A combinatorial proof of Gotzmann’s persistence theorem for monomial ideals

Satoshi Murai

Department of Pure and Applied Mathematics, Graduate School of Information Science and Technology,
Osaka University, Toyonaka, Osaka, 560-0043, Japan

Received 27 April 2005; accepted 7 July 2006
Available online 17 January 2007

Abstract

Gotzmann proved the persistence for minimal growth of Hilbert functions of homogeneous ideals. His theorem is called Gotzmann’s persistence theorem. In this paper, based on the combinatorics of binomial coefficients, a simple combinatorial proof of Gotzmann’s persistence theorem in the special case of monomial ideals is given.

© 2006 Elsevier Ltd. All rights reserved.

0. Introduction

In this paper, we will give a combinatorial proof of Gotzmann’s persistence theorem in the special case of monomial ideals. Let $K$ be an arbitrary field, $R = K[x_1, x_2, \ldots, x_n]$ the polynomial ring with $\deg(x_i) = 1$ for $i = 1, 2, \ldots, n$. Let $M = \{x_1, x_2, \ldots, x_n\}$ and $M^d$ the set of all monomials in $R$ of degree $d$, where $M^0 = \{1\}$. For a monomial $u \in R$ and for a subset $V \subset M^d$, we define $uV = \{uv \mid v \in V\}$ and $MV = \{x_i v \mid v \in V, i = 1, 2, \ldots, n\}$, and write $|V|$ for the cardinality of $V$. Let $\gcd(V)$ denote the greatest common divisor of the monomials belonging to $V$.

Let $n$ and $h$ be positive integers. Then $h$ can be written uniquely in the form, called the $n$th binomial representation of $h$,

$$h = \binom{h(n) + n}{n} + \binom{h(n - 1) + n - 1}{n - 1} + \cdots + \binom{h(i) + i}{i},$$

E-mail address: s-murai@ist.osaka-u.ac.jp.
where \( h(n) \geq h(n-1) \geq \cdots \geq h(i) \geq 0 \) and \( i \geq 1 \). See ([3], Lemma 4.2.6). If \( h = \sum_{j=i}^{n} \binom{h(j)+n}{j} \) is the \( n \)th binomial representation of a positive integer \( h \), write

\[
\begin{aligned}
\ h(n) &= \binom{h(n)+n+1}{n} + \cdots + \binom{h(i)+i+1}{i}, \\
\ h(n) &= \binom{h(n)+n}{n-1} + \cdots + \binom{h(i)+i}{i-1}, \\
\ h(n) &= \binom{h(n)+n-1}{n-1} + \cdots + \binom{h(i)+i-1}{i-1}.
\end{aligned}
\]

Set \( 0^{(n)} = 0_{(n)} = 0_{\langle n \rangle} = 0, 1^{(n)} = 1_{\langle n \rangle} = 1 \) and \( 1_{(n)} = 0 \).

The following inequality (1) was proved by Macaulay [7] (see also [2] and [6] for further information): Let \( V \subset R \) be a set of monomials of the same degree. Then one has

\[
\left| MV \right| \geq \left| V \right|^{(n-1)}. \tag{1}
\]

In 1978, Gotzmann [4] proved a so-called persistence theorem. In the special case of monomial ideals, the persistence theorem says that

**Theorem 0.1 (Persistence Theorem for Monomial Ideals).** Let \( V \) be a set of monomials of degree \( d \). If \( \left| MV \right| = \left| V \right|^{(n-1)} \) then \( \left| M^{i+1}V \right| = \left| M^i V \right|^{(n-1)} \) for all \( i \geq 0 \).

Let \( A = (a_1, a_2, \ldots, a_n) \) and \( B = (b_1, b_2, \ldots, b_n) \) be elements of \( \mathbb{Z}^n_{\geq 0} \). The lexicographic order \( \prec_{\text{lex}} \) on \( \mathbb{Z}^n_{\geq 0} \) is defined by \( A \prec_{\text{lex}} B \) if the leftmost nonzero entry of \( B - A \) is positive. Also the lexicographic order on the set of monomials in \( R \) is defined by \( x_1^{a_1}x_2^{a_2} \cdots x_n^{a_n} \prec_{\text{lex}} x_1^{b_1}x_2^{b_2} \cdots x_n^{b_n} \) if \( A \prec_{\text{lex}} B \) on \( \mathbb{Z}^n_{\geq 0} \).

Let \( V \) be a set of monomials of degree \( d \). Then

(i) \( V \) is called a **Gotzmann** if \( V \) satisfies \( \left| MV \right| = \left| V \right|^{(n-1)} \).

(ii) \( V \) is called a **lexsegment** if \( V \) is a set of first \( \left| V \right| \) monomials of degree \( d \) w.r.t. the lexicographic order. Denote the lexsegment set \( V \) in \( K[x_1, \ldots, x_n] \) of degree \( d \) with \( \left| V \right| = a \) by \( \text{Lex}(n, d, a) \).

It is known that lexsegment sets are Gotzmann. See [2, Section 4.2] or [6]. Also, in [8], all integers \( a > 0 \) such that every Gotzmann set with \( \left| V \right| = a \) and with \( \gcd(V) = 1 \) is lexsegment up to permutations of variables are determined. Related works on Gotzmann’s theorem were done by Aramova, Herzog and Hibi [1]. They proved Gotzmann’s theorem for exterior algebras. In addition, Furedi and Griggs [3] determined all integers \( a > 0 \) such that every squarefree Gotzmann set with \( \left| V \right| = a \) is squarefree lexsegment up to permutations of variables.

