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On existence and local attractivity of solutions of a quadratic Volterra integral equation of fractional order

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ABSTRACT

In the paper we study the existence of solutions of a nonlinear quadratic Volterra integral equation of fractional order. This equation is considered in the Banach space of real functions defined, continuous and bounded on an unbounded interval. Moreover, we show that solutions of this integral equation are locally attractive.

1. Introduction

Differential and integral equations of fractional order play a very important role in describing some real world problems. For example some problems in physics, mechanics and other fields can be described with the help of differential and integral equations of fractional order (cf. [10,11,13,14,16–19]).

The theory of differential and integral equations of fractional order has recently received a lot of attention and now constitutes a significant branch of nonlinear analysis. Numerous research papers and monographs have appeared devoted to differential and integral equations of fractional order (cf. [1,5,7,8,10–20], for example). These papers contain various types of existence results for equations of fractional order [5,7,8,11,12].

The aim of this paper is to study the existence of solutions of a nonlinear quadratic Volterra integral equation of fractional order in the space of real functions defined, continuous and bounded on an unbounded interval. Moreover, we will investigate an important property of the solutions which is called the local attractivity of solutions. This property is a generalization of the global attractivity of solutions introduced in [12] and is also a variant of the property of asymptotic stability of solutions considered in [3,4,6].

It is worthwhile mentioning that up to now integral equations of fractional order have only been studied in the space of real functions defined on a bounded interval.

The result obtained in this paper generalizes several ones obtained earlier by many authors. Also we hope that the concept of local attractivity considered here may be a stimulant for further investigations concerning local and global attractivity of solutions of nonlinear integral equations of other types.

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2. Notation, definitions and auxiliary facts

In this section we collect some definitions and results which will be needed later.

First we recall a few facts concerning fractional calculus [14–16]. Denote by $L^1(a,b)$ the space of real functions defined and Lebesgue integrable on the interval (a,b), which is equipped with the standard norm. Let $x \in L^1(a,b)$ and let $\alpha > 0$ be a fixed number. The Riemann–Liouville fractional integral of order α of the function $x(t)$ is defined by the formula

$$I^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{x(s)}{(t-s)^{1-\alpha}}\, ds, \quad t \in (a,b),$$

where $\Gamma(\alpha)$ denotes the gamma function.

It may be shown that the fractional integral operator I^α transforms the space $L^1(a,b)$ into itself and has some other properties (see [13–16]).

Next we present some facts concerning measures of noncompactness [2].

Let $(E, \|\cdot\|)$ be an infinite dimensional Banach space with the zero element θ. Denote by $B(x,r)$ the closed ball centered at x and with radius r. The symbol B_r stands for the ball $B(\theta, r)$. If X is a subset of X we write $\overline{X}$, $\operatorname{Conv} X$ in order to denote the closure and convex closure of X, respectively. Moreover, we denote by $\mathfrak{M}_E$ the family of all nonempty and bounded subsets of E and by $\mathfrak{N}_E$ its subfamily consisting of all relatively compact sets.

We use the following definition of the concept of a measure of noncompactness [2].

Definition 1. A mapping $\mu : \mathfrak{M}_E \to \mathbb{R}_+ = [0, \infty)$ is said to be a measure of noncompactness in E if it satisfies the following conditions:

1^o The family $\ker \mu = \{X \in \mathfrak{M}_E\colon\ \mu(X) = 0\}$ is nonempty and $\ker \mu \subset \mathfrak{N}_E$.
2^o $X \subset Y \Rightarrow \mu(X) \leqslant \mu(Y)$.
3^o $\mu(\overline{X}) = \mu(X)$.
4^o $\mu(\operatorname{Conv} X) = \mu(X)$.
5^o $\mu(\lambda X + (1-\lambda)Y) \leqslant \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0,1]$.
6^o If (X_n) is a sequence of closed sets from $\mathfrak{M}_E$ such that $X_{n+1} \subset X_n$ $(n = 1, 2, \ldots)$ and if $\lim_{n\to\infty} \mu(X_n) = 0$, then the intersection $X_\infty = \bigcap_{n=1}^\infty X_n$ is nonempty.

The family $\ker \mu$ described in 1^o is said to be *the kernel of the measure of noncompactness* μ. Let us observe that the intersection set X_∞ from 6^o belongs to $\ker \mu$. In fact, since $\mu(X_\infty) \leqslant \mu(X_n)$ for every n then we have that $\mu(X_\infty) = 0$. This simple observation will be essential later.

Other facts concerning measures of noncompactness and their properties may be found in [2].

In what follows we will work in the Banach space $BC(\mathbb{R}_+)$ consisting of all real functions defined, continuous and bounded on $\mathbb{R}_+$. This space is furnished with the standard norm

$$\|x\| = \sup\{|x(t)|\colon\ t \geqslant 0\}.$$

We will use a measure of noncompactness in the space $BC(\mathbb{R}_+)$ which was introduced in [2]. In order to define this measure let us fix a nonempty bounded subset X of the space $BC(\mathbb{R}_+)$ and a positive number T. For $x \in X$ and $\varepsilon \geqslant 0$ denote by $\omega^T(x,\varepsilon)$ *the modulus of continuity* of the function x on the interval $[0,T]$, i.e.

$$\omega^T(x,\varepsilon) = \sup\{|x(t) - x(s)|\colon\ t, s \in [0,T],\ |t-s| \leqslant \varepsilon\}.$$

