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Frobenius maps☆

Yan Wang[a], Jin Ho Kwak[b]

[a] *Mathematics, Yan Tai University, Yan Tai 264005, PR China*
[b] *Mathematics, Pohang University of Science and Technology, Pohang 790-784, Korea*

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Abstract

A graph is called Frobenius if it is a connected orbital regular graph of a Frobenius group. A Frobenius map is a regular Cayley map whose underlying graph is Frobenius. In this paper, we show that almost all low-rank Frobenius graphs admit regular embeddings and enumerate non-isomorphic Frobenius maps for a given Frobenius graph. For some Frobenius groups, we classify all Frobenius maps derived from these groups. As a result, we construct some Frobenius maps with trivial exponent groups as a partial answer of a question raised by Nedela and Škoviera (Exponents of orientable maps, Proc. London Math. Soc. 75(3) (1997) 1–31).

Keywords: Frobenius group; Frobenius graph; Cayley map; Regular; Exponent

1. Introduction

A *map* on a surface is a cellular decomposition of a closed surface into 0-cells called *vertices*, 1-cells called *edges* and 2-cells called *faces*. The vertices and the edges of a map form its *underlying graph*. Every edge of a graph gives rise to a pair of opposite arcs (or darts). A map is *orientable* if the supporting surface is orientable, and *non-orientable* otherwise. Throughout this paper, we deal with only orientable maps. Typically, a map on a surface is constructed by a 2-cell embedding of a connected graph into a surface. Graphs considered in this paper are finite, connected, undirected and simple.

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E-mail addresses: yanwang@ytu.edu.cn (Y. Wang), jinkwak@postech.ac.kr (J.H. Kwak).

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In a combinatorial way, an *oriented* map $\mathscr{M}$ can be described as a triple $(D;\ R, L)$, where $D = D(\mathscr{M})$ is a non-empty finite set of *darts* which are incident vertex–edge pairs, and R and L are two permutations of D such that L is an involution and the generated group $\langle R, L\rangle$ acts transitively on D. The group $\langle R, L\rangle$ is called the oriented *monodromy group* of $\mathscr{M}$, and denoted by $\mathrm{Mon}(\mathscr{M})$. The permutations R and L are called the *rotation* and the *dart-reversing involution* of $\mathscr{M}$, respectively. The orbits of the group $\langle R\rangle$ are the *vertices* of $\mathscr{M}$, and the elements of an orbit v of $\langle R\rangle$ are the darts *emanating* from the vertex v, that is, v is their initial vertex. In the cyclic decomposition of the permutation R, the cycle permuting the darts emanating from a vertex v is called the *local rotation* R_v at v. The orbits of $\langle L\rangle$ and $\langle RL\rangle$ are the *edges* and the *faces* of $\mathscr{M}$, respectively. The incidence between vertices, edges and faces is given by a non-trivial set intersection. The vertices and the edges define the *underlying graph* of $\mathscr{M}$, which is always connected due to the transitive action of the monodromy group.

On the other hands, for a given underlying graph Γ, its 2-cell embedding into an orientable surface can be described by a *rotation* R which cyclically permutes the arcs initiated at each vertex in Γ, because the arc-reversing involution L is determined as a permutation interchanging oppositely directed arcs arising from the same edge.

Given a finite group G and a generating set S of G such that $S = S^{-1}$ and $1 \notin S$, the *Cayley graph* $\Gamma = \mathscr{C}(G, S)$ on G *relative to* S has vertex set G and edge set $\{\{g, gs\} \mid g \in G, s \in S\}$. For any cyclic permutation ρ on S, one can define the Cayley map $\mathscr{M} = \mathscr{CM}(G, S, \rho)$ to be a 2-cell embedding of Γ into an orientable surface, with the same local orientation induced by the permutation ρ at every vertex.

For a graph Γ, every edge of Γ gives rise to a pair of opposite arcs. Let $V(\Gamma)$, $E(\Gamma)$, $A(\Gamma)$ and $\mathrm{Aut}\,(\Gamma)$ denote the vertex set, the edge set, the arc set and the full automorphism group of Γ, respectively. A graph Γ is said to be *vertex-transitive*, *edge-transitive* or *arc-transitive* if $\mathrm{Aut}\,(\Gamma)$ acts transitively on $V(\Gamma)$, $E(\Gamma)$ or $A(\Gamma)$, respectively. It is well known that Cayley graphs are vertex-transitive with the regular G-action on the vertices by left multiplication.

Given two oriented maps $\mathscr{M} = (D;\ R, L)$ and $\mathscr{M}' = (D';\ R', L')$, a *map homomorphism* $\psi : \mathscr{M} \to \mathscr{M}'$ is a function $\psi : D \to D'$ such that

$$\psi R = R'\psi \quad \text{and} \quad \psi L = L'\psi.$$

Since graphs are assumed to be connected, a map homomorphism ψ is surjective. If it is also one-to-one, it is called an *isomorphism* of the maps. Furthermore, if $\mathscr{M} = \mathscr{M}'$, an isomorphism of the map is an *automorphism* of $\mathscr{M}$. The set of all automorphisms of $\mathscr{M}$ forms a group under composition, called the *automorphism group* of $\mathscr{M}$ and denoted by $\mathrm{Aut}\,(\mathscr{M})$. By the definition, the automorphism group $\mathrm{Aut}\,(\mathscr{M})$ is the centralizer of the monodromy group $\mathrm{Mon}(\mathscr{M})$ in the symmetry group S_D.

Since the monodromy group $\mathrm{Mon}(\mathscr{M})$ acts transitively and the automorphism group $\mathrm{Aut}\,(\mathscr{M})$ acts semiregularly on $D(\mathscr{M})$, we have $|\mathrm{Mon}\,(\mathscr{M})| \geqslant |D(\mathscr{M})| \geqslant |\mathrm{Aut}\,(\mathscr{M})|$ for any oriented map $\mathscr{M}$. It is well known that the first equality holds if and only if the second equality holds, so that if one equality holds then both groups $\mathrm{Mon}(\mathscr{M})$ and $\mathrm{Aut}\,(\mathscr{M})$ act regularly (i.e., transitively and semiregularly) on $D(\mathscr{M})$. In this case, the map $\mathscr{M}$ is said to be *regular*. The corresponding embedding of the underlying graph into a surface is also said to be *regular*. History of regular maps includes the discovery of Klein (1878) who described a 3-valent heptagonal regular map on the orientable surface of genus 3. In its early times, the study of regular maps was closely connected with group theory. One can see it in Coxeter and Moser's book [4, Chapter 8]. The present-time interest in regular maps extends to their connection to Dyck's triangle groups, Riemann surfaces, algebraic curves, Galois groups and other areas. Many of these links are surveyed in the recent paper of Jones [11]. One can also refer to McMullen and Schulte's book [12] for more information on regular maps. The classification problem of regular maps has been pursued along the following two main directions:

(1) Classifying regular maps by genus [3,4,6],
(2) Classifying regular maps by underlying graphs [10,13,14].