Inequality (1) and Theorem 0.1 are true for more general cases. Indeed, Gotzmann [4] proved persistence for minimal growth of the Hilbert function of a homogeneous ideal in a polynomial ring (see [6, Theorem C.17]). Green refined Gotzmann’s proof (see [2, Theorem 4.3.3]). Green also gave a simple proof in [5, Theorem 3.8] using generic initial ideals. On the other hand, in the special case of monomial ideals, in [4] Gotzmann proved the persistence theorem more easily than in the general case using his version of the theory of Castelnuovo–Mumford regularity. All of these proofs are completely algebraic. In the present paper we will give a combinatorial proof of the persistence theorem for monomial ideals. The advantage of our proof is that we only use the combinatorics of binomial coefficients.
In Section 1, we will introduce some lemmas about binomial representations. In Section 2, we will give a combinatorial proof of persistence for monomial ideals.

1. Binomial representations

In this section we study some properties about binomial representations. Let \( h(1) s(1) + \cdots + h(i) s(i) \) be a sum of binomials, where \( h(j) \geq s(j) \) for \( j = 1, 2, \ldots, i \). We write
\[
\left\{ \binom{h(1)}{s(1)} + \cdots + \binom{h(i)}{s(i)} \right\}^{[+1]} = \binom{h(1) + 1}{s(1)} + \cdots + \binom{h(i) + 1}{s(i)}.
\]

First, we recall some easy properties about binomial representations.

**Lemma 1.1** ([3, Lemma 4.2.7]). Let \( a, a' \) and \( n \) be positive integers, and let \( a = \sum_{k=1}^{n} \binom{h(k)}{k} \) and \( a' = \sum_{k=1}^{n} \binom{h'(k)}{k} \) be the \( n \)th binomial representations. Set \( h(k) = 0 \) for \( 1 \leq k < i \) and \( h'(k) = 0 \) for \( 1 \leq k < j \). Then one has \( a < a' \) if and only if
\[
(h(n), h(n-1), \ldots, h(1)) <_{\text{lex}} (h'(n), h'(n-1), \ldots, h'(1)).
\]

**Lemma 1.2.** Let \( h \) and \( n \) be integers with \( h \geq 0 \) and \( n > 0 \). Then, for any integer \( 1 \leq \alpha \leq h \), one has
\[
\binom{h+n}{n} = \binom{\alpha - 1 + n}{n} + \binom{\alpha + n - 1}{n-1} + \binom{\alpha + 1 + n - 1}{n-1} + \cdots + \binom{h + n - 1}{n - 1}
\]
and
\[
\binom{h+n}{n}^{[+1]} = \left\{ \binom{\alpha - 1 + n}{n} + \binom{\alpha + n - 1}{n-1} + \binom{\alpha + 1 + n - 1}{n-1} + \right.
\]
\[
\left. \cdots + \binom{h + n - 1}{n - 1} \right\}^{[+1]}.
\]

**Proof.** Use \( \binom{h+n}{n} = \binom{h-1+n}{n} + \binom{h-1+n}{n-1} \) on the leftmost binomial coefficient repeatedly, then we have
\[
\binom{h+n}{n} = \binom{h-2+n}{n} + \binom{h-1+n-1}{n-1} + \binom{h+n-1}{n-1}
\]
\[
\vdots
\]
\[
= \binom{\alpha - 1 + n}{n} + \binom{\alpha + n - 1}{n-1} + \binom{\alpha + 1 + n - 1}{n-1} + \cdots + \binom{h + n - 1}{n - 1},
\]
as desired. \( \square \)

**Lemma 1.3.** Let \( h \) and \( n \) be positive integers. Then,
\[
h^{(n)} = h + h^{(n)}.
\]
Proof. Let \( h = \sum_{j=1}^{n} \binom{h(j)+j}{j} \) be the \( n \)th binomial representation of \( h \). Since \( \binom{h+n}{n} = \binom{h-1+n}{n-1} + \binom{h-1+n}{n-1} \), one has

\[
h + h(n) = \sum_{j=1}^{n} \left\{ \binom{h(j)+j}{j} + \binom{h(j)+j}{j-1} \right\} = \sum_{j=1}^{n} \binom{h(j)+j+1}{j} = h(n),
\]
as desired. \( \Box \)

In the rest of this section, we introduce some lemmas which will be used in the proof of the main theorem.

Lemma 1.4. Let \( a, b \) and \( m \) be positive integers. One has

\[
a^{(m)} + b^{(m)} > (a + b)^{(m)}.
\]

Proof. Let \( n \geq 2 \) and \( M = \{x_1, \ldots, x_n\} \). Take an integer \( d \) with \( |M^d| > a + b \). Let \( V_a = \text{Lex}(n, d, a) \), \( V_b = \text{Lex}(n, d, b) \) and \( u \) be the minimal monomial in \( V_a \) w.r.t. the lexicographic order. Let \( V = x_1^{d+1}V_a \cup ux_nV_b \). Since \( ux_1^{d+1} > \text{lex} ux_1^{d+1}x_n \) and \( n \geq 2 \), \( x_1^{d+1}V_a \cup ux_nV_b \) is a disjoint union. Since \( x_1^{d+1}x_nu \in Mx_1^{d+1}V_a \cap Mux_nV_b \), we have \( Mx_1^{d+1}V_a \cap Mux_nV_b = \emptyset \). By (1), for any positive integer \( n \geq 2 \), we have

\[
(a + b)^{(n-1)} \leq |MV| < |M V_a| + |M V_b| = a^{(n-1)} + b^{(n-1)},
\]
as desired. \( \Box \)