Further, let us put

$$\omega^T(X,\varepsilon) = \sup\{\omega^T(x,\varepsilon)\colon\ x \in X\},$$
$$\omega_0^T(X) = \lim_{\varepsilon\to 0} \omega^T(X,\varepsilon), \qquad \omega_0(X) = \lim_{T\to\infty} \omega_0^T(X).$$

If t is a fixed number from $\mathbb{R}_+$, let us denote

$$X(t) = \{x(t)\colon\ x \in X\}$$

and

$$\operatorname{diam} X(t) = \sup\{|x(t) - y(t)|\colon\ x, y \in X\}.$$

Finally, consider the function μ defined on the family $\mathfrak{M}_{BC(\mathbb{R}_+)}$ by the formula

$$\mu(X) = \omega_0(X) + \limsup_{t\to\infty} \operatorname{diam} X(t). \tag{2.1}$$

It can be shown that the function μ is a measure of noncompactness in the space $BC(\mathbb{R}_+)$. The kernel $\ker\mu$ of this measure consists of nonempty and bounded sets X such that functions from X are locally equicontinuous on $\mathbb{R}_+$ and the thickness of the bundle formed by functions from X tends to zero at infinity. This property will permit us to characterize solutions of the integral equation considered in the next section.

Now, let us assume that Ω is a nonempty subset of the space $BC(\mathbb{R}_+)$ and Q is an operator defined on Ω with values in $BC(\mathbb{R}_+)$.

Consider the following operator equation:

$$x(t) = (Qx)(t), \quad t \geqslant 0. \tag{2.2}$$

Definition 2. We say that solutions of Eq. (2.2) are locally attractive if there exists a ball $B(x_0, r)$ in the space $BC(\mathbb{R}_+)$ such that for arbitrary solutions $x(t)$ and $y(t)$ of Eq. (2.2) belonging to $B(x_0, r) \cap \Omega$ we have that

$$\lim_{t\to\infty} \bigl(x(t) - y(t)\bigr) = 0. \tag{2.3}$$

In the case when the limit (2.3) is uniform with respect to the set $B(x_0, r) \cap \Omega$, i.e. when for each $\varepsilon > 0$ there exists $T > 0$ such that

$$\bigl|x(t) - y(t)\bigr| \leqslant \varepsilon \tag{2.4}$$

for all $x, y \in B(x_0, r) \cap \Omega$ and for $t \geqslant T$, we will say that solutions of Eq. (2.2) are uniformly locally attractive.

Observe that the concept of uniform local attractivity of solutions, which is defined above, is equivalent to the concept of asymptotic stability of solutions introduced in the paper [3] (cf. also [4]).

Finally, let us recall the definition of the concept of global attractivity of solutions introduced in the paper [12].

Definition 3. The solution $x = x(t)$ of Eq. (2.2) is said to be globally attractive if (2.3) holds for each solution $y = y(t)$ of Eq. (2.2).

In other words we may say that solutions of Eq. (2.2) are *globally attractive* if for arbitrary solutions $x(t)$ and $y(t)$ of that equation condition (2.3) is satisfied.

Observe that global attractivity of solutions imply local attractivity. The converse implication is not valid and we will show this later.

3. Main result

In this section we will investigate the following quadratic Volterra integral equation of fractional order:

$$x(t) = p(t) + \frac{f(t, x(t))}{\Gamma(\alpha)} \int_0^t \frac{u(t, s, x(s))}{(t-s)^{1-\alpha}}\, ds, \tag{3.1}$$

where $t \in \mathbb{R}_+$ and α is a fixed number, $\alpha \in (0, 1)$.

Eq. (3.1) will be considered under the following assumptions:

(i) The function $p : \mathbb{R}_+ \to \mathbb{R}$ is continuous and bounded on $\mathbb{R}_+$.

(ii) The function $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a function $m : \mathbb{R}_+ \to \mathbb{R}_+$ being continuous on $\mathbb{R}_+$ and such that

$$\bigl|f(t, x) - f(t, y)\bigr| \leqslant m(t)|x - y|$$

for any $t \in \mathbb{R}_+$ and for all $x, y \in \mathbb{R}$.

(iii) The function $u(t, s, x) = u : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuous. Moreover, there exist a function $n : \mathbb{R}_+ \to \mathbb{R}_+$ being continuous on $\mathbb{R}_+$ and a function $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ being continuous and nondecreasing on $\mathbb{R}_+$ with $\Phi(0) = 0$ and such that

$$\bigl|u(t, s, x) - u(t, s, y)\bigr| \leqslant n(t)\Phi\bigl(|x - y|\bigr)$$

for all $t, s \in \mathbb{R}_+$ such that $s \leqslant t$ and for all $x, y \in \mathbb{R}$.

For further purposes let us define the function $u_1 : \mathbb{R}_+ \to \mathbb{R}_+$ by putting

$$u_1(t) = \max\bigl\{\bigl|u(t, s, 0)\bigr| : \ 0 \leqslant s \leqslant t\bigr\}.$$

Obviously the function u_1 is continuous on $\mathbb{R}_+$.

In what follows we will assume additionally that the following conditions are satisfied:

(iv) The functions $a, b, c, d : \mathbb{R}_+ \to \mathbb{R}_+$ defined by the formulas

$$\begin{aligned} a(t) &= m(t)n(t)t^{\alpha}, \\ b(t) &= m(t)u_1(t)t^{\alpha}, \\ c(t) &= n(t)\big|f(t,0)\big|t^{\alpha}, \\ d(t) &= u_1(t)\big|f(t,0)\big|t^{\alpha} \end{aligned}$$

are bounded on $\mathbb{R}_+$ and the functions $a(t), c(t)$ vanish at infinity i.e. $\lim_{t\to\infty} a(t) = \lim_{t\to\infty} c(t) = 0$.