This paper is related to the second direction. Since a connected graph Γ having a regular embedding must be arc-transitive, a regular embedding of a Cayley graph has been of particular interest for more than a hundred years. Among the many articles devoted to Cayley maps, let us mention at least the following few and the references therein: [2,8,9,15,16]. A *Frobenius group* is a transitive permutation group G on a set V which is not regular on V, but has the property that the only element of G which fixes more than one point is the identity element. Throughout this paper, we assume that all groups considered are *finite*. It was shown by Thompson [17,18] that a finite Frobenius group G has a nilpotent normal subgroup K, called the *Frobenius kernel*, which acts regularly on V. Thus, K is the direct product of its Sylow subgroups and G is the semidirect product $K : H$, where H is the stabilizer of a point of V. A stabilizer H is not unique, but any two of them are conjugate

Table 1
Frobenius groups with non-p-group Frobenius kernels

H	K	$\lvert\mathrm{Aut}(\Gamma)\rvert$	$\lvert\mathrm{Aut}(\Gamma)_1\rvert$	H	K	$\lvert\mathrm{Aut}(\Gamma)\rvert$	$\lvert\mathrm{Aut}(\Gamma)_1\rvert$
$\mathbb{Z}_{18}$	$\mathbb{Z}_{19}\times\mathbb{Z}_{37}$	12654	18	$\mathbb{Z}_{15}$	$\mathbb{Z}_2^4\times\mathbb{Z}_{31}$	14880	30
$\mathbb{Z}_{12}$	$\mathbb{Z}_{13}\times\mathbb{Z}_5^2$	3900	12	$\mathbb{Z}_{12}$	$\mathbb{Z}_{13}\times\mathbb{Z}_{37}$	5772	12
$\mathbb{Z}_{10}$	$\mathbb{Z}_{11}\times\mathbb{Z}_{31}$	3410	10	$\mathbb{Z}_{10}$	$\mathbb{Z}_{11}\times\mathbb{Z}_{41}$	4510	10
$\mathbb{Z}_8$	$\mathbb{Z}_3^2\times\mathbb{Z}_{17}$	1224	8	$^*\mathbb{Z}_8$	$\mathbb{Z}_3^2\times\mathbb{Z}_5^2$	1800	8
$\mathbb{Z}_8$	$\mathbb{Z}_3^2\times\mathbb{Z}_{41}$	2952	8	$\mathbb{Z}_7$	$\mathbb{Z}_2^3\times\mathbb{Z}_{29}$	3248	14
$\mathbb{Z}_7$	$\mathbb{Z}_2^3\times\mathbb{Z}_{43}$	4816	14	$\mathbb{Z}_6$	$\mathbb{Z}_7\times\mathbb{Z}_{13}$	546	6
$\mathbb{Z}_6$	$\mathbb{Z}_7\times\mathbb{Z}_{19}$	798	6	$\mathbb{Z}_6$	$\mathbb{Z}_7\times\mathbb{Z}_5^2$	1050	6
$\mathbb{Z}_6$	$\mathbb{Z}_7\times\mathbb{Z}_{31}$	1302	6	$\mathbb{Z}_6$	$\mathbb{Z}_7\times\mathbb{Z}_{37}$	1554	6
$\mathbb{Z}_6$	$\mathbb{Z}_{13}\times\mathbb{Z}_{19}$	1482	6	$\mathbb{Z}_5$	$\mathbb{Z}_{11}\times\mathbb{Z}_2^4$	1760	10
$\mathbb{Z}_4$	$\mathbb{Z}_5\times\mathbb{Z}_3^2$	180	4	$\mathbb{Z}_4$	$\mathbb{Z}_5\times\mathbb{Z}_{13}$	260	4
$\mathbb{Z}_4$	$\mathbb{Z}_5\times\mathbb{Z}_{17}$	340	4	$\mathbb{Z}_4$	$\mathbb{Z}_5\times\mathbb{Z}_{29}$	580	4
$\mathbb{Z}_4$	$\mathbb{Z}_5\times\mathbb{Z}_{37}$	740	4	$\mathbb{Z}_4$	$\mathbb{Z}_3^2\times\mathbb{Z}_{13}$	468	4
$\mathbb{Z}_4$	$\mathbb{Z}_3^2\times\mathbb{Z}_{17}$	612	4	$\mathbb{Z}_3$	$\mathbb{Z}_2^2\times\mathbb{Z}_7$	168	6
$\mathbb{Z}_3$	$\mathbb{Z}_2^2\times\mathbb{Z}_{13}$	312	6	$\mathbb{Z}_3$	$\mathbb{Z}_2^2\times\mathbb{Z}_{19}$	456	6
$^*\mathbb{Z}_3$	$\mathbb{Z}_2^2\times\mathbb{Z}_5^2$	1200	12	$\mathbb{Z}_3$	$\mathbb{Z}_2^2\times\mathbb{Z}_{31}$	744	6
$\mathbb{Z}_3$	$\mathbb{Z}_2^2\times\mathbb{Z}_{37}$	888	6	$\mathbb{Z}_3$	$\mathbb{Z}_7\times\mathbb{Z}_{13}$	546	6
$\mathbb{Z}_3$	$\mathbb{Z}_7\times\mathbb{Z}_4^2$	672	6	$\mathbb{Z}_3$	$\mathbb{Z}_7\times\mathbb{Z}_{19}$	798	6

Table 2
Frobenius groups with p-group Frobenius kernels

H	K	$\lvert\mathrm{Aut}\,(\Gamma)\rvert$	$\lvert\mathrm{Aut}\,(\Gamma)_1\rvert$	H	K	$\lvert\mathrm{Aut}\,(\Gamma)\rvert$	$\lvert\mathrm{Aut}\,(\Gamma)_1\rvert$
$\mathbb{Z}_{46}$	$\mathbb{Z}_{47^2}$	101614	46	$\mathbb{Z}_{42}$	$\mathbb{Z}_{43^2}$	77658	42
$\mathbb{Z}_{40}$	$\mathbb{Z}_{41^2}$	67240	40	$\mathbb{Z}_{36}$	$\mathbb{Z}_{37^2}$	49284	36
$\mathbb{Z}_{30}$	$\mathbb{Z}_{31^2}$	28830	30	$\mathbb{Z}_{28}$	$\mathbb{Z}_{29^2}$	23548	28
$\mathbb{Z}_{22}$	$\mathbb{Z}_{23^2}$	11638	22	$\mathbb{Z}_{18}$	$\mathbb{Z}_{19^2}$	6498	18
$\mathbb{Z}_{16}$	$\mathbb{Z}_{17^2}$	4624	16	$\mathbb{Z}_{12}$	$\mathbb{Z}_{13^2}$	2028	12
$\mathbb{Z}_{11}$	$\mathbb{Z}_{23^2}$	11638	22	$\mathbb{Z}_{10}$	$\mathbb{Z}_{11^2}$	1210	10
$\mathbb{Z}_{9}$	$\mathbb{Z}_{19^2}$	6498	18	$\mathbb{Z}_{8}$	$\mathbb{Z}_{17^2}$	2312	8
$\mathbb{Z}_{6}$	$\mathbb{Z}_{7^2}$	294	6	$\mathbb{Z}_{6}$	$\mathbb{Z}_{13^2}$	1014	6
$\mathbb{Z}_{5}$	$\mathbb{Z}_{11^2}$	1210	10	$\mathbb{Z}_{4}$	$\mathbb{Z}_{25}$	100	4
$\mathbb{Z}_{4}$	$\mathbb{Z}_{13^2}$	676	4	$\mathbb{Z}_{3}$	$\mathbb{Z}_{7^2}$	294	6

