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Generalising connected components

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ABSTRACT

Let $I : \mathbb{C} \to \mathbb{M}$ be a reflection of a category $\mathbb{C}$ with pullbacks into a full subcategory $\mathbb{M}$ of $\mathbb{C}$. We introduce an additional structure on $\mathbb{C}$ involving a pullback-preserving functor $U : \mathbb{C} \to \mathbb{S}$, which allows us to prove that the reflection I is: (a) semi-left-exact if and only if it makes all connected components connected in an appropriate sense; (b) a reflection with stable units if and only if certain pullbacks of connected components are connected. This was previously done in the case where $\mathbb{S}$ is the category of sets.

1. Introduction

Semi-left-exact reflections and reflections with stable units were originally introduced by Cassidy et al. [1] as reflections that preserve certain pullbacks. We describe an additional structure on a reflection $I : \mathbb{C} \to \mathbb{M}$, involving a pullback-preserving functor $U : \mathbb{C} \to \mathbb{S}$, which allows us to simplify these preservation conditions by reducing them to the preservation of very special pullbacks (see Theorems 2.1 and 2.2 below). Furthermore, we show under stronger assumptions that every simple reflection, in the sense of [1] again, is semi-left-exact (Theorem 2.3). As suggested by topological examples, and in agreement with the terminology of categorical Galois theory, the pullbacks used in Theorem 2.1 are called connected components (see also Definition 2.1), while Theorem 2.2 in fact uses their pullbacks. For the special case where $\mathbb{S}$ is the category of sets, these results were obtained in [3], and applied to reflections of: (i) varieties of universal algebras into subvarieties of idempotent algebras; (ii) (the category of) compact Hausdorff spaces into Stone spaces. The case considered here allows us to find many other examples, including the reflection of categories into preorders and orders, as briefly shown at the end of the paper.

2. Ground structure and connected components

Consider an adjunction $H \vdash I : \mathbb{C} \to \mathbb{M}$, with unit $\eta : 1_{\mathbb{C}} \to HI$, such that the category $\mathbb{C}$ has pullbacks and the right adjoint H is a full inclusion of $\mathbb{M}$ in $\mathbb{C}$, that is, I is a reflection of a category with pullbacks into a full subcategory. Our ground structure also involves a category $\mathbb{S}$ equipped with a class $\mathcal{E}$ of its morphisms, a functor $U : \mathbb{C} \to \mathbb{S}$, and a class $\mathcal{T}$ of objects in $\mathbb{M}$. We will assume that this structure satisfies the following conditions:

(a) U preserves pullbacks;
(b) $\mathcal{E}$ is pullback stable and closed under composition in $\mathbb{S}$, and if $f'f$ is in $\mathcal{E}$ so is f', provided f is in $\mathcal{E}$;[1]
(c) every map $U(\eta_C) : U(C) \to UHI(C)$ belongs to $\mathcal{E}$, $C \in \mathbb{C}$;
(d) a morphism $g : N \to M$ in $\mathbb{M}$ is an isomorphism whenever $UH(g)$ is in $\mathcal{E}$ and there exists $f : A \to UH(N)$ in $\mathcal{E}$ such that, for every morphism $c : T \to M$ in $\mathbb{M}$ with T in $\mathcal{T}$, there exists a commutative diagram of the form

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[1] This is the case if $\mathcal{E}$ is the left-hand class of a stable (pre)factorisation system (in the sense of [1]).

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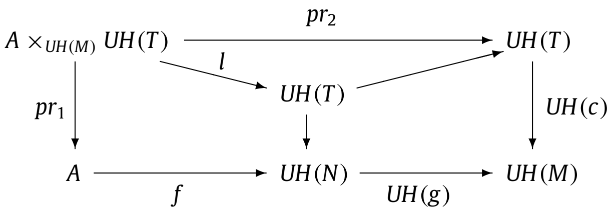
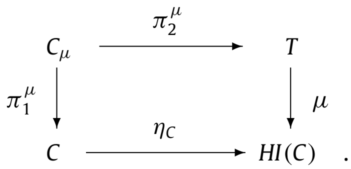


(1)

with l in $\mathcal{E}$.