Keeping in mind assumption (iv) we may define the following finite constants:

$$\begin{aligned} A &= \sup\big\{a(t):\ t \in \mathbb{R}_+\big\}, \\ B &= \sup\big\{b(t):\ t \in \mathbb{R}_+\big\}, \\ C &= \sup\big\{c(t):\ t \in \mathbb{R}_+\big\}, \\ D &= \sup\big\{d(t):\ t \in \mathbb{R}_+\big\}. \end{aligned}$$

Now we formulate the last assumption:

(v) There exists a positive solution r_0 of the inequality

$$\|p\| + \big(1/\Gamma(\alpha+1)\big)\big[Ar\Phi(r) + Br + C\Phi(r) + D\big] \leqslant r.$$

Moreover, $A\Phi(r_0) + B < \Gamma(\alpha+1)$.

Now, let us consider the operators F, U, V defined on the space $BC(\mathbb{R}_+)$ by the formulas:

$$\begin{aligned} (Fx)(t) &= f\big(t, x(t)\big), \\ (Ux)(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t \frac{u(t,s,x(s))}{(t-s)^{1-\alpha}}\,ds, \\ (Vx)(t) &= p(t) + (Fx)(t)(Ux)(t). \end{aligned}$$

Then we have the following lemma.

Lemma 1. *Under the above assumptions the operator V transforms the ball B_{r_0} in the space $BC(\mathbb{R}_+)$ into itself, where r_0 is a number appearing in assumption* (v)*. Moreover, all solutions of Eq.* (3.1) *belonging to the space $BC(\mathbb{R}_+)$ are fixed points of the operator V.*

Proof. Observe that in view of our assumptions, for any function $x \in BC(\mathbb{R}_+)$ the function Fx is continuous on $\mathbb{R}_+$. We show that the same assertion holds also for the operator U. To do this take an arbitrary function $x \in BC(\mathbb{R}_+)$ and fix $T > 0$ and $\varepsilon > 0$. Next assume that $t_1, t_2 \in [0, T]$ are such that $|t_2 - t_1| \leqslant \varepsilon$. Without loss of generality we can assume that $t_1 < t_2$. Then, taking into account our assumptions we get

$$\begin{aligned} \big|(Ux)(t_2) - (Ux)(t_1)\big| &= \frac{1}{\Gamma(\alpha)}\left| \int_0^{t_1} \frac{u(t_2,s,x(s))}{(t_2-s)^{1-\alpha}}\,ds + \int_{t_1}^{t_2} \frac{u(t_2,s,x(s))}{(t_2-s)^{1-\alpha}} - \int_0^{t_1} \frac{u(t_1,s,x(s))}{(t_1-s)^{1-\alpha}}\,ds \right| \\ &\leqslant \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \left| \frac{u(t_2,s,x(s))}{(t_2-s)^{1-\alpha}} - \frac{u(t_1,s,x(s))}{(t_2-s)^{1-\alpha}} \right| ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \left| \frac{u(t_1,s,x(s))}{(t_2-s)^{1-\alpha}} - \frac{u(t_1,s,x(s))}{(t_1-s)^{1-\alpha}} \right| ds + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \frac{|u(t_2,s,x(s))|}{(t_2-s)^{1-\alpha}}\,ds \\ &\leqslant \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \frac{|u(t_2,s,x(s)) - u(t_1,s,x(s))|}{(t_2-s)^{1-\alpha}}\,ds \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\bigl|u\bigl(t_1,s,x(s)\bigr)\bigr|\biggl[\frac{1}{(t_1-s)^{1-\alpha}}-\frac{1}{(t_2-s)^{1-\alpha}}\biggr]ds\\
&+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\frac{|u(t_2,s,x(s))|}{(t_2-s)^{1-\alpha}}\,ds\leqslant\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\omega_1^T\bigl(u,\varepsilon;\|x\|\bigr)\frac{1}{(t_2-s)^{1-\alpha}}\,ds\\
&+\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\bigl[\bigl|u\bigl(t_1,s,x(s)\bigr)-u(t_1,s,0)\bigr|+\bigl|u(t_1,s,0)\bigr|\bigr]\cdot\biggl[\frac{1}{(t_1-s)^{1-\alpha}}-\frac{1}{(t_2-s)^{1-\alpha}}\biggr]ds\\
&+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\frac{|u(t_2,s,x(s))-u(t_2,s,0)|+|u(t_2,s,0)|}{(t_2-s)^{1-\alpha}}\,ds\\
\leqslant{}&\frac{\omega_1^T(u,\varepsilon;\|x\|)}{\Gamma(\alpha)}\cdot\frac{t_2^\alpha-(t_2-t_1)^\alpha}{\alpha}\\
&+\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\bigl[n(t_1)\Phi\bigl(|x(s)|\bigr)+u_1(t_1)\bigr]\biggl[\frac{1}{(t_1-s)^{1-\alpha}}-\frac{1}{(t_2-s)^{1-\alpha}}\biggr]ds\\
&+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\frac{n(t_2)\Phi(|x(s)|)+u_1(t_2)}{(t_2-s)^{1-\alpha}}\,ds\\
\leqslant{}&\frac{\omega_1^T(u,\varepsilon;\|x\|)}{\Gamma(\alpha+1)}t_1^\alpha+\frac{n(t_1)\Phi(\|x\|)+u_1(t_1)}{\Gamma(\alpha+1)}\bigl[t_1^\alpha-t_2^\alpha+(t_2-t_1)^\alpha\bigr]\\
&+\frac{n(t_2)\Phi(\|x\|)+u_1(t_2)}{\Gamma(\alpha+1)}(t_2-t_1)^\alpha\\
\leqslant{}&\frac{1}{\Gamma(\alpha+1)}\bigl\{t_1^\alpha\omega_1^T\bigl(u,\varepsilon;\|x\|\bigr)+(t_2-t_1)^\alpha\bigl[n(t_1)\Phi\bigl(\|x\|\bigr)+u_1(t_1)\bigr]\\
&+(t_2-t_1)^\alpha\bigl[n(t_2)\Phi\bigl(\|x\|\bigr)+u_1(t_2)\bigr]\bigr\},
\end{aligned}
\tag{3.2}
$$