Table 3
Frobenius groups of rank $\leqslant 50$ which need further investigation

$\lvert H\rvert$	H	K	$\lvert H\rvert$	H	K
31	$\mathbb{Z}_{31}$	$K/\mathbb{Z}_2^5 \cong \mathbb{Z}_2^5$	26	$\mathbb{Z}_{26}$	$\mathbb{Z}_9^3$
24	?	$\mathbb{Z}_{25}^2$	15	?	$K/\mathbb{Z}_2^4 \cong \mathbb{Z}_2^4$
8	?	$\mathbb{Z}_9^2$	7	$\mathbb{Z}_7$	$K/\mathbb{Z}_2^3 \cong \mathbb{Z}_2^3$
3	$\mathbb{Z}_3$	$K/\mathbb{Z}_2^2 \cong \mathbb{Z}_2^2$			

because of the vertex transitivity of the action. Such a subgroup H is called a *Frobenius complement* of K in G. Gorenstein ([7, pp. 38, 339]) showed that every element of $H\backslash\{1\}$ induces an automorphism of K by conjugation which fixes only the identity element of K. For a group-theoretic terminology not defined in this paper, we refer the reader to [7,20]. The *rank* of the Frobenius group G, denoted by $r(G)$, is the number of orbits of H in K, that is $r(G) = 1 + (|K| - 1)/|H|$. When the Frobenius kernel is not elementary abelian (that is, it is not isomorphic to $\mathbb{Z}_p^m$ for any $m \geqslant 1$ and any prime p), Wang et al. [19] classified all the Frobenius groups $G = K : H$ with $6 \leqslant r(G) \leqslant 50$, which are listed in Tables 1–3. For a classification of Frobenius groups of $r(G) \leqslant 5$, we refer the reader to [5]. Fang et al. [5] introduced a Frobenius graph as an orbital regular graph of a Frobenius group and showed that a Frobenius graph is a Cayley graph of the Frobenius kernel. InTables 1 and 2, for each Frobenius group $G = K : H$, we use Γ to denote a Frobenius graph derived from G. And, by $\mathrm{Aut}\,(\Gamma)_1$, we denote the point stabilizer of the identity.

This paper is organized as follows. In Section 2, we discuss some properties of Frobenius graphs and their embeddings into orientable surfaces as Cayley maps. In Section 3, we prove that almost all low-rank Frobenius graphs admit regular embeddings, called *Frobenius maps* in this paper. And we classify the Frobenius maps up to isomorphism. For the results, see Theorems 3.7 and 3.11. Using these results, in Section 5, we construct Frobenius maps with trivial exponent groups, see Theorems 5.1 and 5.2, as a partial answer of a question raised by Nedela and Škoviera in [14]. In Section 4, for some Frobenius groups, we classify all the Frobenius maps derived from these groups, see Theorems 4.2, 4.3 and Corollary 4.4.

2. Properties of Frobenius graphs

Given a permutation group G on a set V, the G-action on V induces a natural action on $V \times V$ by $(x, y)^g = (x^g, y^g)$ for $(x, y) \in V \times V$ and $g \in G$. The orbits of G in the action on $V \times V$ are called *orbitals*. Note that the set $\Delta = \{(x, x) | x \in V\}$ is G-invariant as well as the set $\Delta^c = \{(x, y) | x, y \in V, x \neq y\}$. A G-orbit in Δ is called a *trivial orbital* and that in Δ^c is called a *non-trivial orbital*. Let Γ be a connected graph with vertex set V, and let $G \leqslant \mathrm{Aut}(\Gamma)$. Then Γ is said to be a *G-orbital regular graph* if G is regular on each of its orbitals in Δ^c, and there is a non-trivial G-orbital O such that the edge set is $E(\Gamma) = \{\{x, y\} | (x, y) \in O\}$. A graph Γ is *orbital regular* if it is G-orbital regular for some $G \leqslant \mathrm{Aut}(\Gamma)$.

Fang et al. [5] introduced a Frobenius graph as follows:

Definition 2.1. Let G be a Frobenius group on a set V. A G-Frobenius graph is defined to be a connected graph Γ with vertex set $V(\Gamma) = V$ and edge set $E(\Gamma) = \{\{x, y\} | (x, y) \in O\}$ for some non-trivial G-orbital O in Δ^c.

Let $G = K : H$ be a Frobenius group on a set V and let Γ be a G-Frobenius graph. Since K is regular on the vertex set V of Γ, we may identify V with K in such a way that K acts by the left multiplication.

Example 2.1. For any prime number p, the group $G = \mathbb{Z}_p : \mathbb{Z}_{p-1}$ is a Frobenius group, where $K = \mathbb{Z}_p$ and $H = \mathbb{Z}_{p-1}$. Here, the group G acts on K in such a way that K acts on itself by translation and H acts on K by multiplication. Thus, G acts regularly on $(K \times K)^c$ and the G-Frobenius graph is isomorphic to the complete graph K_p.

Clearly, every Frobenius graph is orbital regular. Fang et al. showed that almost all orbital regular graphs are Frobenius.

Lemma 2.1 (*Fang et al.* [5]). *Let Γ be a graph with n vertices and let $G \leqslant \mathrm{Aut}(\Gamma)$. Then Γ is G-orbital regular if and only if one of the following holds*:

(1) *Γ is a G-Frobenius graph or,*
(2) *$\Gamma = C_n$, a cycle of length n, and $G = \mathbb{Z}_n$ for $n \geqslant 3$ or,*
(3) *$\Gamma = K_{1,n-1}$, a bipartite graph and $G = \mathbb{Z}_{n-1}$ for $n \geqslant 3$.*

Let $\mathscr{C}(G, S)$ be a Cayley graph. By an ordered pair $(g, x) \in G \times S$, we denote an arc with initial vertex g and terminal vertex gx, and say that the arc (g, x) has *color x*. Clearly, the number of arcs in a Cayley graph $\mathscr{C}(G, S)$ is equal to $|G| \cdot |S|$. For elements x, y in a

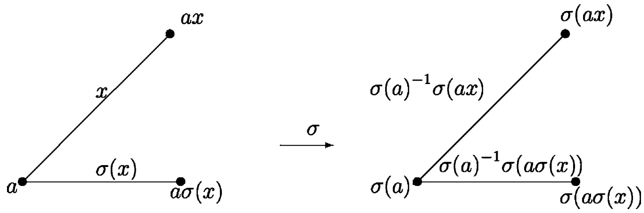

Fig. 1. The automorphism σ preserving the color of arcs.

group G, we shall write x^y to denote the conjugate $y^{-1}xy$ and ${}^y x$ to denote yxy^{-1}. If H is a subgroup of G and $x \in G$, then $x^H = \{x^h | h \in H\}$.

Let $G = K : H$ be a Frobenius group with Frobenius kernel K and Frobenius complement H. The next lemma shows that all Frobenius graphs are Cayley graphs.

Lemma 2.2 (*Fang et al.* [5, Theorem 1.4]). *Let $G = K : H$ be a Frobenius group with Frobenius kernel K and Frobenius complement H. Then a G-Frobenius graph is a Cayley graph $\mathscr{C}(K, S)$ for K and for some generating subset S of the form*

$$S = \begin{cases} x^H & \text{if } |H| \text{ is even or } |x| = 2, \\ x^H \cup (x^{-1})^H & \text{if } |H| \text{ is odd and } |x| \neq 2, \end{cases} \tag{1}$$

where $x \in K$ such that $\langle x^H \rangle = K$. Conversely, if $x \in K$ satisfies $\langle x^H \rangle = K$, then $\mathscr{C}(K, S)$ is G-Frobenius with S defined in Eq. (1).

From now on, whenever we say $S = x^H \cup (x^{-1})^H$, it is assumed that $x^H \neq (x^{-1})^H$.

Jajcay characterized the Cayley graphs admitting regular Cayley maps.

Lemma 2.3 (*Jajcay* [9, Theorem 2]). *Let $\Gamma = \mathscr{C}(G, S)$ be a Cayley graph. Suppose that there exists a graph automorphism σ of Γ fixing* 1, *acting cyclically on S and satisfying*

$$\sigma(\sigma(a)^{-1}\sigma(ax)) = \sigma(a)^{-1}\sigma(a\sigma(x)) \tag{2}$$

for any $a \in G$ and $x \in S$. Then, Γ admits a regular Cayley map $\mathscr{CM}(G, S, \sigma|_S)$. Conversely, if Γ admits a regular Cayley map, then such a graph automorphism σ of Γ exists.