Definition 1.5. Let \( h \) be a positive integer and \( h = \sum_{j=1}^{n} \binom{h(j)+j}{j} \) the \( n \)th binomial representation of \( h \). Let \( \alpha = \max\{0, \max\{\alpha \in \mathbb{Z} \mid h - \binom{\alpha+n}{n} > 0\}\} \). Define \( \psi_n(h) = h - \binom{\alpha+n}{n} \), in other words,

(i) \( \psi_n(h) = 0 \) if \( h = 1 \);

(ii) \( \psi_n(h) = \binom{h(n)+n-1}{n-1} \) if \( h > 1 \) and \( i = n \);

(iii) \( \psi_n(h) = \sum_{j=1}^{n-1} \binom{h(j)+j}{j} \) if \( h > 1 \) and \( i < n \).

Notice that this construction says \( \psi_n(h) \leq \binom{\alpha+n}{n-1} \) and \( h(n) = \binom{\alpha+n}{n} + \psi_n(h(n-1)) \). Furthermore, if \( h > 1 \) then \( \psi_n(h) \geq 1 \).

Lemma 1.6. Let \( a, b, c \) and \( \alpha \) be positive integers. If \( \binom{\alpha+n}{n} + a = b + c \) and \( a, b, c < \binom{\alpha+n}{n} \), then one has

\[
\binom{\alpha+n}{n} ^{(n)} + a ^{(n)} \leq b ^{(n)} + c ^{(n)}.
\]

Moreover, if \( \binom{\alpha+n}{n} ^{(n)} + a ^{(n)} = b ^{(n)} + c ^{(n)} \) then one has

\[
\left\{ \binom{\alpha+n}{n} ^{(n)} \right\} ^{(n)} + a ^{(n-1)} = \{b ^{(n)} \} ^{(n)} + \{c ^{(n)} \} ^{(n)}.
\]
Proof. We use induction on \( n \). First, we consider the case \( n = 1 \). If \( h \) is a positive integer, then 
\[
 h^{(1)} = \left( \frac{h+1}{1} \right) = h + 1.
\]
Then we have 
\[
 (\alpha+1)^{(1)} + a^{(1)} = b + 1 + c + 1 = b^{(1)} + c^{(1)}.
\]
Thus we may assume \( n > 1 \).

Let \( a = \left( a(n) + n \right) \) and \( c = \left( c(n) + n \right) \). Then we have 
\[
 \bar{a} = \psi_n(a), \quad b = \left( b(n) + n \right) + \psi_n(b) \quad \text{and} \quad c = \left( c(n) + n \right) \psi_n(c) \text{ be the form of Definition 1.5.}
\]
Set \( \bar{a} = \psi_n(a), \bar{b} = \psi_n(b) \) and \( \bar{c} = \psi_n(c) \).

First, we note the following fundamental inequalities.

(\( \alpha \)) \( a < b, a < c, \alpha > b(n), \alpha > c(n) \) and \( a(n) \leq c(n) \);
(\( \beta \)) \( \bar{b} \geq 1 \) and \( \bar{c} \geq 1 \);
(\( \gamma \)) \( \bar{b} < \left( \frac{\alpha + n - 1}{n-1} \right) \) and \( \bar{c} < \left( \frac{\alpha + n - 1}{n-1} \right) \).

Statement (\( \alpha \)) follows from the assumption and Lemma 1.1, and statement (\( \beta \)) follows from 
\( 1 \leq a < b \) and \( 1 \leq a < c \). We consider statement (\( \gamma \)). By Definition 1.5 and statement (\( \alpha \)),
we have \( \bar{b} \leq \left( \frac{b(n) + n}{n-1} \right) \leq \left( \frac{\alpha + n - 1}{n-1} \right) \). However, if \( \bar{b} = \left( \frac{b(n) + n}{n-1} \right) \) then \( b(n) < \alpha - 1 \) since
\[
 b = \left( \frac{b(n) + 1 + n}{n} \right) < \left( \frac{\alpha + n}{n} \right). \quad \text{Thus statement (\( \gamma \)) follows.}
\]

Next, by Lemma 1.2, we can write \( \left( \frac{\alpha + n}{n} \right) \) and \( \left( \frac{c(n) + n}{n} \right) \) as follows:
\[
 \left( \frac{\alpha + n}{n} \right) = \left( \frac{b(n) + n}{n} \right) + \sum_{i=b(n)+1}^{\alpha} \left( \frac{i + n - 1}{n - 1} \right);
\]
\[
 \left( \frac{c(n) + n}{n} \right) = \left( \frac{a(n) + n}{n} \right) + \sum_{i=a(n)+1}^{c(n)} \left( \frac{i + n - 1}{n - 1} \right).
\]