where we denoted

$$\omega_1^T\bigl(u,\varepsilon;\|x\|\bigr)=\sup\bigl\{\bigl|u(t_2,s,y)-u(t_1,s,y)\bigr|:\ s,t_1,t_2\in[0,T],\ s\leqslant t_1,\ s\leqslant t_2,\ |t_2-t_1|\leqslant\varepsilon,\ |y|\leqslant\|x\|\bigr\}.$$

Obviously, in view of the uniform continuity of the function $u(t,s,y)$ on the set $[0,T]\times[0,T]\times[-\|x\|,\|x\|]$ we have that $\omega_1^T(u,\varepsilon;\|x\|)\to 0$ as $\varepsilon\to 0$.

In what follows let us denote

$$\bar{n}(T)=\max\bigl\{n(t):\ t\in[0,T]\bigr\},$$

$$\overline{u_1}(T)=\max\bigl\{u_1(t):\ t\in[0,T]\bigr\}.$$

Then, keeping in mind the estimate (3.2) we obtain

$$\omega^T(Ux,\varepsilon)\leqslant\frac{1}{\Gamma(\alpha+1)}\bigl\{T^\alpha\omega_1^T\bigl(u,\varepsilon;\|x\|\bigr)+2\varepsilon^\alpha\bigl[\bar{n}(T)\Phi\bigl(\|x\|\bigr)+\overline{u_1}(T)\bigr]\bigr\}.$$

From the above inequality we infer that the function Ux is continuous on the interval $[0,T]$ for any $T>0$. This yields the continuity of Ux on $\mathbb{R}_+$.

Finally we deduce that the function Vx is continuous on $\mathbb{R}_+$.

Next, let us take an arbitrary function $x\in BC(\mathbb{R}_+)$. Then, using our assumptions, for a fixed $t\in\mathbb{R}_+$ we have

$$
\begin{aligned}
\bigl|(Vx)(t)\bigr|&\leqslant\bigl|p(t)\bigr|+\frac{1}{\Gamma(\alpha)}\bigl[\bigl|f\bigl(t,x(t)\bigr)-f(t,0)\bigr|+\bigl|f(t,0)\bigr|\bigr]\int_0^t\frac{|u(t,s,x(s))-u(t,s,0)|+|u(t,s,0)|}{(t-s)^{1-\alpha}}\,ds\\
&\leqslant\|p\|+\frac{m(t)|x(t)|+|f(t,0)|}{\Gamma(\alpha)}\int_0^t\frac{n(t)\Phi(|x(s)|)+u_1(t)}{(t-s)^{1-\alpha}}
\end{aligned}
$$

$$\begin{aligned}
&\leqslant \|p\| + \frac{m(t)\|x\| + |f(t,0)|}{\Gamma(\alpha)}\big[n(t)\Phi\big(\|x\|\big) + u_1(t)\big]\int_0^t \frac{ds}{(t-s)^{1-\alpha}} \\
&\leqslant \|p\| + \frac{1}{\Gamma(\alpha+1)}\big[m(t)n(t)t^\alpha\|x\|\Phi\big(\|x\|\big) + m(t)u_1(t)t^\alpha\|x\| + n(t)\big|f(t,0)\big|t^\alpha\Phi\big(\|x\|\big) + \big|f(t,0)\big|u_1(t)t^\alpha\big] \\
&= \|p\| + \frac{1}{\Gamma(\alpha+1)}\big[a(t)\|x\|\Phi\big(\|x\|\big) + b(t)\|x\| + c(t)\Phi\big(\|x\|\big) + d(t)\big].
\end{aligned} \tag{3.3}$$

Hence, in view of assumption (iv) we infer that the function Vx is bounded on $\mathbb{R}_+$. This assertion in conjunction with the continuity of Vx on $\mathbb{R}_+$ allows us to conclude that $Vx \in BC(\mathbb{R}_+)$. Moreover, from the estimate (3.3) we obtain

$$\|Vx\| \leqslant \|p\| + \frac{1}{\Gamma(\alpha+1)}\big[A\|x\|\Phi\big(\|x\|\big) + B\|x\| + C\Phi\big(\|x\|\big) + D\big].$$

Linking this estimate with assumption (v) we deduce that there exists $r_0 > 0$ such that the operator V transforms the ball B_{r_0} into itself.

Finally, let us notice that the second assertion of our lemma is obvious in the light of the fact that the operator V transforms the space $BC(\mathbb{R}_+)$ into itself. The proof is complete. □

Now we are prepared to formulate our main existence result.