One can see that the graph automorphism σ in Lemma 2.3 satisfies Eq. (2) if and only if σ preserves the color of arcs, as shown in Fig. 1.

3. Regular embeddings of Frobenius graphs

Let $G = K : H$ be a Frobenius group and let $\Gamma = \mathscr{C}(K, S)$ be a Frobenius graph with S defined in Eq. (1). In this section, we determine some conditions under which the Frobenius graph Γ can be regularly embedded into an orientable surface and classify all of such embeddings up to isomorphism.

Lemma 3.1. *Let $G = K : H$ be a Frobenius group and let $\Gamma = \mathscr{C}(K, S)$ be a Frobenius graph with S defined in Eq.* (1). *Then, H is isomorphic to a subgroup, also denoted by H, of* Aut (Γ). *Moreover,*
(1) *if $S = x^H$, H acts regularly on the arcs emanating from the identity*;
(2) *if $S = x^H \cup (x^{-1})^H$, H acts semi-regularly on the arcs emanating from the identity*.

Proof. Because $G = K : H$ is a Frobenius group, H is a subgroup of Aut (K). In fact, H acts semi-regularly on $K\backslash\{1\}$ by conjugation. For each $h \in H$, according to the definition of S, one can naturally extend the action of h on K to the action of h on Γ as follows: for any arc $(k, y) \in D(\Gamma) = K \times S$, $h(k, y) = (k^h, y^h)$. The action preserves the adjacency of the graph Γ, so that $h \in$ Aut (Γ). It means that H can be considered as a subgroup of Aut (Γ). For any $h \in H$, the only element in K that is fixed under the action of h is the identity element. Thus, if $S = x^H$, H acts regularly on the arcs emanating from the identity, and if $S = x^H \cup (x^{-1})^H$, H acts semi-regularly on the same set. □

Let $G = K : H$ be a Frobenius group and let $\Gamma = \mathscr{C}(K, S)$ be a Frobenius graph with $S = x^H \cup (x^{-1})^H$, so that K is assumed to be abelian. Define $\beta : G \to G$ as follows:

(i) For any $h \in H$, $\beta(h) = h$;
(ii) For any $k \in K$, $\beta(k) = k^{-1}$;
(iii) For any $g=kh \in G$, let $\beta(g) = \beta(k)\beta(h) = k^{-1}h$. (3)

It is easy to see that β is a bijection on the group G, $\beta(H) = H$, $\beta(K) = K$ and $\beta(1) = 1$. For any $g_1, g_2 \in G$, let $g_1 = k_1h_1$ and $g_2 = k_2h_2$. Then $g_1g_2 = k_1({}^{h_1}k_2)h_1h_2$. A direct calculation shows that $\beta(g_1g_2) = k_1^{-1}({}^{h_1}(k_2^{-1}))h_1h_2 = \beta(g_1)\beta(g_2)$. So, $\beta \in$ Aut (G).

Moreover, the automorphism $\beta \in$ Aut (G) can be extended to an automorphism of the Frobenius graph Γ as follows: (use the same notation β for an extended automorphism for notational convenience). For any arc $(k, y) \in D(\Gamma) = K \times S$, define $\beta(k, y) = (\beta(k), \beta(y)) = (k^{-1}, y^{-1})$. Clearly, β preserves the adjacency relation of Γ. Since Γ is assumed to be a simple graph without semiedges, loops or multiple edges, $\beta \in$ Aut (Γ).

Let $A =$ Aut (Γ) to simplify a notation and let A_1 denote the stabilizer of the identity element 1.

Lemma 3.2. *The automorphism $\beta \in A$ satisfies the following properties*:
(1) *the order of β is $|\beta| = 2$*;
(2) *$\beta(1, y) = (1, y^{-1})$ for any $y \in S$*;
(3) *$\beta \cdot h = h \cdot \beta$ for any $h \in H$*;
(4) *if $|A| = 2|K||H|$, then $A \cong K : (H \times \langle\beta\rangle)$ and $A_1 \cong H \times \langle\beta\rangle$*;
(5) *$H \times \langle\beta\rangle$ acts transitively on the arcs emanating from the identity.*

Proof. (1) and (2) follow immediately from the definition of β.

(3) For any arc (k, y), $\beta \cdot h(k, y) = \beta(k^h, y^h) = ((k^{-1})^h, (y^{-1})^h)$ and $h \cdot \beta(k, y) = h(k^{-1}, y^{-1}) = ((k^{-1})^h, (y^{-1})^h)$.

(4) For any $k_1, k_2 \in K$, $\beta k_2 \beta(k_1, y) = \beta k_2(k_1^{-1}, y^{-1}) = \beta(k_2k_1^{-1}, y^{-1}) = (k_2^{-1}k_1, y) = k_2^{-1}(k_1, y)$. So, $\langle\beta\rangle \leqslant N_A(K)$. According to (2), (3) and $|A| = 2|K||H|$, we have $A \cong K : (H \times \langle\beta\rangle)$ and $A_1 \cong H \times \langle\beta\rangle$.

(5) Given two arcs $(1, x^{h_i})$ and $(1, (x^{-1})^{h_j})$, we have $(h_i^{-1}h_j)\cdot\beta(1, x^{h_i})=(1, (x^{-1})^{h_j})$. Since H acts transitively on x^H and $(x^{-1})^H$, respectively, one can get the conclusion. □

Theorem 3.3. *Let $G = K : H$ be a Frobenius group and let $\Gamma = \mathscr{C}(K, S)$ be a Frobenius graph with S defined in Eq.* (1). *When $S = x^H \cup (x^{-1})^H$, we assumed that K is abelian. Then, Γ is arc-transitive*.

Proof. Since the Frobenius graph $\Gamma = \mathscr{C}(K, S)$ is vertex-transitive, we only need to show that the point stabilizer A_1 acts transitively on the arcs emanating from the identity element 1. If $S = x^H$, by Lemma 3.1, $H \leqslant A_1$ and it acts regularly on the arcs emanating from 1; if $S = x^H \cup (x^{-1})^H$, by Lemma 3.2, $H \times \langle\beta\rangle \leqslant A_1$ which acts regularly on the same arc set. □

In the following, we divide our discussions into two cases, according to the choices of S in Eq. (1).

Case 1. $|H|$ is even or $|x| = 2$. In this case, $S = x^H$, $x \in K$ and $\langle x^H\rangle = K$.

Theorem 3.4. *Let $G = K : H$ be a Frobenius group and let $\Gamma = \mathscr{C}(K, S)$ be a Frobenius graph with $S = x^H$. If $H \cong \langle h\rangle$ is cyclic and* Aut $(\Gamma) \cong K : H$, *then Γ admits a regular Cayley map $\mathscr{CM}(K, S, \rho)$ if and only if $\rho = h^t|_S = (x\ x^{h^t} \cdots x^{h^{(m-1)t}})$ for some integer t with $(t, m) = 1$, where $m = |H|$.*

Proof. According to Lemma 2.3, Γ admits a regular Cayley map if and only if there exists a graph automorphism σ fixing the identity element 1 of K, acting cyclically on S and satisfying the condition: $\sigma(\sigma(k)^{-1}\sigma(ky)) = \sigma(k)^{-1}\sigma(k\sigma(y))$ for any $k \in K$ and $y \in S$. From the condition Aut $(\Gamma) \cong K : H$, if such a graph automorphism σ exists, it belongs to H. Let $\sigma = h^t$. Then, one of the orbits of σ acting on S is $\{x, x^{h^t}, \ldots, x^{h^{(m-1)t}}\}$. Therefore, σ acts cyclically on S if and only if $(t, m)=1$. A direct calculation shows that $\sigma(\sigma(k)^{-1}\sigma(ky))= \sigma(k)^{-1}\sigma(k\sigma(y)) = y^{h^{2t}}$. Let $\rho = h^t|_S$, then $\mathscr{CM}(K, S, \rho)$ is a regular Cayley map. □

A Cayley map $\mathscr{CM}(G, S, \rho)$ is *balanced* if $\rho(x^{-1}) = \rho(x)^{-1}$ for every $x \in S$, and *antibalanced* if $\rho(x^{-1}) = (\rho^{-1}(x))^{-1}$ for every $x \in S$. Škoviera and Širáň [16] showed that a Cayley map $\mathscr{CM}(G, S, \rho)$ is regular and balanced if and only if there exists a group automorphism $\alpha : G \to G$ such that $\alpha|_S = \rho$.

