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An extension of a Bourgain–Lindenstrauss–Milman inequality

Omer Friedland, Sasha Sodin *

School of Mathematical Sciences, Tel Aviv University, Ramat Aviv, Tel Aviv 69978, Israel

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Abstract

Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$. Averaging $\|(\varepsilon_1 x_1, \ldots, \varepsilon_n x_n)\|$ over all the 2^n choices of $\vec{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_n) \in \{-1, +1\}^n$, we obtain an expression $|||x|||$ which is an unconditional norm on $\mathbb{R}^n$. Bourgain, Lindenstrauss and Milman [J. Bourgain, J. Lindenstrauss, V.D. Milman, Minkowski sums and symmetrizations, in: Geometric Aspects of Functional Analysis (1986/1987), Lecture Notes in Math., vol. 1317, Springer, Berlin, 1988, pp. 44–66] showed that, for a certain (large) constant $\eta > 1$, one may average over ηn (random) choices of $\vec{\varepsilon}$ and obtain a norm that is isomorphic to $|||\cdot|||$. We show that this is the case for any $\eta > 1$.

Keywords: Bourgan–Lindenstrauss–Milman inequality; Kahane–Khinchin averages; Unconditional

1. Introduction

Let $(E, \|\cdot\|)$ be a normed space, and let $v_1, \ldots, v_n \in E \setminus \{0\}$. Define a norm $|||\cdot|||$ on $\mathbb{R}^n$ as follows:

$$|||x||| = \mathbb{E}\left\|\sum \varepsilon_i x_i v_i\right\|, \tag{1}$$

* Corresponding author.
E-mail addresses: omerfrie@post.tau.ac.il (O. Friedland), sodinale@post.tau.ac.il (S. Sodin).

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where the expectation is over the choice of n independent random signs $\varepsilon_1, \ldots, \varepsilon_n$. This is an *unconditional* norm; that is,

$$\left|\!\left|\!\left|(x_1, x_2, \ldots, x_n)\right|\!\right|\!\right| = \left|\!\left|\!\left|\big(|x_1|, |x_2|, \ldots, |x_n|\big)\right|\!\right|\!\right|.$$

The following theorem states that it is sufficient to average $O(n)$, rather than 2^n, terms in (1), in order to obtain a norm that is isomorphic to $|||\cdot|||$ (and in particular approximately unconditional).

Theorem. *Let $N = (1+\xi)n$, $\xi > 0$, and let*

$$\{\varepsilon_{ij} \mid 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant N\}$$

be a collection of independent random signs. Then

$$\mathbb{P}\left\{\forall x \in \mathbb{R}^n\ c(\xi)\, |||x||| \leqslant \frac{1}{N}\sum_{j=1}^{N}\left\|\sum_{i=1}^{n}\varepsilon_{ij}x_i v_i\right\| \leqslant C(\xi)|||x|||\right\} \geqslant 1 - e^{-c'\xi n},$$

where

$$c(\xi) = \begin{cases} c\xi^2, & 0 < \xi < 1, \\ c, & 1 \leqslant \xi < C'', \\ 1 - C'/\xi^2, & C'' \leqslant \xi, \end{cases} \qquad C(\xi) = \begin{cases} C, & 0 < \xi < C'', \\ 1 + C''/\xi^2, & C'' \leqslant \xi, \end{cases}$$

and $c, c', C, C', C'' > 0$ are universal constants (such that $1 - C'/C''^2 \geqslant c$, $1 + C'/C''^2 \leqslant C$).

This extends a result due to Bourgain, Lindenstrauss and Milman [3], who considered the case of large ξ ($\xi \geqslant C''$); their proof makes use of the Kahane–Khinchin inequality. Their argument yields the upper bound for the full range of ξ, so the innovation is in the lower bound for small ξ.

With the stated dependence on ξ, the corresponding result for the scalar case $\dim E = 1$ was proved by Rudelson [6], improving previous bounds on $c(\xi)$ in [1,2,4]; see below. This is one of the two main ingredients of our proof, the second one being Talagrand's concentration inequality [8] (which, as shown by Talagrand, also implies the Kahane–Khinchin inequality).