Definition 2.1. Consider any morphism $\mu : T \to HI(C)$ from an object $T \in \mathcal{T}$ into $HI(C)$, for some $C \in \mathbb{C}$. The connected component of the morphism μ is the pullback $C_\mu = C \times_{HI(C)} T$ in the following pullback square:

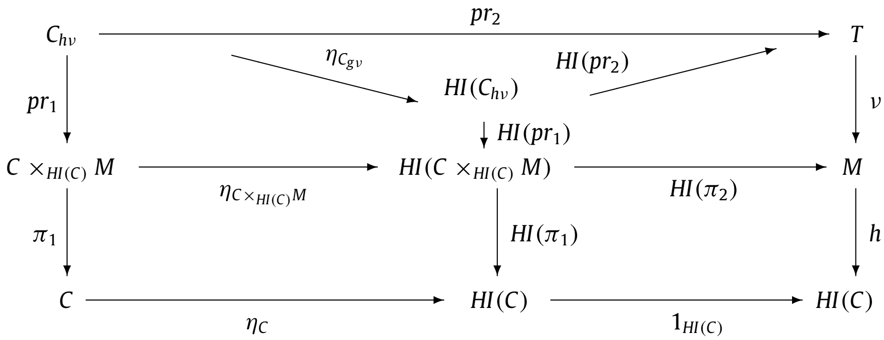


(2)

Theorem 2.1. *$I \dashv H$ is semi-left-exact if and only if $HI(C_\mu) \cong T$, for every connected component C_μ.*

Proof. If $I \dashv H$ is semi-left-exact, then $I(C \times_{HI(C)} M)$ must be isomorphic to $I(M)$ in the pullback square of the unit morphism $\eta_C : C \to HI(C)$ with a morphism $h : M \to HI(C)$, where M is in $\mathbb{M}$. In particular, $I(C \times_{HI(C)} M) \cong I(T)$ if $M \cong T$.

Suppose now that every connected component is connected, that is, $I(C_\mu) \cong T$ for every $\mu : T \to HI(C)$, $C \in \mathbb{C}$, $T \in \mathcal{T}$, and consider the diagram

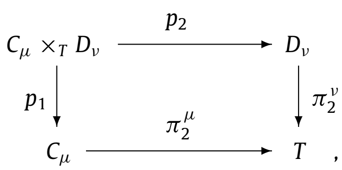


(3)

where π_1, π_2, pr_1, and pr_2 are suitable pullback projections, and h and ν arbitrary morphisms in $\mathbb{M}$ with T in $\mathcal{T}$. We have to show that $HI(\pi_2)$ is an isomorphism.

The upper rectangle $\nu pr_2 = HI(\pi_2)\eta_{C \times_{HI(C)} M} pr_1$ in diagram (3) is a pullback square; therefore the outer rectangle in diagram (3) is in fact a pullback square of the form of (2), and $C_{h\nu}$ is the connected component associated with $h\nu : T \to HI(C)$. Then, as conditions (a)–(d) above hold, $I(\pi_2)$ is an isomorphism since $HI(C_{h\nu}) \cong T$, for any morphisms $h : M \to HI(C)$, with $M \in \mathbb{M}$, and $\nu : T \to M$, with $T \in \mathcal{T}$. □

Theorem 2.2. *$I \dashv H$ has stable units if and only if $HI(C_\mu \times_T D_\nu) \cong T$, for every pair of connected components C_μ, D_ν, and $T \in \mathcal{T}$, where $C_\mu \times_T D_\nu$ $(= C \times_{(\eta_C, \mu)} T \times_{(\nu, \eta_D)} D)$ is the pullback object in any pullback of the form*

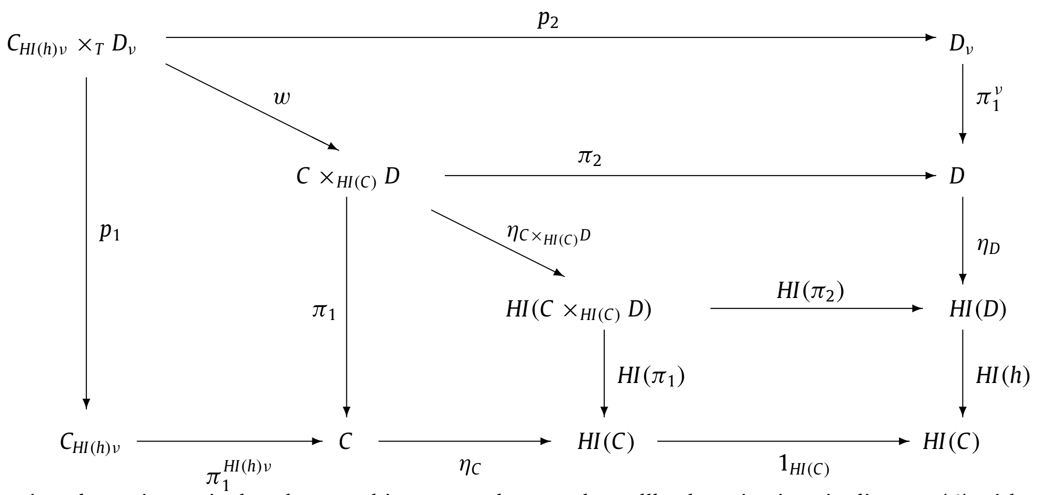


(4)