We substitute the above equations into \( \left( \frac{\alpha + n}{n} \right) + a = b + c \). Then we have
\[
 \left\{ \sum_{i=b(n)+1}^{\alpha} \left( \frac{i + n - 1}{n - 1} \right) \right\} + \bar{a} = \bar{b} + \bar{c} \left\{ \sum_{i=a(n)+1}^{c(n)} \left( \frac{i + n - 1}{n - 1} \right) \right\}.
\]

In the same way, Lemma 1.2 also implies that the inequality \( \left( \frac{\alpha + n}{n} \right)^{(n)} + a^{(n)} \leq b^{(n)} + c^{(n)} \) is equivalent to
\[
 \left\{ \sum_{i=b(n)+1}^{\alpha} \left( \frac{i + n - 1}{n - 1} \right) \right\}^{[n+1]} + \bar{a}^{(n-1)} \\
\leq \bar{b}^{(n-1)} + \bar{c}^{(n-1)} + \left\{ \sum_{i=a(n)+1}^{c(n)} \left( \frac{i + n - 1}{n - 1} \right) \right\}^{[n+1]}.
\]

Thus, instead of considering \( \left( \frac{\alpha + n}{n} \right) + a = b + c \) and \( \left( \frac{\alpha + n}{n} \right)^{(n)} + a^{(n)} \leq b^{(n)} + c^{(n)} \), it is enough to consider (3) and (4). We will consider two cases.

Case I. Let \( \bar{c} \geq \bar{a} \) and \( n > 1 \). Set \( P_{c(n)+1} = \bar{c}, t_0 = c(n) + 2, d_0 = \bar{a} \) and \( d_{\alpha - b(n)} = \bar{b} \). We will prove that, for \( i = 0, 1, \ldots, \alpha - (b(n) + 1) \), \( \left( \frac{\alpha - i + n - 1}{n - 1} \right) \) can be written in the form
\[
 \left( \frac{\alpha - i + n - 1}{n - 1} \right) = -d_i + \sum_{j=t_i+1}^{t_{i+1}} P_j + d_{i+1},
\]

(5)
where $P_j = \binom{j+n-1}{n-1}$ for $j = a(n) + 1, \ldots, c(n), t_i + 1 < t_i \leq c(n) - i + 2$ and $0 \leq d_i < P_{t_i-1}$.

We use induction on $i$. For $i = 0$, since $\bar{c} - \bar{a} = \binom{\alpha+n-1}{n-1}$, there exists an integer $t_1 \leq c(n) + 1$ such that

$$\sum_{i=t_1}^{c(n)} \binom{i+n-1}{n-1} + \bar{c} - \bar{a} \leq \binom{\alpha+n-1}{n-1} \leq \sum_{i=t_1}^{c(n)+1} \binom{i+n-1}{n-1} + \bar{c} - \bar{a}.$$ 

Thus we have

$$\binom{\alpha+n-1}{n-1} = \bar{c} - \bar{a} + \sum_{j=t_1}^{c(n)} P_j + d_1 = -d_0 + \sum_{j=t_1}^{c(n)+1} P_j + d_1$$

with $0 \leq d_1 < P_{t_1-1}$. Assume (5) holds for $i = 0, \ldots, s - 1$. Since $\alpha > c(n)$, the assumption of induction says $\binom{\alpha-s+n-1}{n-1} \geq \binom{c(n)-s+1+n-1}{n-1} \geq \binom{t_{s-1}+n-1}{n-1} = P_{t_s-1}$. Thus

$$\binom{\alpha-s+n-1}{n-1} \geq -d_{s+1} + P_{t_s-1} and t_{s+1} < t_s.$$ 

In the same way as in the case $i = 0$, it follows that (5) holds for $i = s$. Especially, if $s = \alpha - (b(n) + 1)$ then (3) says that

$$\binom{b(n)+n}{n-1} = -d_s + \sum_{j=a(n)+1}^{t_s-1} \binom{j+n-1}{n-1} + \bar{b}.$$ 

Thus each $\binom{\alpha-i+n-1}{n-1}$ can be written in the form (5).

Then, by using (5), the induction hypothesis and Lemma 1.4 say

$$\binom{\alpha-i-1+n-1}{n-1}^{[+1]} \leq -d_{i+1} \binom{n-1}{n-1}^{(n-1)} + \left( \sum_{j=t_i}^{n-1} P_j \right)^{(n-1)} + d_{i+2} \binom{n-1}{n-1}$$

$$\leq -d_{i+1} \binom{n-1}{n-1}^{(n-1)} + \sum_{j=t_{i+1}}^{n-1} P_j \binom{n-1}{n-1}^{(n-1)} + d_{i+2} \binom{n-1}{n-1}.$$ 

Summing (5) on both sides for $i = 0, \ldots, \alpha - (b(n) + 1)$ yields (3), and summing (7) in both sides for $i = 0, \ldots, \alpha - (b(n) + 1)$ yields (4). Thus the first statement of Lemma 1.6 follows. On the other hand, Lemma 1.4 says that if (7) is equal then $t_i - 1 = t_{i+1}$. Since (4) is equal if and only if (7) are equal for all $i$, the induction hypothesis says that if (4) is equal then (2) is satisfied.