Corollary 3.5. *The Cayley maps $\mathscr{CM}(K, S, \rho)$ given in Theorem* 3.4 *are balanced*.

Proof. Because the graph automorphism σ mentioned in the proof of Theorem 3.4 is a group automorphism, the Cayley maps given in Theorem 3.4 are all balanced. □

The *genus* of a map $\mathscr{M}$ is defined as the genus of its supporting surface.

Corollary 3.6. *If K is abelian and $|H|$ is a multiple of* 4, *then the genus of the Cayley map $\mathscr{CM}(K, S, \rho)$ given in Theorem* 3.4 *is* $g = \frac{1}{4}(4 - 4|K| + |K||H|)$.

Proof. According to Theorem 3.4, Corollary 3.5 and the assumption of $|H|$, one can assume $|H| = 4n$ for some integer n and $\rho = (x_1\ x_2 \ldots x_{2n}\ x_1^{-1}\ x_2^{-1} \ldots x_{2n}^{-1})$. Take an arc $(1, x_1)$ and consider the orbit of *RL* including $(1, x_1)$. Because K is abelian, the orbit has the following $|H|$ arcs:

$$(1, x_1), (x_2, x_2^{-1}), (x_2 x_3^{-1}, x_3), \ldots, (x_2 x_3^{-1} \cdots x_{2n}, x_{2n}^{-1}),$$
$$(x_2 x_3^{-1} \cdots x_{2n} x_1, x_1^{-1}),\ (x_2 x_3^{-1} \cdots x_{2n} x_1 x_2^{-1}, x_2), \ldots,$$
$$(x_2 \cdots x_{2n} x_1 x_2^{-1} \cdots x_{2n}^{-1}, x_{2n}).$$

Hence, the 2-cells in the Cayley map $\mathscr{CM}(K, S, \rho)$ are all topological $|H|$-gons. Let $|V|, |E|, |F|$ denote the number of vertices, edges and faces of the map, respectively. Then, in case of the Cayley maps given in Theorem 3.4, $|V| = |K|$, $|E| = \frac{1}{2}(|K||H|)$ and $|F||H| = |K||H|$, that is $|F| = |K|$. From the Euler–Poincaré characteristic, $2 - 2g = |V| - |E| + |F|$, one can get $g = \frac{1}{4}(4 - 4|K| + |K||H|)$. □

Definition 3.1. Let $G = K : H$ be a Frobenius group and let $\mathscr{C}(K, S)$ be a Frobenius graph. If a Cayley map $\mathscr{CM}(K, S, \rho)$ is regular, it is called a *Frobenius map*.

Example 3.1. Consider the Frobenius group $G = \mathbb{Z}_5 : \mathbb{Z}_4$, where $K = \mathbb{Z}_5$ and $H = \mathbb{Z}_4$. Take $S = \{\pm 1, \pm 2\}$ and let $\rho = (1\# - 2\# - 1\#2)$, then $\mathscr{CM}(K, S, \rho)$ is a Frobenius map. In fact, it is a regular embedding of the complete graph K_5 into the torus.

Recall that the Euler's totient function $\phi : \mathbb{N} \to \mathbb{N}$ is defined as $\phi(1) = 1$ and for $n \geqslant 2$, $\phi(n)$ is the number of positive integers less than n and relatively prime to n.

Theorem 3.7. *Let $G = K : H$ be a Frobenius group with $K = P_1 \times \cdots \times P_t$, where P_i are the distinct Sylow subgroups of K, and let $H \cong \langle h \rangle$ be cyclic, say $m = |H|$. Let $\Gamma = \mathscr{C}(K, S)$ be a Frobenius graph with $S = x^H$ for some $x \in K$ such that $\langle x^H \rangle = K$. If* $\mathrm{Aut}\,(\Gamma) \cong K : H$ *and there exists at least one Sylow subgroup, say P_1, such that* $\mathrm{Aut}\,(P_1)$ *is abelian, then Γ admits $\phi(m)$ non-isomorphic Frobenius maps.*

Proof. By Theorem 3.4, a Cayley map $\mathscr{CM}(K, S, \rho)$ is regular if and only if $\rho = (x\ x^{h^t} \cdots x^{h^{(m-1)t}})$ for some integer t with $(t, m) = 1$. Therefore, to prove the theorem, it is sufficient to show that for any two distinct integers $1 \leqslant t_1 \neq t_2 \leqslant m$, with $(t_i, m) = 1$, $i = 1, 2$, the Cayley maps $\mathscr{CM}(K, S, \rho_1)$ and $\mathscr{CM}(K, S, \rho_2)$ are not isomorphic, where

$$\rho_1 = (x\ \ x^{h^{t_1}} \cdots x^{h^{(m-1)t_1}}) \quad \text{and} \quad \rho_2 = (x\ \ x^{h^{t_2}} \cdots x^{h^{(m-1)t_2}}).$$

Suppose that $\mathscr{CM}(K, S, \rho_1) \cong \mathscr{CM}(K, S, \rho_2)$. Because $\mathscr{CM}(K, S, \rho_1)$ and $\mathscr{CM}(K, S, \rho_2)$ are balanced regular Cayley maps, by the results in [15, Corollary 5.3], there exists a group automorphism $\alpha \in \mathrm{Aut}\,(K)$ such that $\alpha(x^{h^{rt_1}}) = x^{h^{(r+j)t_2}}$ for any $0 \leqslant r \leqslant m - 1$ and a fixed j with $0 \leqslant j \leqslant m - 1$.

Since $K = P_1 \times \cdots \times P_t$ and $\mathrm{Aut}\,(K) = \mathrm{Aut}\,(P_1) \times \cdots \times \mathrm{Aut}\,(P_t)$, one may assume that $x = x_1 \ldots x_t$ with $x_i \in P_i$, $\langle x_i^H \rangle = P_i$ and $\alpha = \alpha_1 \ldots \alpha_t$ with $\alpha_i \in \mathrm{Aut}\,(P_i)$. Therefore, from $\alpha(x^{h^{rt_1}}) = x^{h^{(r+j)t_2}}$, we get $\alpha_i(x_i^{h^{rt_1}}) = x_i^{h^{(r+j)t_2}}$ for each $1 \leqslant i \leqslant t$. When $i = 1$, it follows that $\alpha_1(x_1^{h^{rt_1}}) = x_1^{h^{(r+j)t_2}}$. On the other hand, by the assumption that Aut (P_1) is abelian, we have $\alpha_1(x_1^{h^{rt_1}}) = (\alpha_1(x_1))^{h^{rt_1}} = x_1^{h^{rt_1+jt_2}}$. Therefore, $x_1^{h^{rt_1+jt_2}} = x_1^{h^{(r+j)t_2}}$. It means that $m | r(t_1 - t_2)$ for any $0 \leqslant r \leqslant m-1$, we get $t_1 = t_2$. This completes the proof. □

Case 2. $|H|$ is odd and $|x| \neq 2$. In this case, $S = x^H \cup (x^{-1})^H$ for some $x \in K$ and $\langle x^H \rangle = K$. In this case, K is assumed to be abelian. From the definition of β given in Eq. (3), it is easy to see that $\beta \in \text{Aut}(K)$.