where π_2^μ and π_2^ν are the second projections in pullback diagrams of the form (2).

Proof. If $I \dashv H$ has stable units then the functor I preserves the pullback diagrams (4), since the right corner at the bottom is $T \in \mathbb{M}$; therefore, $HI(C_\mu \times_T D_\nu) \cong T$ because $HI(\pi_2^\mu) : HI(C_\mu) \cong T \cong HI(D_\nu) : HI(\pi_2^\nu)$, for every pair of connected components C_μ, D_ν.

Suppose now that every pullback (4) of two connected components with respect to the same object T is connected, that is, $HI(C_\mu \times_T D_\nu) \cong T$ for every pair of morphisms $\mu : T \to HI(C)$ and $\nu : T \to HI(D)$, with $C, D \in \mathbb{C}$, and $T \in \mathcal{T}$, and consider the diagram

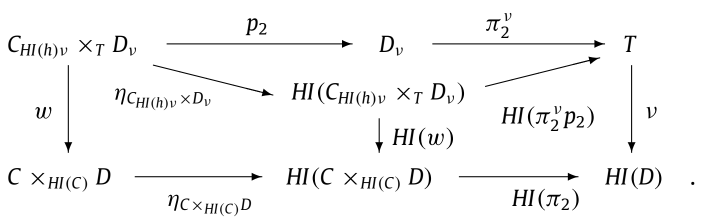


(5)

in the notation above; in particular, the morphisms p_1 and p_2 are the pullback projections in diagram (4) with $\mu = HI(h)\nu$. We have to prove again that $HI(\pi_2)$ is an isomorphism. The morphism w is the unique morphism which makes diagram (5) commute; it is well defined since

$$HI(h)\eta_D\pi_1^\nu p_2 = HI(h)\nu\pi_2^\nu p_2 = HI(h)\nu\pi_2^{HI(h)\nu}p_1 = \eta_C\pi_1^{HI(h)\nu}p_1.$$

$UHI(\pi_2)$ is obviously in $\mathcal{E}$, for $UHI(\pi_2)U(\eta_{C\times_{HI(C)}D}) = U(\eta_D)U(\pi_2)$ and $U(\eta_{C\times_{HI(C)}D})$, and $U(\eta_D)$ and $U(\pi_2)$ are all in $\mathcal{E}$ by the assumptions. Then, $I(\pi_2)$ is an isomorphism if the outer rectangle in the following diagram is a pullback square, for every morphism $\nu : T \to HI(D)$, $T \in \mathcal{T}$ (cf. diagram (1)):



(6)

Proving that the outer rectangle in diagram (6) is a pullback square is a straightforward calculation. □

Theorem 2.3. *In addition to conditions* (a)–(d) *used above, assume that:*

(e) *the map* $I_{T,C} : \mathbb{C}(T, C) \to \mathbb{M}(T, I(C))$ *is a surjection, for all objects* $C \in \mathbb{C}$ *and* $T \in \mathcal{T}$.

Then the reflection $I \dashv H$ *is semi-left-exact if and only if it is simple.*

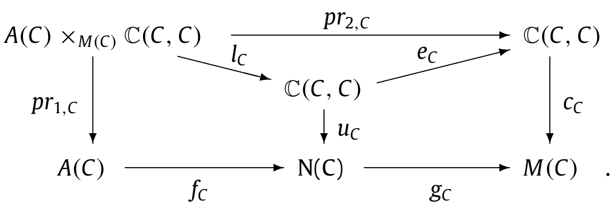
Proof. Suppose that $I \dashv H$ is a simple reflection, that is, $I(w)$ is an isomorphism in every pullback diagram of a unit morphism $\eta_B : B \to HI(B)$ with $HI(f) : HI(A) \to HI(B)$, where w is $\langle f, \eta_A\rangle : A \to B \times_{HI(B)} HI(A)$. Consider the pullback square (2) in Definition 2.1, and let $w : T \to C_\mu$ be the unique morphism such that $\pi_1^\mu w = \nu$ and $\pi_2^\mu w = 1_T$, where ν is such that $HI(\nu) = \mu$ (ν exists by (e) in the statement). The composite $I(\pi_2^\mu)I(w)$ is the isomorphism 1_T. Hence, $I(\pi_2^\mu)$ is an isomorphism, and we can apply Theorem 2.1. □