Case II. Let $\bar{c} < \bar{a}$ and $n > 1$. Set $d_0 = \bar{b}, t_0 = c(n) + 1$ and $t_{\alpha-b(n)} = a(n) + 1$. We will prove that, for $i = 0, 1, \ldots, \alpha - (b(n) + 1), \binom{\alpha-i+n-1}{n-1}$ can be written in the form

$$\binom{\alpha-i+n-1}{n-1} = d_i + \sum_{j=t_i}^{n-1} \binom{j+n-1}{n-1} - d_{i+1}$$

and

$$\bar{a} = \bar{c} + d_{\alpha-b(n)}.$$ 

where $0 \leq d_i < \binom{t_i+n-1}{n-1}$ and $t_{i+1} < t_i \leq c(n) - i + 1$. 

For \( i = 0 \), since \( \left( \frac{\alpha+n-1}{n-1} \right) > \bar{b} \), in the same way as Case I we have

\[
\left( \frac{\alpha+n-1}{n-1} \right) = \bar{b} + \sum_{j=t_1}^{c(n)} \left( \frac{j+n-1}{n-1} \right) - d_1.
\]

Also, if we have Eq. (8) for \( i = 0, 1, \ldots, s-1 \), then we have \( \left( \frac{\alpha-s+n-1}{n-1} \right) \geq \left( \frac{c(n)-(s-1)+n-1}{n-1} \right) \geq \left( \frac{t_s+n-1}{n-1} \right) > d_s \). Thus we have \( t_s < t_s \) and we have Eq. (8) for \( i = s \) in the same way as in the case \( i = 0 \). Finally, since \( \tilde{a} - \tilde{c} < \bar{a} \leq \left( \frac{a(n)+n-1}{n-1} \right) \) by the definition of \( \bar{a} \), we have \( \tilde{a} = \tilde{c} + d_{a-b(n)} \) and \( t_{a-b(n)} = a(n)+1 \).

Then, by using (8), the induction hypothesis and Lemma 1.4

\[
\left( \frac{\alpha-i+n-1}{n-1} \right)^{[+1]} \leq d_i^{(n-1)} + \sum_{j=i+1}^{n-1} \left( \frac{j+n-1}{n-1} \right)^{(n-1)} - d_{i+1}^{(n-1)}. \tag{10}
\]

Furthermore, since \( \tilde{c} > 0 \) and \( d_{a-b(n)} > 0 \), Lemma 1.4 says

\[
\tilde{a}^{(n-1)} < c^{(n-1)} + d_{s+1}^{(n-1)}. \tag{11}
\]

Then, by summing (8) and (9), Eq. (3) follows. Also, by summing (10) and (11), we have

\[
\left\{ \sum_{i=h(n)+1}^{\alpha} \left( \frac{i+n-1}{n-1} \right)^{[+1]} \right\} + \tilde{a}^{(n-1)} < \tilde{b}^{(n-1)} + \tilde{c}^{(n-1)}
\]

\[
+ \left\{ \sum_{i=a(n)+1}^{c(n)} \left( \frac{i+n-1}{n-1} \right)^{[+1]} \right\},
\]

as desired. Thus the first statement of Lemma 1.6 follows. In this case the above equation says that (4) is not equal. Thus we do not need to consider the second statement. \( \square \)

**Lemma 1.7.** Let \( h \) and \( n \) be positive integers. Then one has

\[
h^{(n)} < h^{(n+1)}.
\]

**Proof.** Let \( h = \left( \frac{h(n+1)+n+1}{n+1} \right) + \psi_{n+1}(h) \). Then we have \( h^{(n+1)} = \left( \frac{h(n+1)+n+1}{n+1} \right)^{(n+1)} + \psi_{n+1}(h)^{(n)} \). By Lemma 1.2, we have

\[
\left( \frac{h(n+1)+n+1}{n+1} \right) = \left( \frac{n+1}{n+1} \right) + \sum_{i=1}^{h(n+1)} \left( \frac{i+n}{n} \right).
\]

Since \( \left( \frac{n+1}{n} \right)^{[+1]} > \left( \frac{n}{n} \right)^{[+1]} \), by Lemma 1.4, we have

\[
\left( \frac{h(n+1)}{n+1} \right)^{(n+1)} + \psi_{n+1}(h)^{(n)} > \left( \frac{n}{n} \right)^{(n)} + \sum_{i=1}^{h(n+1)} \left( \frac{i+n}{n} \right)^{(n)} + \psi_{n+1}(h)^{(n)}
\]

\[
\geq \left\{ \left( \frac{n}{n} \right) + \sum_{i=1}^{h(n+1)} \left( \frac{i+n}{n} \right) + \psi_{n+1}(h) \right\}^{(n)} = h^{(n)},
\]

as desired. \( \square \)
2. A combinatorial proof of persistence for monomial ideals

Let $V$ be a set of monomials of degree $d$ and $u = \gcd(V)$. If $|V| > 1$, we define $K_i(V) = \{ v \in V \mid x_i u \text{ divides } v \}$ and $D_i(V) = V \setminus K_i(V)$ for $i = 1, 2, \ldots, n$. If $|V| = 1$, then we define $K_i(V) = V$ and $D_i(V) = \emptyset$. Note that if $|V| > 1$, then $D_i(V) = \emptyset$. Set $\overline{M}_i = M \setminus \{x_i\}$ for $i = 1, 2, \ldots, n$.