Theorem 1. *Under assumptions* (i)–(v) *Eq.* (3.1) *has at least one solution $x = x(t)$ which belongs to the space $BC(\mathbb{R}_+)$. Moreover, solutions of Eq.* (3.1) *are uniformly locally attractive.*

Proof. Let us take a nonempty set $X \subset B_{r_0}$, where B_{r_0} is a ball in the space $BC(\mathbb{R}_+)$ described in Lemma 1. Then, for $x, y \in X$ and for an arbitrarily fixed $t \in \mathbb{R}_+$, in view of assumptions (ii)–(iv) we obtain

$$\begin{aligned}
\big|(Vx)(t) - (Vy)(t)\big| &\leqslant \left|\frac{f(t,x(t))}{\Gamma(\alpha)}\int_0^t \frac{u(t,s,x(s))}{(t-s)^{1-\alpha}}\,ds - \frac{f(t,y(t))}{\Gamma(\alpha)}\int_0^t \frac{u(t,s,y(s))}{(t-s)^{1-\alpha}}\,ds\right| \\
&\leqslant \frac{1}{\Gamma(\alpha)}\big|f\big(t,x(t)\big) - f\big(t,y(t)\big)\big|\int_0^t \frac{|u(t,s,x(s))|}{(t-s)^{1-\alpha}}\,ds + \frac{|f(t,y(t))|}{\Gamma(\alpha)}\int_0^t \frac{|u(t,s,x(s)) - u(t,s,y(s))|}{(t-s)^{1-\alpha}}\,ds \\
&\leqslant \frac{1}{\Gamma(\alpha)}m(t)\big|x(t) - y(t)\big|\int_0^t \frac{|u(t,s,x(s)) - u(t,s,0)| + |u(t,s,0)|}{(t-s)^{1-\alpha}}\,ds \\
&\quad + \frac{1}{\Gamma(\alpha)}\big[\big|f\big(t,y(t)\big) - f(t,0)\big| + \big|f(t,0)\big|\big]\int_0^t \frac{n(t)\Phi(|x(s) - y(s)|)}{(t-s)^{1-\alpha}}\,ds \\
&\leqslant \frac{m(t)|x(t) - y(t)|}{\Gamma(\alpha)}\int_0^t \frac{n(t)\Phi(|x(s)|) + u_1(t)}{(t-s)^{1-\alpha}}\,ds + \frac{[m(t)|y(t)| + |f(t,0)|]n(t)}{\Gamma(\alpha)}\int_0^t \frac{\Phi(|x(s) - y(s)|)}{(t-s)^{1-\alpha}}\,ds \\
&\leqslant \frac{m(t)n(t)(|x(t)| + |y(t)|)}{\Gamma(\alpha)}\int_0^t \frac{\Phi(|x(s)|)}{(t-s)^{1-\alpha}}\,ds + \frac{m(t)u_1(t)}{\Gamma(\alpha)}\big|x(t) - y(t)\big|\int_0^t \frac{ds}{(t-s)^{1-\alpha}} \\
&\quad + \frac{m(t)n(t)|y(t)|}{\Gamma(\alpha)}\int_0^t \frac{\Phi(|x(s)| + |y(s)|)}{(t-s)^{1-\alpha}}\,ds + \frac{n(t)|f(t,0)|}{\Gamma(\alpha)}\int_0^t \frac{\Phi(|x(s)| + |y(s)|)}{(t-s)^{1-\alpha}}\,ds \\
&\leqslant \frac{2m(t)n(t)r_0\Phi(r_0)}{\Gamma(\alpha)}\int_0^t \frac{ds}{(t-s)^{1-\alpha}} + \frac{m(t)u_1(t)}{\Gamma(\alpha)}\operatorname{diam} X(t)\int_0^t \frac{ds}{(t-s)^{1-\alpha}} \\
&\quad + \frac{m(t)n(t)r_0\Phi(2r_0)}{\Gamma(\alpha)}\int_0^t \frac{ds}{(t-s)^{1-\alpha}} + \frac{n(t)|f(t,0)|\Phi(2r_0)}{\Gamma(\alpha)}\int_0^t \frac{ds}{(t-s)^{1-\alpha}} \\
&= \frac{2a(t)}{\Gamma(\alpha+1)}r_0\Phi(r_0) + \frac{a(t)}{\Gamma(\alpha+1)}r_0\Phi(2r_0) + \frac{c(t)}{\Gamma(\alpha+1)}\Phi(2r_0) + \frac{b(t)}{\Gamma(\alpha+1)}\operatorname{diam} X(t).
\end{aligned}$$

From the above estimate we derive the following inequality:

$$\operatorname{diam}(VX)(t) \leqslant \frac{2a(t)}{\Gamma(\alpha+1)} r_0 \Phi(r_0) + \frac{a(t)}{\Gamma(\alpha+1)} r_0 \Phi(2r_0) + \frac{c(t)}{\Gamma(\alpha+1)} \Phi(2r_0) + \frac{b(t)}{\Gamma(\alpha+1)} \operatorname{diam} X(t).$$

Hence, by assumption (iv) we get

$$\limsup_{t\to\infty} \operatorname{diam}(VX)(t) \leqslant k \limsup_{t\to\infty} \operatorname{diam} X(t), \tag{3.4}$$

where we denoted

$$k = \frac{A\Phi(r_0) + B}{\Gamma(\alpha+1)}.$$

Obviously, in view of assumption (v) we have that $k < 1$.