2. Proof of Theorem

Let us focus on the case $\xi < 1$; the same method works (in fact, in a simpler way) for $\xi \geqslant 1$.

Denote $|||x|||_N = \frac{1}{N}\sum_{j=1}^{N}\|\sum_{i=1}^{n}\varepsilon_{ij}x_i v_i\|$; this is a random norm depending on the choice of ε_{ij}. Let $S^{n-1}_{|||\cdot|||} = \{x \in \mathbb{R}^n\colon\ |||x||| = 1\}$ be the unit sphere of $(\mathbb{R}^n, |||\cdot|||)$; we estimate

$$\begin{aligned} &\mathbb{P}\big\{\forall x \in S^{n-1}_{|||\cdot|||},\ c\xi^2 \leqslant |||x|||_N \leqslant C\big\} \\ &\quad\geqslant 1 - \mathbb{P}\big\{\exists x \in S^{n-1}_{|||\cdot|||},\ |||x|||_N > C\big\} \\ &\qquad - \mathbb{P}\big\{\big(\forall y \in S^{n-1}_{|||\cdot|||},\ |||y|||_N \leqslant C\big) \wedge \big(\exists x \in S^{n-1}_{|||\cdot|||}, |||x|||_N < c\xi^2\big)\big\}. \end{aligned} \tag{2}$$

Upper bound. Let us estimate the first term

$$\mathbb{P}\left\{\exists x \in S^{n-1}_{|||\cdot|||},\ |||x|||_N > C\right\}.$$

Remark. As we mentioned, the needed estimate follows from the argument in [3]; for completeness, we reproduce a proof in the similar spirit.

Theorem. *(See Talagrand [8].) Let $w_1, \dots, w_n \in E$ be vectors in a normed space $(E, \|\cdot\|)$, and let $\varepsilon_1, \dots, \varepsilon_n$ be independent random signs. Then for any $t > 0$*

$$\mathbb{P}\left\{\left|\left\|\sum_{i=1}^n \varepsilon_i w_i\right\| - \mathbb{E}\left\|\sum_{i=1}^n \varepsilon_i w_i\right\|\right| \geqslant t\right\} \leqslant C_1 e^{-c_1 t^2/\sigma^2}, \tag{3}$$

where $c_1, C_1 > 0$ are universal constants, and

$$\sigma^2 = \sigma^2(w_1, \dots, w_n) = \sup\left\{\sum_{i=1}^n \varphi(w_i)^2 \;\middle|\; \varphi \in E^*,\ \|\varphi\|^* \leqslant 1\right\}.$$

Remark. Talagrand has proved (3) with the median $\mathrm{Med}\,\|\sum_{i=1}^n \varepsilon_i w_i\|$ rather than the expectation; one can however replace the median by the expectation according to the proposition in Milman and Schechtman [5, Appendix V].

For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, denote

$$\sigma^2(x) = \sigma^2(x_1 v_1, \dots, x_n v_n).$$

Claim 1. *σ is a norm on $\mathbb{R}^n$ and $\sigma(x) \leqslant C_2 |||x|||$ for any $x \in \mathbb{R}^n$.*

Proof. The first statement is trivial. For the second one, note that

$$|||x||| = \mathbb{E}\left\|\sum \varepsilon_i x_i v_i\right\| \geqslant \mathbb{E}\left|\varphi\left(\sum \varepsilon_i x_i v_i\right)\right| = \mathbb{E}\left|\sum \varepsilon_i \varphi(x_i v_i)\right|, \quad \|\varphi\|^* \leqslant 1.$$

Now, by the classical Khinchin inequality,

$$\sqrt{\sum y_i^2} \geqslant \mathbb{E}\left|\sum \varepsilon_i y_i\right| \geqslant C_2^{-1}\sqrt{\sum y_i^2} \tag{4}$$