Lemma 3.8. *Let $G = K : H$ be a Frobenius group and $\Gamma = \mathscr{C}(K, S)$ be a Frobenius graph with $S = x^H \cup (x^{-1})^H$. If K is abelian and $H \cong \langle h \rangle$ is cyclic, say $|H| = m$, then $h^i \beta^j$ is a cyclic permutation of S if and only if $(i, m) = 1$ and $j = 1$.*

Proof. If $j = 0$, clearly h^i is not a cyclic permutation of S for any $1 \leqslant i \leqslant m$; if $j = 1$, because m is odd, one of the orbits under the action of $h^i \beta$ on S is

$$\{x, (x^{-1})^{h^i}, x^{h^{2i}}, \ldots, x^{h^{(m-1)i}}, x^{-1}, \ldots, (x^{-1})^{h^{(m-1)i}}\}.$$

Therefore, $h^i \beta$ is a cyclic permutation on S if and only if $x^{h^{ti}} \neq x^{h^{ri}}$ whenever $t \neq r$, or equivalently, $(i, m) = 1$. □

Theorem 3.9. *Let $G = K : H$ be a Frobenius group and let $\Gamma = \mathscr{C}(K, S)$ be a Frobenius graph with $S = x^H \cup (x^{-1})^H$. If K is abelian, $H \cong \langle h \rangle$ is a cyclic group and* Aut $(\Gamma) \cong K : (H \times \langle \beta \rangle)$ *with the β given in Eq.* (3), *then Γ admits a regular Cayley map $\mathscr{CM}(K, S, \rho)$ if and only if $\rho = h^i \beta|_S$ for some integer i with $(i, |H|) = 1$.*

Proof. By Lemma 2.3, Γ admits a regular Cayley map if and only if there exists a function $\sigma \in \text{Aut}(\Gamma)$ fixing the identity element 1, acting cyclically on S and satisfying the condition: $\sigma(\sigma(k)^{-1}\sigma(ky)) = \sigma(k)^{-1}\sigma(k\sigma(y))$ for any $k \in K$ and $y \in S$. Because Aut $(\Gamma) \cong K : (H \times \langle \beta \rangle)$, if such a function σ exists, it must belong to $H \times \langle \beta \rangle$. From Lemma 3.8, one can assume that $\sigma = h^i \beta$ for some integer i, with $(i, |H|) = 1$. For any $k \in K$, $y \in S$, a direct calculation shows that $\sigma(\sigma(k)^{-1}\sigma(ky)) = y^{h^{2i}} = \sigma(k)^{-1}\sigma(k\sigma(y))$. Let $\rho = h^i \beta|_S$, then $\mathscr{CM}(K, S, \rho)$ is a regular Cayley map. □

By a method similar to Corollaries 3.5 and 3.6, one can get the following one.

Corollary 3.10. *The Frobenius maps $\mathscr{CM}(K, S, \rho)$ given in Theorem* 3.9 *are all balanced. Moreover, if $|H|$ is even, their genera are $g = \frac{1}{2}(2 - 2|K| + |K||H|)$.*

Theorem 3.11. *Let $G = K : H$ be a Frobenius group with an abelian Frobenius kernel $K = P_1 \times \cdots \times P_t$, where P_i are the distinct Sylow subgroups of K, and $H \cong \langle h \rangle$, $|H| = m$, is a cyclic group. Let $\Gamma = \mathscr{C}(K, S)$ be a Frobenius graph with $S = x^H \cup (x^{-1})^H$ for some $x \in K$ such that $\langle x^H \rangle = K$. If* Aut $(\Gamma) \cong K : (H \times \langle \beta \rangle)$ *with the β given in Eq.* (3) *and there exists at least one Sylow subgroup, say P_1, such that* Aut (P_1) *is abelian, then Γ admits $\phi(m)$ non-isomorphic Frobenius maps.*

Proof. By Theorem 3.9, a Cayley map $\mathscr{CM}(K, S, \rho)$ is regular if and only if $\rho=(x\ (x^{-1})^{h^t} x^{h^{2t}} \cdots x^{h^{(m-1)t}} x^{-1} \cdots (x^{-1})^{h^{(m-1)t}})$ for some integer t with $(t, m)=1$. Therefore, to prove the theorem, it suffices to show that for any two distinct integers $1 \leqslant t_1 \neq t_2 \leqslant m$, with $(t_1, m)=1$ and $(t_2, m)=1$, the Cayley maps $\mathscr{CM}(K, S, \rho_1)$ and $\mathscr{CM}(K, S, \rho_2)$ are not isomorphic, where

$$\rho_1 = (x\ (x^{-1})^{h^{t_1}} x^{h^{2t_1}} \cdots x^{h^{(m-1)t_1}} x^{-1} \cdots (x^{-1})^{h^{(m-1)t_1}})$$

and

$$\rho_2 = (x\ (x^{-1})^{h^{t_2}} x^{h^{2t_2}} \cdots x^{h^{(m-1)t_2}} x^{-1} \cdots (x^{-1})^{h^{(m-1)t_2}}).$$

Suppose that $\mathscr{CM}(K, S, \rho_1) \cong \mathscr{CM}(K, S, \rho_2)$ for some integers $1 \leqslant t_1, t_2 \leqslant m$ with $(t_1, m)=1$ and $(t_2, m) = 1$. Since $\mathscr{CM}(K, S, \rho_1)$ and $\mathscr{CM}(K, S, \rho_2)$ are balanced regular Cayley maps, there exists a group automorphism $\alpha \in \mathrm{Aut}\,(K)$ such that $\alpha((x^{\varepsilon})^{h^{rt_1}}) = (x^{\varepsilon'})^{h^{(r+j)t_2}}$ for any $0 \leqslant r \leqslant m-1$ and a fixed j, $0 \leqslant j \leqslant m-1$, where $\varepsilon, \varepsilon' = \pm 1$ and $\varepsilon' = (-1)^j \varepsilon$. Since $K = P_1 \times \cdots \times P_t$ and $\mathrm{Aut}\,(K) = \mathrm{Aut}\,(P_1) \times \cdots \times \mathrm{Aut}\,(P_t)$, one may assume that $x = x_1 \cdots x_t$ with $x_i \in P_i$, $\langle x_i^H \rangle = P_i$ and $\alpha = \alpha_1 \cdots \alpha_t$ with $\alpha_i \in \mathrm{Aut}\,(P_i)$. Therefore, from $\alpha((x^{\varepsilon})^{h^{rt_1}}) = (x^{\varepsilon'})^{h^{(r+j)t_2}}$, we get $\alpha_i((x_i^{\varepsilon})^{h^{rt_1}}) = (x_i^{\varepsilon'})^{h^{(r+j)t_2}}$ for each $1 \leqslant i \leqslant t$.When $i = 1$, it follows that $\alpha_1((x_1^{\varepsilon})^{h^{rt_1}}) = (x_1^{\varepsilon'})^{h^{(r+j)t_2}}$. While by the assumption that $\mathrm{Aut}\,(P_1)$ is abelian, we have $\alpha_1((x_1^{\varepsilon})^{h^{rt_1}}) = (\alpha_1(x_1^{\varepsilon}))^{h^{rt_1}} = (x_1^{\varepsilon'})^{h^{rt_1+jt_2}}$. Therefore, $(x_1^{\varepsilon'})^{h^{rt_1+jt_2}} = (x_1^{\varepsilon'})^{h^{(r+j)t_2}}$. It means that $m | r(t_1 - t_2)$ for any $0 \leqslant r \leqslant m-1$, implying that $t_1 = t_2$. This completes the proof. □

Note: From Tables 1 to 3, one can see that for Frobenius groups with rank $6 \leqslant r \leqslant 50$, except for those listed in Table 3 and the two Frobenius groups marked by a "*" in Table 1, they satisfy the assumptions in Theorems 3.4, 3.7, 3.9 and 3.11. Therefore, the conditions in Theorems 3.4, 3.7, 3.9 and 3.11 are not very restrictive. An interesting result is that for each of these Frobenius graphs, its automorphism group is equal to that of the Frobenius maps it admits.

