isolated components of A_i and C_i is the intersection of the embedded components of A_i (if there are no embedded components, set $C_i = R_i$). Then $(B_i) \cap (C_i) = (A_i) \subseteq L$, so $(B_i) \subseteq L$ or $(C_i) \subseteq L$. However $n < \mathrm{uht}((C_i))$ and $n = \mathrm{uht}(L)$, so this forces $(B_i) \subseteq L$. For each $i \in I$, let V_i be the height n prime ideals of R_i containing B_i. Since B_i is the intersection of primary ideals, each having an associated prime in V_i, it follows that there exists $f_i > 0$ such that $\bigcap_{P \in V_i} P^{f_i} \subseteq B_i$. Since $(B_i) \subseteq L$ we may reduce now to the case that $A = (A_i) \subseteq L$ has the property that for each $i \in I$, there exists $f_i > 0$ such that if V_i is the (finite) set of height n prime ideals containing A_i, then $A_i = \bigcap_{P \in V_i} P^{f_i}$.

Let V be the subset of $\mathcal{L}_n(R^*)$ induced by the family $\{V_i\}$ and let $Q \in \mathrm{Max}_n(R^*)$ such that $L \subseteq Q$. By Lemma 3.2(v) $\mathbf{J}(V) = (\mathbf{J}(V_i)) = (\sqrt{A_i})$, since by design, $\mathbf{J}(V_i) = \sqrt{A_i}$ for all i. Thus by Lemma 6.7 $\mathbf{J}(V) \subseteq Q$, so $V \in \mathcal{F}(Q)$. For each $i \in I$, define a function $e_i \in E_i$ from the set of height n prime ideals to $\mathbb{Z}_{\geqslant 0}$ by $e_i(P) = f_i$ for all $P \in V_i$ and $e_i(P) = 0$ for every height n prime ideal not in V_i. Let $e = (e_i)_{i \in I} \in E$. We claim that $Q_e \subseteq L$. It suffices to show that for all $U \in \mathcal{F}(Q)$ with $U \subseteq V$, $\bigcap_{P \in U} P^e \subseteq L$. Let $U \in \mathcal{F}(Q)$ be induced by a family $\{U_i\}$ of sets $U_i \subseteq V_i$. For each $i \in I$, let $B_i = \bigcap_{P \in U_i} P^{f_i}$, and set $B = (B_i)$. Then $B = \bigcap_{P \in U} P^e$. Consider $W = V \setminus U$. By Lemma 3.2(i) the set W is finitely induced by the family $\{W_i\}$, where for each $i \in I$, $W_i = V_i \setminus U_i$. For each $i \in I$, set $C_i = \bigcap_{P \in W_i} P^{f_i}$, and let $C = (C_i)$. Then $C = \bigcap_{P \in W} P^e$. Hence $B \cap C = A \subseteq L$, so $B \subseteq L$ or $C \subseteq L$. If $C \subseteq L \subseteq Q$, then by Lemma 6.7 $\mathbf{J}(W) = (\mathbf{J}(W_i)) = (\sqrt{C_i}) \subseteq Q$ so that $W \in \mathcal{F}(Q)$. However by assumption $U \in \mathcal{F}(Q)$ and $U \cap W = \phi$, so since $\mathcal{F}(Q)$ is a maximal (and hence proper) filter by Theorem 5.4, we have a contradiction. Thus $B = \bigcap_{P \in U} P^e \subseteq L$, as claimed. This proves $Q_e \subseteq L$. That Q is the unique such prime ideal follows from Theorem 5.4. $\square$

7. Prime ideals of ultra-height 1

Standing hypotheses for Section 7. In this section R^* is the usual ultraproduct and n denotes a nonnegative integer.

In this last section we describe prime ideals of ultra-height 1 in ultraproducts of certain classes of rings. We derive some of these descriptions from the following more general theorem.

Theorem 7.1. *Suppose that for each $i \in I$, the Krull dimension of R_i is at least n and every ideal of height n is contained in only finitely many height n prime ideals. Let $N \geqslant 0$ and suppose that for $\mathcal{U}$-many i, R_i has exactly N prime ideals P of height n such that $(R_i)_P$ is not a valuation domain. Then for exactly N many $Q \in \mathrm{Max}_n(R^*)$, R^*_Q is not a valuation domain.*