(see Szarek [7] for the optimal constant $C_2 = \sqrt{2}$). Therefore

$$|||x||| \geqslant C_2^{-1} \sup_{\|\varphi\|^* \leqslant 1} \sqrt{\sum \varphi(x_i v_i)^2} = C_2^{-1}\sigma(x). \qquad \square$$

By the claim and Talagrand's inequality, for every (fixed) $x \in S^{n-1}_{|||\cdot|||}$

$$\mathbb{P}\left\{\left\|\sum_{i=1}^n \varepsilon_i x_i\right\| \geqslant t\right\} \leqslant C_1 \exp\left(-c_2 t^2\right).$$

Together with a standard argument (based on the exponential Chebyshev inequality), this implies (for t large enough)

$$\mathbb{P}\left\{\frac{1}{N}\sum_{j=1}^{N}\left\|\sum_{i=1}^{n}\varepsilon_{ij}x_i\right\| \geqslant t\right\} \leqslant \exp\bigl(-c_3t^2N\bigr).$$

In particular, for $t = C_3 \geqslant \sqrt{4/c_3}$ the left-hand side is smaller than $12^{-N} < 6^{-n}2^{-N}$.

The following fact is well known, and follows for example from volume estimates (cf. [5]).

Claim 2. *For any $\theta > 0$, there exists a θ-net $\mathcal{N}_\theta$ with respect to $\|\!|\cdot|\!\|$ on $S^{n-1}_{\|\!|\cdot|\!\|}$ of cardinality $\#\mathcal{N}_\theta \leqslant (3/\theta)^n$.*

For now we only use this for $\theta = 1/2$. By the above, with probability greater than $1 - 2^{-N}$, we have: $\|\!|x|\!\|_N \leqslant C_3$ simultaneously for all $x \in \mathcal{N}_{1/2}$.

Representing an arbitrary unit vector $x \in S^{n-1}_{\|\!|\cdot|\!\|}$ as

$$x = \sum_{k=1}^{\infty} a_k x^{(k)}, \quad |a_k| \leqslant 1/2^{k-1},\ x^{(k)} \in \mathcal{N}_{1/2},$$

we deduce: $\|\!|x|\!\|_N \leqslant 2C_3$, and hence finally,

$$\mathbb{P}\bigl\{\exists x \in S^{n-1}_{\|\!|\cdot|\!\|},\ \|\!|x|\!\|_N > C\bigr\} \leqslant 2^{-N} \tag{5}$$

(for $C = 2C_3$).

Lower bound. Now we turn to the second term

$$\mathbb{P}\bigl\{\bigl(\forall y \in S^{n-1}_{\|\!|\cdot|\!\|}, \|\!|y|\!\|_N \leqslant C\bigr) \wedge \bigl(\exists x \in S^{n-1}_{\|\!|\cdot|\!\|}, \|\!|x|\!\|_N < c\xi^2\bigr)\bigr\}.$$

For σ_0 (that we choose later), let us decompose $S^{n-1}_{\|\!|\cdot|\!\|} = U \uplus V$, where

$$U = \bigl\{x \in S^{n-1}_{\|\!|\cdot|\!\|} \bigm| \sigma(x) \geqslant \sigma_0\bigr\}, \qquad V = \bigl\{x \in S^{n-1}_{\|\!|\cdot|\!\|} \bigm| \sigma(x) < \sigma_0\bigr\}.$$

Recall the following result (mentioned in the introduction); we use the lower bound that is due to Rudelson [6].

Theorem. *(See [1,2,4,6].) Let $N = (1+\xi)n$, $0 < \xi < 1$, and let*

$$\{\varepsilon_{ij} \mid 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant N\}$$

be a collection of independent random signs. Then

$$\mathbb{P}\left\{\forall y \in \mathbb{R}^n,\ c_4\xi^2\,|y| \leqslant \frac{1}{N}\sum_{j=1}^{N}\left|\sum_{i=1}^{n}\varepsilon_{ij}y_i\right| \leqslant C_4|y|\right\} \geqslant 1 - e^{-c_4'\xi n},$$

where $c_4, c_4', C_4 > 0$ are universal constants, and $|\cdot|$ is the standard Euclidean norm.