4. Frobenius maps of some Frobenius groups

Given a Frobenius group $G = K : H$ with Frobenius kernel K, Lemma 2.2 says that a graph Γ is G-Frobenius if and only if it is isomorphic to a Cayley graph $\mathscr{C}(K, S)$ for some $S = S_0 \cup S_0^{-1}$ (possible $S_0 = S_0^{-1}$) and S_0 runs over all H-orbits in K each of which generates K. Let

$$\Lambda = \{S_0 \cup S_0^{-1} | S_0 \text{ is an } H\text{-orbit in } K \text{ and } \langle S_0 \rangle = K\}.$$

Given $S = S_0 \cup S_0^{-1} \in \Lambda$, let $\mathscr{M}_S$ denote the set of all non-isomorphic Frobenius maps with underlying graph $\mathscr{C}(K, S)$.

Lemma 4.1 (*Fang et al.* [5, Theorem 3.5]). *Let $G = K : H$ be a Frobenius group with Frobenius kernel K, say $K = P_1 \times \cdots \times P_t$, where P_i are the distinct Sylow subgroups of*

K. If, for each $i = 1, \ldots, t$, the normalizer $N_{\mathrm{Aut}(K)}(H)$ is transitive on the set of H-orbits in P_i each of which generates P_i, then all G-Frobenius graphs are isomorphic.

Theorem 4.2. *Let $G = K : H$ be a Frobenius group with Frobenius kernel $K = P_1 \times \cdots \times P_t$ and a cyclic Frobenius complement H. Assume that there exists an $S = S_0 \cup S_0^{-1} \in \Lambda$ such that $S_0 = S_0^{-1}$ and $\mathrm{Aut}(\mathscr{C}(K, S)) \cong K : H$. If for each $i = 1, \ldots, t$, the centralizer $C_{\mathrm{Aut}(K)}(H)$ is transitive on the set of H-orbits in P_i each of which generates P_i, then up to isomorphism, $\mathscr{M}_S = \mathscr{M}_{S'}$ for any two $S, S' \in \Lambda$.*

Proof. By Lemma 4.1, under the assumption that $C_{\mathrm{Aut}(K)}(H)$ is transitive on the set of H-orbits in P_i each of which generates P_i, we know that for any two elements in Λ, say S_1 and S_2, $\mathscr{C}(K, S_1) \cong \mathscr{C}(K, S_2)$. Consequently, $\mathrm{Aut}(\mathscr{C}(K, S_1)) \cong \mathrm{Aut}(\mathscr{C}(K, S_2))$; and if there exists an $S = S_0 \cup S_0^{-1} \in \Lambda$ such that $S_0 = S_0^{-1}$, then for any $S' = S_0' \cup S_0'^{-1} \in \Lambda$, we have $S_0' = S_0'^{-1}$.

Let $\mathscr{CM}(K, S_j, \rho_j)$ for $j = 1, 2$ be two G-Frobenius maps with $S_1 = x^H$ and $S_2 = y^H$ for some $x, y \in K$. Let $H \cong \langle h \rangle$ be cyclic and let $|H| = m$. By Theorem 3.4, one can assume that

$$\rho_1 = (x \;\; x^{h^{r_1}} \cdots x^{h^{(m-1)r_1}}) \quad \text{and} \quad \rho_2 = (y \;\; y^{h^{r_2}} \cdots y^{h^{(m-1)r_2}}),$$

for integers $0 \leqslant r_j \leqslant m - 1$ and $(r_j, m) = 1$, $j = 1, 2$. Since $C_{\mathrm{Aut}(K)}(H)$ is transitive on Λ and $S_1, S_2 \in \Lambda$, there exists a $\sigma \in C_{\mathrm{Aut}(K)}(H)$ such that $\sigma(x) = y$. It follows that $\sigma(x^{h^{jr_1}}) = (\sigma(x))^{h^{jr_1}} = y^{h^{jr_1}}$ for any $0 \leqslant j \leqslant m-1$. Thus, if $r_1 = r_2$, then $\mathscr{CM}(K, S_1, \rho_1) \cong \mathscr{CM}(K, S_2, \rho_2)$. From Theorems 3.4 and 3.7, we have $\mathscr{M}_{S_1} = \mathscr{M}_{S_2}$. □

By a method similar to Theorem 4.2 and by Theorems 3.9 and 3.11, one can get the following theorem.

Theorem 4.3. *Let $G = K : H$ be a Frobenius group with an abelian Frobenius kernel $K = P_1 \times \cdots \times P_t$ and a cyclic Frobenius complement H. Assume that there exists an $S = S_0 \cup S_0^{-1} \in \Lambda$ such that $S_0 \neq S_0^{-1}$ and $\mathrm{Aut}(\mathscr{C}(K, S)) \cong K : (H \times \langle \beta \rangle)$ with the β given in Eq. (3). If for each $i = 1, \ldots, t$, the centralizer $C_{\mathrm{Aut}(K)}(H)$ is transitive on the set of H-orbits in P_i each of which generates P_i, then up to isomorphism, $\mathscr{M}_S = \mathscr{M}_{S'}$ for any $S, S' \in \Lambda$.*

Corollary 4.4. *Let p be a prime and $n = p_1^{a_1} \cdots p_t^{a_t}$, where p_i are distinct primes. Let $G = K : H$ be a Frobenius group with cyclic Frobenius complement H and Frobenius kernel K which is one of the following two cases:*
(1) $K \cong \mathbb{Z}_n$,
(2) $K \cong \mathbb{Z}_p^r \times \mathbb{Z}_n$ *with* $(p, n) = 1$ *and* $H \cong \mathbb{Z}_{p^r-1}$, *where* $p_i^{a_i} \equiv 1 (\mathrm{mod}\ p^r - 1)$.
In each case, if there exists a Frobenius graph $\mathscr{C}(K, S)$ whose automorphism group satisfies the conditions in Theorem 4.2 or Theorem 4.3 according to the choices of S, then $\mathscr{M}_S = \mathscr{M}_{S'}$ up to isomorphism for any $S, S' \in \Lambda$.

Proof. By Theorems 4.2 and 4.3, in either case, we need only to show that $C_{\mathrm{Aut}(K)}(H)$ acts transitively on the set of H-orbits in P_i each of which generates P_i.

(1) Since K is cyclic, $\mathrm{Aut}(K)$ is abelian and transitive on generators for K. Note that H is isomorphic to a subgroup of $\mathrm{Aut}\,(K)$, so $C_{\mathrm{Aut}(K)}(H) = \mathrm{Aut}\,(K)$.