Further, let us take arbitrary numbers $T > 0$ and $\varepsilon > 0$. Next, fix arbitrarily a function $x \in X$ and $t_1, t_2 \in [0, T]$ such that $|t_1 - t_2| \leqslant \varepsilon$. Without loss of generality we may assume that $t_1 < t_2$. Then, taking into account our assumptions and using the previously obtained estimate (3.2) we get

$$\begin{aligned}
\big|(Vx)(t_2) - (Vx)(t_1)\big| &\leqslant \big|p(t_2) - p(t_1)\big| + \big|(Fx)(t_2)(Ux)(t_2) - (Fx)(t_1)(Ux)(t_2)\big| + \big|(Fx)(t_1)(Ux)(t_2) - (Fx)(t_1)(Ux)(t_1)\big| \\
&\leqslant \omega^T(p,\varepsilon) + \big|f\big(t_2, x(t_2)\big) - f\big(t_1, x(t_1)\big)\big| \cdot \frac{1}{\Gamma(\alpha)} \int_0^{t_2} \frac{|u(t_2,s,x(s))|}{(t_2-s)^{1-\alpha}}\,ds \\
&\quad + \frac{|f(t_1,x(t_1))|}{\Gamma(\alpha+1)} \big\{T^\alpha \omega_1^T(u,\varepsilon;r_0) + 2\varepsilon^\alpha\big[\bar{n}(T)\Phi(r_0) + \overline{u_1}(T)\big]\big\} \\
&\leqslant \omega^T(p,\varepsilon) + \frac{|f(t_2,x(t_2)) - f(t_2,x(t_1))| + |f(t_2,x(t_1)) - f(t_1,x(t_1))|}{\Gamma(\alpha)} \\
&\quad \cdot \int_0^{t_2} \frac{|u(t_2,s,x(s)) - u(t_2,s,0)| + |u(t_2,s,0)|}{(t_2-s)^{1-\alpha}}\,ds \\
&\quad + \frac{|f(t_1,x(t_1)) - f(t_1,0)| + |f(t_1,0)|}{\Gamma(\alpha+1)} \big\{T^\alpha \omega_1^T\big(u,\varepsilon;\|x\|\big) + 3\varepsilon^\alpha\big[\bar{n}(T)\Phi(r_0) + \overline{u_1}(T)\big]\big\} \\
&\leqslant \omega^T(p,\varepsilon) + \frac{m(t_2)|x(t_2) - x(t_1)| + \omega_1^T(f,\varepsilon)}{\Gamma(\alpha)} \int_0^{t_2} \frac{n(t_2)\Phi(|x(s)|) + u_1(t_2)}{(t_2-s)^{1-\alpha}} \\
&\quad + \frac{m(t_1)|x(t_1)| + |f(t_1,0)|}{\Gamma(\alpha+1)} \big\{T^\alpha \omega_1^T\big(u,\varepsilon;\|x\|\big) + 3\varepsilon^\alpha\big[\bar{n}(T)\Phi(r_0) + \overline{u_1}(T)\big]\big\} \\
&\leqslant \omega^T(p,\varepsilon) + \frac{[m(t_2)\omega^T(x,\varepsilon) + \omega_1^T(f,\varepsilon)]t_2^\alpha[n(t_2)\Phi(r_0) + u_1(t_2)]}{\Gamma(\alpha+1)} \\
&\quad + \frac{\overline{m}(T)r_0 + \overline{f}(T)}{\Gamma(\alpha+1)} \big\{T^\alpha \omega_1^T\big(u,\varepsilon;\|x\|\big) + 3\varepsilon^\alpha\big[\bar{n}(T)\Phi(r_0) + \overline{u_1}(T)\big]\big\} \\
&\leqslant \omega^T(p,\varepsilon) + \frac{[m(t_2)n(t_2)t_2^\alpha\Phi(r_0) + m(t_2)u_1(t_2)t_2^\alpha]\omega^T(x,\varepsilon)}{\Gamma(\alpha+1)} \\
&\quad + \frac{\omega_1^T(f,\varepsilon)T^\alpha[\bar{n}(T)\Phi(r_0) + \overline{u_1}(T)]}{\Gamma(\alpha+1)} \\
&\quad + \frac{\overline{m}(T)r_0 + \overline{f}(T)}{\Gamma(\alpha+1)} \big\{T^\alpha \omega_1^T(u,\varepsilon;r_0) + 3\varepsilon^\alpha\big[\bar{n}(T)\Phi(r_0) + \overline{u_1}(T)\big]\big\} \\
&\leqslant \omega^T(p,\varepsilon) + \frac{A\Phi(r_0) + B}{\Gamma(\alpha+1)}\omega^T(x,\varepsilon) + \frac{\omega_1^T(f,\varepsilon)T^\alpha[\bar{n}(T)\Phi(r_0) + \overline{u_1}(T)]}{\Gamma(\alpha+1)} \\
&\quad + \frac{\overline{m}(T)r_0 + \overline{f}(T)}{\Gamma(\alpha+1)} \big\{T^\alpha \omega_1^T(u,\varepsilon;r_0) + 2\varepsilon^\alpha\big[\bar{n}(T)\Phi(r_0) + \overline{u_1}(T)\big]\big\},
\end{aligned} \tag{3.5}$$

where we denoted

$$\omega_1^T(f,\varepsilon) = \sup\big\{\big|f(t_2,x) - f(t_1,x)\big|:\ t_1, t_2 \in [0,T],\ |t_2 - t_1| \leqslant \varepsilon,\ x \in [-r_0, r_0]\big\},$$

$$\overline{m}(T) = \max\big\{m(t):\ t \in [0,T]\big\},$$

$$\bar{f}(T)=\max\{|f(t,0)|:\ t\in[0,T]\}.$$

Moreover, let us mention that other notations used in the above estimate were introduced earlier.