Remark. By the Khinchin inequality (4), this is indeed the scalar case of Theorem 1 for $0 < \xi < 1$.

Thence with probability $\geqslant 1 - e^{-c_4'\xi n}$ the following inequality holds for all $x \in U$ (simultaneously):

$$\begin{aligned} |||x|||_N &\geqslant \sup_{\|\varphi\|^* \leqslant 1} \frac{1}{N} \sum_{j=1}^N \left| \varphi\left(\sum_{i=1}^n \varepsilon_{ij} x_i v_i \right) \right| = \sup_{\|\varphi\|^* \leqslant 1} \frac{1}{N} \sum_{j=1}^N \left| \sum_{i=1}^n \varepsilon_{ij} \varphi(x_i v_i) \right| \\ &\geqslant \sup_{\|\varphi\|^* \leqslant 1} c_4 \xi^2 \sqrt{\sum_{i=1}^n \varphi(x_i v_i)^2} = c_4 \xi^2 \sigma(x) \geqslant c_4 \xi^2 \sigma_0. \end{aligned} \tag{6}$$

Now let us deal with vectors $x \in V$. Let $\mathcal{N}_\theta$ be a θ-net on $S^{n-1}_{|||\cdot|||}$ (where θ will be also chosen later). For $x' \in \mathcal{N}_\theta$ such that $|||x - x'||| \leqslant \theta$, $\sigma(x') \leqslant \sigma_0 + C_2\theta$ by Claim 1. Therefore by Talagrand's inequality (3),

$$\mathbb{P}\left\{ \left\| \sum_{i=1}^n \varepsilon_i x_i' v_i \right\| < 1/2 \right\} \leqslant C_1 \exp\bigl(-c_1/\bigl(4(\sigma_0 + C_2\theta)^2\bigr)\bigr),$$

and hence definitively

$$\begin{aligned} \mathbb{P}\left\{ \frac{1}{N} \sum_{j=1}^N \left\| \sum_{i=1}^n \varepsilon_{ij} x_i' v_i \right\| < 1/4 \right\} &\leqslant 2^N \left\{ C_1 \exp\left(-\frac{c_1}{4(\sigma_0 + C_2\theta)^2} \right) \right\}^{N/2} \\ &= \exp\left\{ -\left(\frac{c_1}{8(\sigma_0 + C_2\theta)^2} - \log(2\sqrt{C_1}) \right) N \right\}. \end{aligned}$$

Let $\sigma_0 = C_2\theta$, and choose $0 < \theta < 1/(8C)$ so that

$$\frac{c_1}{32 C_2^2 \theta^2} - \log(2\sqrt{C_1}) > \log 2 + \log(3/\theta).$$

Then the probability above is not greater than $2^{-N}(\theta/3)^N < 2^{-N}/\#\mathcal{N}_\theta$ (by Claim 2). Therefore with probability $\geqslant 1 - 2^{-N}$ we have

$$|||x'|||_N \geqslant 1/4 \quad \text{for } x' \in \mathcal{N}_\theta \text{ such that } |||x - x'||| < \theta \text{ for some } x \in V.$$

Using the upper bound (5), we infer:

$$\begin{aligned} |||x|||_N &\geqslant |||x'|||_N - |||x' - x|||_N \geqslant 1/4 - C/8C \\ &= 1/4 - 1/8 = 1/8, \quad x \in V. \end{aligned} \tag{7}$$

The juxtaposition of (2) and (5)–(7) concludes the proof.

Remark. Another way to conclude the proof for $x \in V$ would be to choose a θ-net $\mathcal{N}_\theta \subset V$ for V such that $\#\mathcal{N}_\theta \leqslant (3/\theta)^n$. This simplifies the condition on σ_0 and θ and avoids the use of Claim 1; on the other hand, the argument above illustrates the robustness of the method.

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