(2) Since $K \cong \mathbb{Z}_p^r \times \mathbb{Z}_n$, then $K \cong \mathbb{Z}_p^r \times \mathbb{Z}_{p_1^{a_1}} \times \cdots \times \mathbb{Z}_{p_t^{a_t}}$ and $\mathrm{Aut}\,(K) = \mathrm{Aut}\,(\mathbb{Z}_p^r) \times \mathrm{Aut}\,(\mathbb{Z}_n)$. It follows that $\mathrm{Aut}\,(\mathbb{Z}_n)$ is a normal subgroup of $\mathrm{Aut}\,(K)$. Let $X = H\,\mathrm{Aut}\,(\mathbb{Z}_n)$, then $X \leqslant \mathrm{Aut}\,(K)$. Clearly, for each $1 \leqslant i \leqslant t$, $\mathrm{Aut}\,(\mathbb{Z}_n)$ is transitive on generators for $\mathbb{Z}_{p_i^{a_i}}$ and H is transitive on $\mathbb{Z}_p^r \backslash \{1\}$. Thus, X is transitive on the set of H-orbits in $\mathbb{Z}_p^r$ and $\mathbb{Z}_{p_i^{a_i}}$ each of which generates $\mathbb{Z}_p^r$ and $\mathbb{Z}_{p_i^{a_i}}$ respectively. On the other hand, $\mathrm{Aut}\,(\mathbb{Z}_n)$ centralizes H, and so $X \leqslant C_{\mathrm{Aut}(K)}(H)$.

In either case, we have that $C_{\mathrm{Aut}(K)}(H)$ satisfies the conditions in Theorems 4.2 and 4.3. □

5. Frobenius maps with trivial exponent groups

An integer e is called an *exponent* of an orientable regular map $\mathscr{M}$ if $\mathscr{M} = (D;\, R,\, L)$ and $\mathscr{M}^e = (D;\, R^e,\, L)$ are isomorphic. We remake a couple of observations about exponents. Let $\mathscr{M}$ be an orientable regular map of valence r. Firstly, if e is an exponent of $\mathscr{M}$, then $\gcd(r, e) = 1$. Secondly, if e is an integer and k is a multiple of r, then $\mathscr{M}^e = \mathscr{M}^{e+k}$. Thirdly, if e_1 and e_2 are exponents, so is $e_1 e_2$. It follows that the exponents form a subgroup of $\mathbb{Z}_r^*$, the multiplicative group of the ring of integers module r. We call this group the *exponent group* of $\mathscr{M}$ and denote it by $\mathrm{Ex}(\mathscr{M})$. We say that $\mathscr{M}$ has a trivial exponent group if $|\mathrm{Ex}(\mathscr{M})| = 1$.

The exponent group of an orientable regular map gives us information on the degree of symmetry of the map. For example, -1 is in the exponent group of $\mathscr{M}$ if and only if the map is reflexible. Clearly, any cubic irreflexible regular map (or chiral map) has the trivial exponent group, but non-cubic examples are less obvious. When p is a prime, the regular embedding of the complete graph K_p gives such an example.

Generally, for any integer n, Nedela and Škoviera asked [14] for a construction of a regular map of valency n with trivial exponent group. Archdeacon et al. [1] constructed such an example by using lifting techniques under the assumption that the base map has a trivial exponent group.

From Frobenius maps, we provide examples of regular maps with trivial exponent groups, as a partial answer of the question raised by Nedela and Škoviera.

Theorem 5.1. *Let $G = K : H$ be a Frobenius group with a cyclic Frobenius complement $H \cong \langle h \rangle$, say $|H| = m$. Let $\Gamma = \mathscr{C}(K, S)$ be a Frobenius graph with $S = x^H$. If* $\mathrm{Aut}\,(\Gamma) \cong K : H$ *and there exists at least one Sylow subgroup of K, say P, such that* $\mathrm{Aut}\,(P)$ *is abelian, then the Frobenius map $\mathscr{CM}(K, S, \rho)$ has the trivial exponent group, where $\rho = (x \;\; x^{h^t} \cdots x^{h^{(m-1)t}})$ for some integer t with $(t, m) = 1$.*

Proof. If e is an exponent of $\mathscr{CM}(K, S, \rho)$, then $\mathscr{CM}(K, S, \rho) \cong \mathscr{CM}(K, S, \rho^e)$, which implies $e \equiv 1 \pmod{m}$ by Theorem 3.7. □

Similarly, one can obtain the following theorem.

Theorem 5.2. *Let $G = K : H$ be a Frobenius group with an abelian Frobenius kernel K and $H \cong \langle h \rangle$, $|H| = m$, and let $\Gamma = \mathscr{C}(K, S)$ be a Frobenius graph with $S = x^H \cup (x^{-1})^H$. If* Aut $(\Gamma) \cong K : (H \times \langle \beta \rangle)$ *with the β given in Eq.* (3) *and there exists at least one Sylow subgroup of K, say P, such that* Aut (P) *is abelian, then the Frobenius map $\mathscr{C}\mathscr{M}(K, S, \rho)$ has the trivial exponent group, where $\rho = (x \;\; (x^{-1})^{h^i} \;\; x^{h^{2i}} \cdots x^{h^{(m-1)i}} \;\; x^{-1} \cdots (x^{-1})^{h^{(m-1)i}})$ for some integer i with $(i, m) = 1$.*

References

[1] D. Archdeacon, P. Gvozdjak, J. Širáň, Constructing and forbidding automorphisms in lifted maps, Math. Slovaca 47 (1997) 113–129.

[2] N.L. Biggs, A.T. White, Permutation Groups and Combinatorial Structures, Cambridge University Press, Cambridge, 1979.

[3] M. Conder, P. Dobcsányi, Determination of all regular maps of small genus, J. Combin. Theory Ser. B 81 (2001) 224–242.

[4] H.S.M. Coxeter, W.O.J. Moser, Generators and relations for discrete groups, third ed., Springer, New York, 1972.

[5] X.G. Fang, C.H. Li, C.E. Praeger, On orbital regular graphs and Frobenius graphs, Discrete Math. 182 (1998) 85–99.

[6] D. Garbe, Über die regulären Zerlegungen orientierbarer Flächen, J. Reine Angew. Math. 237 (1969) 39–55.

[7] D. Gorenstein, Finite Groups, Harper and Row, New York, 1968.

[8] R. Jajcay, Automorphism groups of Cayley Maps, J. Combin. Theory Ser. B 59 (1993) 297–310.

[9] R. Jajcay, Characterization and construction of Cayley graphs admitting regular Cayley maps, Discrete Math. 158 (1996) 151–160.

[10] L.D. James, G.A. Jones, Regular orientable imbeddings of complete graphs, J. Combin. Theory Ser. B 39 (1985) 353–367.

[11] G.A. Jones, Maps on surfaces and Galois groups, Math. Slovaca 47 (1997) 1–33.

[12] P. McMullen, E. Schulte, Abstract Regular Polytopes, Cambridge University Press, Cambridge, 2002.

[13] R. Nedela, M. Škoviera, Regular maps of canonical double coverings of graphs, J. Combin. Theory Ser. B 67 (1996) 249–277.

[14] R. Nedela, M. Škoviera, Exponents of orientable maps, Proc. London Math. Soc. 75 (3) (1997) 1–31.

[15] R.B. Richter, J. Širáň, R. Jajcay, T.W. Tucker, M.E. Watkins, Cayley maps, J. Combin. Theory Ser. B, to appear.

[16] M. Škoviera, J. Širáň, Regular maps from Cayley graphs, Part 1: balanced Cayley maps, Discrete Math. 109 (1992) 265–276.

[17] J.G. Thompson, Finite groups with fixed-point-free automorphisms of prime order, Math. Acad. Sci. USA 45 (1959) 578–581.

[18] J.G. Thompson, Normal *p*-complements for finite groups, Math. Z. 72 (1960) 332–354.

[19] Y. Wang, X.G. Fang, D.F. Hsu, On the edge-forwarding indices of Frobenius graphs, DIMACS Technical Reports 30, 2002.

[20] H. Wielandt, Finite Permutation Groups, Academic Press, New York, 1964.