Now, keeping in mind the uniform continuity of the function $u=u(t,s,x)$ on the set $[0,T]\times[0,T]\times[-r_0,r_0]$ and the uniform continuity of the function $f=f(t,x)$ on the set $[0,T]\times[-r_0,r_0]$, from the estimate (3.5) we derive the following one:

$$\omega_0^T(VX)\leqslant k\omega_0^T(X).$$

Hence we get

$$\omega_0(VX)\leqslant k\omega_0(X). \tag{3.6}$$

Now observe, that linking (3.4) and (3.6) and keeping in mind the definition of the measure of noncompactness μ given by the formula (2.1), we derive the following inequality:

$$\mu(VX)\leqslant k\mu(X). \tag{3.7}$$

In the sequel let us put $B_{r_0}^1=\operatorname{Conv}V(B_{r_0})$, $B_{r_0}^2=\operatorname{Conv}V(B_r^1)$ and so on. Observe that $B_{r_0}^1\subset B_{r_0}$, which is a simple consequence of Lemma 1. Further notice that this sequence is decreasing i.e. $B_{r_0}^{n+1}\subset B_{r_0}^n$ for $n=1,2,\ldots$. Also the sets of this sequence are closed, convex and nonempty. Moreover, in view of (3.7) we get

$$\mu(B_{r_0}^n)\leqslant k^n\mu(B_{r_0}),$$

for any $n=1,2,\ldots$. We can also easily calculate that $\mu(B_{r_0})=4r_0$. Combining this fact with the above inequality we obtain

$$\lim_{n\to\infty}\mu(B_{r_0}^n)=0.$$

Hence, taking into account Definition 1 we infer that the set $Y=\bigcap_{n=1}^{\infty}B_{r_0}^n$ is nonempty, bounded, closed and convex. Moreover, the set Y is a member of the kernel $\ker\mu$ of the measure of noncompactness μ (cf. remarks made after Definition 1). In particular we have that

$$\limsup_{t\to\infty}\operatorname{diam}Y(t)=\lim_{t\to\infty}\operatorname{diam}Y(t)=0. \tag{3.8}$$

Let us also observe that the operator V transforms the set Y into itself.

In what follows we show that V is continuous on the set Y.

To prove this let us fix a number $\varepsilon>0$ and take arbitrary functions $x,y\in Y$ such that $\|x-y\|\leqslant\varepsilon$. Using (3.8) and the fact that $VY\subset Y$ we deduce that there exists $T>0$ such that for an arbitrary $t\geqslant T$ we get

$$|(Vx)(t)-(Vy)(t)|\leqslant\varepsilon. \tag{3.9}$$

Further, let us assume that $t\in[0,T]$. Then, applying the imposed assumptions and evaluating similarly as above, we obtain

$$\begin{aligned}|(Vx)(t)-(Vy)(t)|&\leqslant\frac{m(t)|x(t)-y(t)|}{\Gamma(\alpha)}\int_0^t\frac{n(t)\Phi(|x(s)|)+u_1(t)}{(t-s)^{1-\alpha}}\,ds+\frac{[m(t)|y(t)|+|f(t,0)|]n(t)}{\Gamma(\alpha)}\int_0^t\frac{\Phi(|x(s)-y(s)|)}{(t-s)^{1-\alpha}}\,ds\\&\leqslant\frac{[m(t)n(t)\Phi(r_0)+m(t)u_1(t)]\varepsilon}{\Gamma(\alpha)}\int_0^t\frac{ds}{(t-s)^{1-\alpha}}+\frac{[m(t)n(t)r_0+|f(t,0)|n(t)]\Phi(\varepsilon)}{\Gamma(\alpha)}\int_0^t\frac{ds}{(t-s)^{1-\alpha}}\\&=\frac{a(t)\Phi(r_0)+b(t)}{\Gamma(\alpha+1)}\varepsilon+\frac{a(t)r_0+c(t)}{\Gamma(\alpha+1)}\Phi(\varepsilon)\leqslant\frac{A\Phi(r_0)+B}{\Gamma(\alpha+1)}\varepsilon+\frac{Ar_0+C}{\Gamma(\alpha+1)}\Phi(\varepsilon).\end{aligned} \tag{3.10}$$

Now, combining (3.9) and (3.10) and assumption (iv) we conclude that the operator V transforms continuously the set Y into itself.

Finally, let us observe that taking into account all the facts concerning the set Y and the operator $V:Y\to Y$ which were established above, in view of the classical Schauder fixed point principle we deduce that V has at least one fixed point x in the set Y. In view of Lemma 1 the function $x=x(t)$ is a solution of the quadratic fractional integral equation (3.1). Moreover, keeping in mind the fact that $Y\in\ker\mu$ and the characterization of sets belonging to $\ker\mu$ (cf. remarks made after formula (2.1) defining the measure μ) we conclude that all solutions of Eq. (3.1) are uniformly locally attractive in the sense of Definition 2. This completes the proof. □

4. An example and remarks

In this section we provide an example illustrating the main existence result contained in Theorem 1.

Example. Consider the following quadratic Volterra integral equation of fractional order:

$$x(t) = te^{-t^2/2} + \frac{t + t^2 x(t)}{\Gamma(2/3)} \int_0^t \frac{e^{-3t-s}\sqrt[3]{x^2(s)} + \frac{1}{10t^{8/3}+1}}{(t-s)^{1/3}}\, ds, \tag{4.1}$$

where $t \in \mathbb{R}_+$.

Observe that the above equation is a special case of Eq. (3.1). Indeed, if we put $\alpha = 2/3$ and

$$p(t) = te^{-t^2/2},$$
$$f(t,x) = t + t^2 x,$$
$$u(t,s,x) = e^{-3t-s}\sqrt[3]{x^2} + \frac{1}{10t^{8/3}+1},$$

then we can easily check that the assumptions of Theorem 1 are satisfied. In fact, we have that the function $p(t)$ is continuous and bounded on $\mathbb{R}_+$. Moreover, $\|p\| = p(1) = e^{-1/2} = 0.60653\ldots$. Thus assumption (i) is satisfied.