Proof. For each $i \in I$, let $U_i = \{P \in \mathrm{Spec}(R_i)\colon\ \mathrm{ht}(P) = n$ and $(R_i)_P$ is not a valuation domain$\}$. Let U be the element of $\mathcal{L}_n(R^*)$ induced by the family $\{U_i\}$. Since $\mathcal{U}$ is an ultrafilter it is not hard to see that U has exactly N members. Let $Q \in \mathrm{Max}_n(R^*)$. It suffices to show that R^*_Q is a valuation domain if and only if $Q \notin U$. If $Q \in U$, then Q is an induced ideal of the form (P_i), where for each $i \in I$, $P_i \in U_i$. But then $R^*_Q \cong \prod_{\mathcal{U}} (R_i)_{P_i}$, and R^*_Q is not a valuation domain since for all i, $(R_i)_{P_i}$ is not a valuation domain (see for example [15]). Conversely, suppose that $Q \notin U$. Let $W \in \mathcal{F}(Q)$. Then $U \cup (W \setminus U) = W \in \mathcal{F}(Q)$. By Theorem 5.4 $\mathcal{F}(Q)$ is a maximal filter, so by Lemma 2.7 $U \in \mathcal{F}(Q)$ or $W \setminus U \in \mathcal{F}(Q)$. If $U \in \mathcal{F}(Q)$, then since U is finite, another application of Lemma 2.7 shows that $\{P\} \in \mathcal{F}(Q)$ for some $P \in U$. In this case $Q = (\mathcal{F}(Q)) = P \in U$, a contradiction. Hence $W \setminus U \in \mathcal{F}(Q)$. For every $P \in W \setminus U$, R^*_P is a valuation domain. Hence by Theorem 2.12 R^*_Q is a valuation domain. □

In a Krull domain R every nonzero element of R is contained in at most finitely many height one prime ideals P of R, and for each such prime ideal P, R_P is a Noetherian valuation domain. Thus the results of the previous two sections apply to ultraproducts of Krull domains (in the case $n = 1$). In particular by Theorem 5.4 there is a one-to-one correspondence between maximal filters on $\mathcal{L}_1(R^*)$ and the members of $\mathrm{Max}_1(R^*)$. Moreover, we have by Theorem 7.1:

Corollary 7.2. *If for each $i \in I$, R_i is a Krull domain, then R^*_Q is a valuation domain for every $Q \in \mathrm{Spec}_1(R^*)$.*

If X and Y are subsets of the quotient field of a domain R, then $(X : Y)$ denotes the set $\{r \in R\colon\ rY \subseteq X\}$. If R is a Noetherian domain with module-finite integral closure $\overline{R}$ and P is a nonzero height 1 prime ideal of R, then $(R : \overline{R})$ is nonzero and $(R : \overline{R})R_P = (R_P : \overline{R}_P)$. Thus if $(R : \overline{R})$ is not contained in P, then $R_P = \overline{R}_P$ and R_P is a Noetherian valuation domain. Since there are at most finitely many height one prime ideals containing $(R : \overline{R})$ (and because R is Noetherian), it follows that if R has module-finite integral closure, then there are at most finitely many height one prime ideals of R such that R_P is not a valuation domain. Hence by Theorem 7.1 we have:

Corollary 7.3. *If R^* is an ultrapower of a Noetherian domain R having finite Krull dimension and module-finite integral closure, then for all but finitely many $Q \in \mathrm{Max}_1(R^*)$, R^*_Q is valuation domain.*

Remark 7.4. If R^* is an ultrapower of a Noetherian domain R with module-finite integral closure, and $Q \in \mathrm{Max}_1(R^*)$ but R^*_Q is not a valuation domain, then it possible to describe the ring R^*_Q and its prime spectrum using the techniques of [16]. For it follows from the proof of Theorem 7.1 that $Q = (P_i)$, where for each $i \in I$, P_i is a prime of R_i such that R_{P_i} is not a valuation domain. Furthermore, since integral closure commutes with localization, it follows that the one-dimensional ring R_P also has module-finite integral closure. Hence R_P is analytically unramified. There are only finitely many such primes ideals P_i of R, so since $\mathcal{U}$ is an ultrafilter, a simple argument shows one may choose a prime ideal P of R such that $P_i = P$ for all $i \in I$. Hence R^*_Q is an ultrapower of the one-dimensional analytically unramified local domain R_P. Such ultrapowers are described in Section 6 of [16].

Using the results of Section 6 we can now describe all the prime ideals of ultra-height one in an ultraproduct of Krull domains.

Theorem 7.5. *Suppose that for each $i \in I$, R_i is a Krull domain. If $P \in \mathrm{Spec}_1(R^*)$, then there is a unique prime ideal Q of R^* such that $P \subseteq Q$ and P is the union of the ideals Q_e, $e \in E$, such that $Q_e \subseteq P$.*

Proof. By Corollary 6.5 each Q_e, $e \in E$, is a prime ideal. Let $a \in P$ and choose $a_i \in R_i$ such that $a = (a_i)$. Let V_i be the (finite) set of height 1 prime ideals of R_i containing a_i, and define V to the be member of $\mathcal{L}_1(R^*)$ induced by the family $\{V_i\}$. Define a mapping $e_i \in E_i$ by $e_i(L) = 0$ if L is a height one prime ideal of R_i not in V_i and $e_i(L) = \mathrm{ord}_L(a_i)$ if $L \in V_i$. Set $e = (e_i)_{i \in I} \in E$. (Note e depends only on a, not the choice of a_i.) We claim that $a \in Q_e \subseteq P$.