New examples:

1. Let $I \dashv H$ be any reflection from the category $\hat{\mathbb{C}} = \mathbf{Set}^{\mathbb{C}^{op}}$ of presheaves into a full subcategory $\mathbb{M}$, such that its unit $\eta : 1_{\hat{\mathbb{C}}} \to HI$ is a surjection componentwise. The functor U is the identity functor $1_{\hat{\mathbb{C}}} : \hat{\mathbb{C}} \to \mathbb{S} = \hat{\mathbb{C}}$, and $\mathcal{E}$ is the class of morphisms which are surjections componentwise. If, furthermore, every hom-functor $\mathbb{C}(-, C)$ is in $\mathbb{M}$,[2] $\mathbb{C}(C, C)$ is a finite

[2] It is known that, provided $\mathbb{X}$ is any cocomplete category, any faithful functor $T : \mathbb{C} \to \mathbb{X}$ determines an adjunction $G \vdash F : \hat{\mathbb{C}} \to \mathbb{X}$ such that the $(\mathcal{E}, \mathcal{M})$-factorisation of its unit $\varphi = \mu\eta$ produces a full reflection $H \vdash I : \hat{\mathbb{C}} \to \mathbb{M}$, with $\mathbb{C}(-, C) \in \mathbb{M}$ for each $C \in \mathbb{C}$ (see [2]), where $\mathbb{M}$ is determined by the presheaves $S : \mathbb{C}^{op} \to \mathbf{Set}$ for which φ_S belongs to $\mathcal{M}$, the class of morphisms which are injections componentwise.

set for each $C \in \mathbb{C}$ (e.g., $\mathbb{C} = \Delta$ the category of positive ordinals), and $\mathcal{T} = \{\mathbb{C}(-, C) \mid C \in \mathbb{C}\}$, then conditions (a), (b) and (c) obviously hold. Condition (d) holds as well, as we are going to show. The following diagram is a pointwise instance of diagram (1) of condition (d), as regards the present example, where c is determined by $c_C(1_C) = m \in M(C)$:



Suppose that g_C is not an injective map, that is, $g_C(n_1) = m = g_C(n_2)$ with $n_1 \neq n_2$. Then, there will be distinct $a_1, a_2 \in A(C)$ such that $n_1 = f_C(a_1)$, $n_2 = f_C(a_2)$, and $u_C l_C(a_1, 1_C) = n_1$, $u_C l_C(a_2, 1_C) = n_2$, which implies $l_C(a_1, 1_C) \neq l_C(a_2, 1_C)$ and $e_C(l_C(a_1, 1_C)) = e_C(l_C(a_2, 1_C)) = 1_C$. This is a contradiction, because e_C being a surjection between finite sets must also be an injection.

Note further that such a reflection $H \vdash I : \hat{\mathbb{C}} \to \mathbb{M}$ is semi-left-exact if and only if it is simple, as follows from Theorem 2.3: the map $I_{\mathbb{C}(-,C),A} : \hat{\mathbb{C}}(\mathbb{C}(-, C), A) \to \mathbb{M}(\mathbb{C}(-, C), I(A))$ is a surjection, by the Yoneda lemma.

2. Parentheses will be used to describe various obvious reflections, which satisfy conditions (a)–(d) above. $H \vdash I$: **Cat** $\to$ (**Pre**)**Ord** is a reflection from the category of small categories into the category of (pre)ordered sets, such that $I(\mathbb{C})$ is the (pre)order in which all morphisms in the same hom-set, and, just in the case $\mathbb{M} =$ **Ord**, objects A, B for which both hom-sets $\mathbb{C}(A, B)$ and $\mathbb{C}(B, A)$ are non-empty, are identified. U : **Cat** $\to$ (**R**)**Graphs** is the forgetful functor into the category of (reflexive) graphs. $\mathcal{T} = \{T\}$, where $T =$ **2** is the (pre)order with two objects and one non-identity morphism. $\mathcal{E}$ is the class of all graph morphisms which are simultaneously surjections on the nodes and on the arrows.

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