Lemma 2.1. Let $V \subseteq M^d$ be a set of monomials of degree $d$ and $u = \gcd(V)$. For each $i = 1, 2, \ldots, n$, we have

(i) $\overline{M}_i D_i(V) \subseteq MV \setminus x_i V$;

(ii) $|MV| \geq |K_i(V)|^{(n-1)} + |D_i(V)|^{(n-2)}$. \hfill (12)

Moreover, in (13), the equality holds if and only if $K_i(V)$ is a Gotzmann set of $K[x_1, x_2, \ldots, x_n]$, $\frac{1}{u} D_i(V)$ is a Gotzmann set of $K[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n]$ and $x_i D_i(V) \subseteq \overline{M}_i K_i(V)$.

Proof. Any element in $\overline{M}_i D_i(V)$ can not be divided by $u x_i$. On the other hand, $\overline{M}_i D_i(V) \subseteq MV$. Thus we have $\overline{M}_i D_i(V) \subseteq MV \setminus x_i V$. Thus statement (i) follows.

On the other hand, it is clear that

$$|MV| = |MK_i(V)| + |\overline{M}_i D_i(V)| - |x_i D_i(V) \setminus MK_i(V)|.\hfill (13)$$

Since $\frac{1}{u} D_i(V) \subseteq K[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n]$, it follows from (1) that

$$|MV| \geq |K_i(V)|^{(n-1)} + |D_i(V)|^{(n-2)} - |x_i D_i(V) \setminus MK_i(V)|.\hfill (13)$$

By the above inequality, we have $|MV| \geq |K_i(V)|^{(n-1)} + |D_i(V)|^{(n-2)}$. Also, it is clear that equality holds if and only if $K_i(V)$ and $\frac{1}{u} D_i(V)$ are Gotzmann sets and $MK_i(V) \supseteq x_i D_i(V)$. However, since $\frac{1}{u} D_i(V) \subseteq K[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n]$ and any monomial in $\frac{1}{u} K_i(V)$ contains $x_i$, $MK_i(V) \supseteq x_i D_i(V)$ implies $\overline{M}_i K_i(V) \supseteq x_i D_i(V)$. \hfill \Box

Next, we determine the range of $|D_i(V)|$ when $V$ is a Gotzmann set.

Lemma 2.2. Let $V \subseteq M^d$ be a Gotzmann set of monomials of degree $d$ and $h = |V|$. Then, for any $i = 1, 2, \ldots, n$, we have

$$\psi_{n-1}(|V|) \leq |D_i(V)| \leq |V|_{(n-1)}$. \hfill (14)$$

Proof. If $|V| = 0$ or $|V| = 1$, then $\psi_{n-1}(|V|) = |D_i(V)| = 0$. Thus we may assume $n > 1$ and $|V| > 1$. First, we consider the second inequality of (14). Lemma 2.1 and (1) say that

$$|D_i(V)|^{(n-2)} \leq |\overline{M}_i D_i(V)| \leq |(MV \setminus x_i V)|.\hfill (16)$$

On the other hand, by Lemma 1.3, we have

$$|(MV \setminus x_i V)| = |MV| - |V| = |V|^{(n-1)} - |V| = |V|_{(n-1)}.\hfill (17)$$

Thus $|D_i(V)|^{(n-2)} \leq |V|_{(n-1)}$. Hence we have $|D_i(V)| \leq |V|_{(n-1)}$.\hfill (18)

Next, we consider the first inequality of (14). If $n = 2$, then $\psi_{n-1}(|V|) = |D_i(V)| = 1$. Thus we may assume $n \geq 3$. Let $|V| = \left(\begin{array}{c} a+n-1 \\ n-1 \end{array}\right) + \psi_{n-1}(|V|)$. If $|D_i(V)| < \psi_{n-1}(|V|)$ then
Lemma 2.1 says that Lemma 1.1. Let \( V \) be the binomial representations. Assume \( V \neq \emptyset \). Thus we may assume that \( V \neq 0 \).

By Lemma 2.1, we have

\[
|MV| \geq |K_i(V)|^{(n-1)} + |D_i(V)|^{(n-2)} = \left(\frac{a+n-1}{n-1}\right)^{(n-1)} + b^{(n-2)} + |D_i(V)|^{(n-2)}.
\]

On the other hand Lemma 1.4 says that

\[
b^{(n-2)} + |D_i(V)|^{(n-2)} > \{b + |D_i(V)|\}^{(n-2)} = (\psi_{n-1}(|V|))^{(n-2)}.
\]

Thus we have

\[
|MV| > \left(\frac{a+n-1}{n-1}\right)^{(n-1)} + (\psi_{n-1}(|V|))^{(n-2)} = |V|^{(n-1)}.
\]

This contradicts the fact that \( V \) is a Gotzmann set. \( \square \)

Lemma 2.3. Let \( V \subset M^d \) be a Gotzmann set with \( \gcd(V) = 1 \) and \( V \neq M^d \). Then there exists an \( i \in \{1, 2, \ldots, n\} \) which satisfies the following conditions:

(i) \( K_i(V) \) is a Gotzmann set of \( K[x_1, \ldots, x_n] \), \( D_i(V) \) is a Gotzmann set of \( K[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n] \) and \( |D_i(V)| < |V|^{(n-1)} \);

(ii) \( x_iD_i(V) \subset M_iK_i(V) \);

(iii) \( |K_i(V)|^{(n-1)} + |D_i(V)|^{(n-2)} = |V|^{(n-1)} \).

Proof. Let \( |V| = a = \sum_{j=p}^{n-1} \binom{a(j)+j}{j} \), \( |D_i(V)| = b = \sum_{j=q}^{n-2} \binom{b(j)+j}{j} \) and \( |K_i(V)| = c = \sum_{j=r}^{n-1} \binom{c(j)+j}{j} \) be the binomial representations. Assume \( V \neq \emptyset \).