Clearly $a \in \bigcap_{L \in V} L^e \subseteq Q_e$, so it remains to show that $Q_e \subseteq P$. For each $i \in I$, $(R_i)_{\mathbf{S}(V_i)}$ is a PID since $(R_i)_{\mathbf{S}(V_i)}$ is an intersection of finitely many Noetherian valuation domains. Hence for all $i \in I$,

$$a_i R_{i\,\mathbf{S}(V_i)} = \left(\bigcap_{L \in V_i} L^{e_i(L)} \right) R_{i\,\mathbf{S}(V_i)}.$$

Thus since $\bigcap_{L \in V} L^e$ is the induced ideal $(\bigcap_{L \in V_i} L^{e_i(L)})$, it follows that $\bigcap_{L \in V} L^e \subseteq a R^*_{\mathbf{S}(V)} \subseteq P R^*_{\mathbf{S}(V)}$, and since $P = R^* \cap P R^*_{\mathbf{S}(V)}$, we have $\bigcap_{L \in V} L^e \subseteq P$.

To show now that $Q_e \subseteq P$, it suffices to show that for all $W \subseteq V$ with $W \in \mathcal{F}(Q)$, we have $\bigcap_{L \in W} L^e \subseteq P$. Let $W \subseteq V$ with $W \in \mathcal{F}(Q)$. Since $W \in \mathcal{L}_1(R^*)$, we have also by Lemma 3.2(i) that $V \setminus W \in \mathcal{L}_1(R^*)$. Now

$$\left(\bigcap_{L \in W} L^e \right) \cap \left(\bigcap_{L \in V \setminus W} L^e \right) = \bigcap_{L \in V} L^e \subseteq P.$$

If $\bigcap_{L\in V\setminus W} L^e \subseteq P$, then by Lemmas 3.2(v) and 6.7, $\mathbf{J}(V \setminus W) \subseteq Q$. Hence $V \setminus W \in \mathcal{F}(Q)$, contrary to the assumption that W is also in the maximal filter $\mathcal{F}(Q)$. Thus $\bigcap_{L\in W} L^e \subseteq P$, and we have proved that for each $a \in P$, there exists $e \in E$ such that $Q_e \subseteq P$. Therefore P is the union of all the ideals Q_e, $e \in E$, such that $Q_e \subseteq P$. $\square$

Theorems 5.4 and 7.5 give a classification of all the prime ideals in an ultraproduct of Dedekind domains.[2] More generally, they give a description of all the primes ideals of ultra-height one in an ultraproduct R^* of Krull domains.

By Corollary 7.2 the localization of R^* at any of these prime ideals is a valuation domain. Theorem 5.4 and Remark 7.4 give the outline for a classification of prime ideals in an ultrapower of a one-dimensional Noetherian domain with module-finite integral closure.

The prime spectrum of ultraproducts (even ultrapowers) of higher-dimensional Noetherian rings is much more complicated. For example let R be a two-dimensional regular local ring, and let R^* be an ultrapower of R with respect to a free ultrafilter $\mathcal{U}$ on an index set I. There is a partition of the set of prime ideals of R^* given by

$$\operatorname{Spec}(R^*) = \operatorname{Spec}_2(R^*) \cup \operatorname{Spec}_1(R^*) \cup \{(0)\}.$$

Since regular local rings are Krull domains, the prime ideals in $\operatorname{Spec}_1(R^*)$ can be described as above. Thus it remains to describe $\operatorname{Spec}_2(R^*)$. The set $\operatorname{Max}_2(R^*) = \operatorname{Max}(R^*)$ consists of a single element, namely the maximal ideal N of R^* induced by the maximal ideal M of R^*. If $P \in \operatorname{Spec}_2(R^*)$, then for each $i \in I$, there exists $e_i > 0$ such that $(M^{e_i}) \subseteq P$. Thus to describe the prime ideals of R^*, it would be sufficient to describe the prime spectra of ultraproducts of Artinian rings of the form R/M^{e_i}. While several papers have studied the prime ideals of ultraproducts of zero-dimensional rings (and in particular Artinian rings), to our knowledge the prime ideals in such ultraproducts have not been completely described [8,9,22]. (Also see Remark 5.8.) In general there are many prime ideals P of R^* containing a prime ideal of the form M_e, $e \in E$, and such a prime ideal P does not have to be of the form M_f for $f \in E$ or even a union of such ideals. For example, if $A = \bigcap_{k=1}^{\infty} M^k$, then R^*/A is the complete regular local ring $R^{(*)}$ of Krull dimension 2 described in [16], and the set of prime ideals containing A is at least as complicated as the set of prime ideals of the completion $\widehat{R}$ of R.

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[2] In Corollary 12 of [11] it is asserted (in our terminology) that if $\mathbb{Z}^*$ is an ultrapower of the ring of integers $\mathbb{Z}$, then each prime ideal in $\mathbb{Z}^*$ is a union of prime ideals of the form Q_e, where Q is a maximal ideal of $\mathbb{Z}^*$ and $e = (e_i)$ with each e_i a constant-valued function. However there is a gap in the proof of this corollary and the assertion that the e_i can be assumed to be constant-valued appears to be unjustified.

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