Further observe that the function $f(t,x)$ satisfies assumption (ii) with $m(t) = t^2$ and $|f(t,0)| = f(t,0) = t$. Next, let us notice that the function $u(t,s,x)$ satisfies assumption (iii), where $n(t) = e^{-3t}$, $\Phi(r) = \sqrt[3]{r^2}$ and $u(t,s,0) = 1/(10t^{8/3}+1)$. Thus $u_1(t) = u(t,s,0)$. To check that assumption (iv) is satisfied let us observe that the functions a, b, c, d appearing in that assumption take the form:

$$a(t) = t^{8/3}e^{-3t},$$
$$b(t) = t^{8/3}/\big(10t^{8/3}+1\big),$$
$$c(t) = t^{5/3}e^{-3t},$$
$$d(t) = t^{5/3}/\big(10t^{8/3}+1\big).$$

Thus, it is easily seen that $a(t) \to 0$ as $t \to \infty$ and $A = a(8/9) = (8/9)^{8/3}e^{-8/3} = 0.0507543\ldots$. Further we have that the function $b(t)$ is bounded on $\mathbb{R}_+$ and $B = 0.1$. It is also easy to check that $c(t) \to 0$ as $t \to \infty$. Moreover, we have that $C = c(5/9) = (5/9)^{5/3}e^{-5/3} = 0,0709235\ldots$. Also we see that $d(t) \to 0$ as $t \to \infty$ and $D = d((1/6)^{3/8}) = 0.1223733\ldots$.

Finally, let us note that the inequality from assumption (v) has the form

$$e^{-1/2} + \frac{1}{\Gamma(5/3)}\big[Ar^{5/3} + Br + Cr^{2/3} + D\big] \leqslant r.$$

Let us write this inequality in the form

$$\Gamma(5/3)e^{-1/2} + Ar^{5/3} + Br + Cr^{2/3} + D \leqslant r\Gamma(5/3). \tag{4.2}$$

Denoting by $L(r)$ the left-hand side of this inequality, i.e.

$$L(r) = \Gamma(5/3)e^{-1/2} + Ar^{5/3} + Br + Cr^{2/3} + D$$

and keeping in mind the above established values of the constants A, B, C, D, for $r = 1$ we obtain

$$L(1) = \Gamma(5/3)e^{-1/2} + A + B + C + D = \Gamma(5/3)0.60653\ldots + 0.344051\ldots.$$

Hence, taking into account that $\Gamma(5/3) > 0.8856$ (cf. [9]), we obtain that the number $r_0 = 1$ is a solution of the inequality (4.2).

Now, based on Theorem 1 we infer that Eq. (4.1) has a solution in the space $BC(\mathbb{R}_+)$ belonging to the ball B_1. Moreover, solutions of Eq. (4.1) are uniformly locally attractive in the sense of Definition 2. That means that for arbitrary solutions $x(t)$ and $y(t)$ of Eq. (4.1) belonging to B_1 we have that

$$\lim_{t\to\infty}\big(x(t) - y(t)\big) = 0$$

uniformly with respect to the ball B_1.

In what follows we compare the result contained in Theorem 1 with the result on existence and global attractivity of solutions contained in the paper [12]. First of all let us observe that the result of Theorem 1 remains true if we take the limit case of Eq. (3.1) with $\alpha = 1$.

Further, let us notice that in the paper [12] the authors studied, among other, the following quadratic Volterra integral equation:

$$x(t) = g\big(t, x(t)\big) + x(t) \int_0^t u\big(t, s, x(s)\big)\, ds, \tag{4.3}$$

where $t \in \mathbb{R}_+$.

To simplify our considerations we will investigate a special case of Eq. (4.3) having the form

$$x(t) = x(t) \int_0^t u\big(t, s, x(s)\big)\, ds. \tag{4.4}$$

We note that Eq. (4.4) is a special limit case of Eq. (3.1), where $f(t,x) = x$ and $\alpha = 1$.

Observe that the main assumption concerning Eq. (4.4) imposed in [12] is

$$\lim_{t\to\infty} \int_0^t \big|u\big(t, s, x(s)\big)\big|\, ds = 0 \tag{4.5}$$

uniformly with respect to $x \in BC(\mathbb{R}_+)$.

On the other hand the essential part of assumptions (iii) and (iv) formulated in Theorem 1 and adapted to Eq. (4.4) (where $m(t) \equiv 1$, $\alpha = 1$) has the form

$$\big|u(t,s,x) - u(t,s,y)\big| \leqslant n(t)\Phi\big(|x-y|\big),$$

where $n : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function and $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous and nondecreasing function. Moreover, we require additionally that $n(t) \to 0$ as $t \to \infty$ and the function $u_1(t) = \max\{|u(t,s,0)|\colon\ 0 \leqslant s \leqslant t\}$ is bounded on $\mathbb{R}_+$.

Observe that these assumptions do not imply that condition (4.5) is satisfied. Indeed, we have

$$\int_0^t \big|u\big(t, s, x(s)\big)\big|\, ds \leqslant tn(t)\Phi\big(\|x\|\big) + tu_1(t).$$

In the light of the assumptions formulated in Theorem 1 and mentioned above we cannot infer that condition (4.5) is satisfied.

This shows that the result obtained in this paper is more general than the result from [12].

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