If \( |V| = 1 \), then \( V = M^0 \) since \( \gcd(V) = 1 \). Also, if \( n = 1 \) then \( |V| = 1 \). Thus we may assume that \( |V| > 1 \) and \( n > 1 \).

By Lemma 2.1, if \( a^{(n-1)} \leq b^{(n-2)} + c^{(n-1)} \) then conditions (i) and (ii) are satisfied. We will show that if \( b < a^{(n-1)} \) then \( a^{(n-1)} \leq b^{(n-2)} + c^{(n-1)} \) and condition (iii) is satisfied.

If \( b < a^{(n-1)} \), then Lemma 1.1 says that there exists the maximal integer \( t \) such that \( n-1 \geq t \geq p \) and

\[
0 \leq b - \sum_{j=t+1}^{n-1} \binom{a(j) + j - 1}{j-1} < \binom{a(t) + t - 1}{t-1}.
\]

Let

\[
a = \sum_{j=t+1}^{n-1} \binom{a(j) + j}{j} + \binom{a(t) + t}{t} + a', \quad (15)
\]

\[
b = \sum_{j=t+1}^{n-1} \binom{a(j) + j - 1}{j-1} + b', \quad (16)
\]

and

\[
c = a - b = \sum_{j=t+1}^{n-1} \binom{a(j) + j - 1}{j} + c'. \quad (17)
\]
Since $0 \leq b' < \binom{a(t)+t-1}{t-1}$ we have $\binom{a(t)+t-1}{t} < c' < \binom{a(t)+t+1}{t}$. Also, we have

$$a^{(n-1)} = \left\{ \sum_{j=t+1}^{n-1} \binom{a(j) + j}{j} \right\}^{[+1]} + \left\{ \binom{a(t) + t}{t} \right\}^{[+1]} + a'(t-1),$$

and $$b^{(n-2)} = \left\{ \sum_{j=t+1}^{n-1} \binom{a(j) + j - 1}{j} \right\}^{[+1]} + b'(t-1).$$

**Case (A).** Assume $b < a_{(n-1)}$ and $c' < \binom{a(t)+t}{t}$.

Let $c'' = c' - \binom{a(t)+t-1}{t-1}$. If $b' = 0$, then $c' \geq \binom{a(t)+t}{t}$. Thus $b' > 0$. On the other hand, we have $c'' > 0$ since $c' > \binom{a(t)+t-1}{t-1}$. Since $c'' < \binom{a(t)+t-1}{t-1}$, $c = \sum_{j=t}^{n-1} \binom{a(j)+j}{j} + \{t-1\}$th binomial representation of $c''$ is the $(n-1)$th binomial representation of $c$. Thus

$$c^{(n-1)} = \left\{ \sum_{j=t}^{n-1} \binom{a(j) + j - 1}{j} \right\}^{[+1]} + c''(t-1).$$

Thus, by (19), we have

$$b^{(n-2)} + c^{(n-1)} = \left\{ \sum_{j=t+1}^{n-1} \binom{a(j) + j}{j} \right\}^{[+1]} + \left\{ \binom{a(t) + t - 1}{t} \right\}^{[+1]} + b'(t-1) + c''(t-1).$$

Since $\binom{a(t)+t}{t} = \binom{a(t)+t-1}{t-1} + \binom{a(t)+t-1}{t-1}$, the equation $a = b + c$ together with (15)–(17) says $b' + c'' = a' + \binom{a(t)+t-1}{t-1}$. Hence, by Lemmas 1.4 and 1.6 together with the facts that $b' > 0$ and $c'' > 0$, we have

$$b'(t-1) + c''(t-1) \geq a'(t-1) + \binom{a(t) + t - 1}{t-1}. $$

(21)

Thus by (18) and (20), we have $a^{(n-1)} \leq b^{(n-2)} + c^{(n-1)}$. It remains to show condition (iii). Since $a^{(n-1)} = b^{(n-2)} + c^{(n-1)}$, (21) is equal. Then, since $c'' < \binom{a(t)+t-1}{t-1}$, Lemma 1.6 says

$$\left\{ b'(t-1)^{(t-1)} + c''(t-1)^{(t-1)} \right\} = \left\{ a'(t-1)^{(t-1)} + \left\{ \binom{a(t) + t - 1}{t-1} \right\}^{(t-1)} \right\}^{(t-1)}.$$ 

Thus, by (18) and (19), we have $a^{(n-1)} = b^{(n-2)} + c^{(n-1)}$.

**Case (B).** Assume $b < a_{(n-1)}$ and $c' \geq \binom{a(t)+t}{t}$.

Let $c'' = c' - \binom{a(t)+t}{t}$ and $\alpha = \max\{i \mid a(i) = a(t)\}$. Since $\sum_{j=t+1}^{\alpha} \binom{a(j)+j}{j} + \binom{a(t)+t}{t} = \frac{\binom{a(t)+\alpha}{\alpha}}{a(\alpha)}$ and $c'' < \binom{a(t)+t-1}{t-1} \leq \frac{\binom{a(t)+\alpha}{\alpha}}{a(\alpha-1)}$, we have
\[ c^{(n-1)} = \left\{ \sum_{j=0}^{n-1} \binom{a(j) + j - 1}{j} \right\}^{[+1]} + \binom{a(\alpha) + \alpha}{\alpha}^{[+1]} + c^n(\alpha-1) \]

Thus, by (19), we have

\[ b^{(n-2)} + c^{(n-1)} = \left\{ \sum_{j=0}^{n-1} \binom{a(j) + j}{j} \right\}^{[+1]} + b^{(t-1)} + c^n(\alpha-1). \] (22)

Equation \( a = b + c \) together with (15)–(17) implies \( a' = b' + c'' \). By Lemmas 1.4 and 1.7, we have

\[ a^{(t-1)} \leq b^{(t-1)} + c^{n(\alpha-1)} \leq b^{(t-1)} + c^n(\alpha-1). \] (23)

Hence, by (18) and (22), we have \( a^{(n-1)} \leq b^{(n-2)} + c^{(n-1)} \). It remains to show condition (iii). Since \( a^{(n-1)} = b^{(n-2)} + c^{(n-1)} \), (23) must be equal. Then Lemmas 1.4 and 1.7 say \( c' = 0 \) or \( b' = 0 \) and \( \alpha = t \). In both cases, we have \( a^{(t-1)} = b^{(t-1)} + c^{n(\alpha-1)} \). Hence we have \( a^{(n-1)} = b^{(n-2)} + c^{(n-1)} \) by (18) and (19).

By Cases (A) and (B), if \( |D_i(V)| < a_{(n-1)} \) for some \( i \), then condition (i), (ii) and (iii) are satisfied. Finally, we will prove that if \( |D_i(V)| = a_{(n-1)} \) for \( i = 1, 2, \ldots, n \), then \( V = M^d \) or \( V = \emptyset \).

Suppose \( |D_i(V)| = a_{(n-1)} \) for all \( i \). By Lemma 2.1, we have \( \overline{M}_i D_i(V) \subseteq M V \setminus x_i V \).

However, (1) says \( |\overline{M}_i D_i(V)| \geq b^{(n-2)} \). On the other hand, we have \( a_{(n-1)} = a^{(n-1)} - a = (MV \setminus x_i V) \) and \( b^{(n-2)} = a_{(n-1)} \). Thus we have \( \overline{M}_i D_i(V) = M V \setminus x_i V \) for all \( i \). We claim the following.

(\#) Assume \( |D_i(V)| = a_{(n-1)} \) for \( i = 1, 2, \ldots, n \). If there exists a monomial \( v \in M^d \) such that \( v \notin V \), then for any \( x_j \) and \( x_i \) with \( x_i | v \), one has \( \frac{x_j}{x_i} v \notin V \).

We will prove (\#). Suppose that \( v \notin V \) and there exist \( x_i \) and \( x_j \) such that \( \frac{x_j}{x_i} v \in V \). Since \( v \notin V \), we have \( x_j v \notin x_j V \). Thus we have \( x_j \frac{x_i}{x_j} v = x_j v \in M V \setminus x_j V = \overline{M}_j D_j(V) \). However, any element in \( \overline{M}_j D_j(V) \) does not contain \( x_j \) since \( \gcd(V) = 1 \). This is a contradiction.

Claim (\#) implies that if there exists a monomial \( v \in M^d \) such that \( v \in V \) then all monomials in \( M^d \) do not belong to \( V \). Hence we have \( V = M^d \) or \( V = \emptyset \). \( \square \)

We are now in a position to finish our combinatorial proof of the persistence theorem for monomial ideals.

**Proof of persistence theorem for monomial ideals.** What we have to prove is that if \( V \) is a Gotzmann set then \( M V \) is also a Gotzmann set.

Let \( V \) be a Gotzmann set of degree \( d \). We use induction on \( |V| \). Notice that, for any monomial \( u \in R, V \) is a Gotzmann set if and only if \( u V \) is a Gotzmann set since \( |V| = |u V| \) and \( |M V| = |u M V| \). Thus we may assume \( \gcd(V) = 1 \).

If \( V = M^d \) then \( M V \) is clearly a Gotzmann set. Also, if \( |V| = 1 \) then \( V = M^0 \).

Assume \( V \neq M^d \) and \( |V| > 1 \). Lemma 2.3(ii) says that there exists an integer \( i \in \{1, 2, \ldots, n\} \) such that \( \overline{M}_i K_i(V) \supset x_i D_i(V) \) and \( M^2 K_i(V) \supset \overline{M}_i^2 K_i(V) \supset x_i \overline{M}_i D_i(V) \). Thus \( |M V| = |M K_i(V)| + |\overline{M}_i D_i(V)| \) and \( |M^2 V| = |M^2 K_i(V)| + |\overline{M}_i^2 D_i(V)| \). By Lemma 2.3(i) and
the induction hypothesis, both $M_K V$ and $\overline{M}_D V$ are Gotzmann sets. Hence Lemma 2.3(iii) says
\[
|M^2 V| = |M^2 K V| + |\overline{M}_D V|
\]
\[
= \{|K V|^{(n-1)}|^{(n-1)} + \{|D V|^{(n-2)}|^{(n-2)}
\]
\[
= \{|V|^{(n-1)}|^{(n-1)}
\]
\[
= \{|MV|^{(n-1)}
\]
This completes the proof. □

Acknowledgement

The author is supported by JSPS Research Fellowships for Young Scientists.

References