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A Test for the Independence of Two Gaussian Processes

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A bivariate Gaussian process with mean **0** and covariance

$$\Sigma(s,t,\rho)=\begin{pmatrix}\Sigma_{11}(s,t) & \rho\Sigma_{12}(s,t)\\ \rho\Sigma_{21}(s,t) & \Sigma_{22}(s,t)\end{pmatrix}$$

is observed in some region Ω of R^r, where $\{\Sigma_{ij}(s,t)\}$ are given functions and ρ an unknown parameter. A test of $H_0:\rho=0$, locally equivalent to the likelihood ratio test, is given for the case when Ω consists of p points. An unbiased estimate of ρ is given. The case where Ω has positive (but finite) Lebesgue measure is treated by spreading the p points evenly over Ω and letting $p\to\infty$. Two distinct cases arise, depending on whether $\Lambda_{2,p}$, the sum of squares of the canonical correlations associated with $\Sigma(s,t,1)$ on Ω^2, remains bounded. In the case of primary interest as $p\to\infty$, $\Lambda_{2,p}\to\infty$, in which case $\hat{\rho}$ converges to ρ and the power of the one-sided and two-sided tests of H_0 tends to 1. (For example, this case occurs when $\Sigma_{ij}(s,t)\equiv\Sigma_{11}(s,t)$.) © 1984 Academic Press, Inc.

1. INTRODUCTION AND SUMMARY

Suppose that a zero mean bivariate Gaussian process $Z=(X,Y)'$ is observed in some region Ω in R^r, with covariance positive definite of the form

$$\Sigma(s,t,\rho)=EZ(s)\,Z(t)'=\begin{pmatrix}\Sigma_{11}(s,t) & \rho\Sigma_{12}(s,t)\\ \rho\Sigma_{21}(s,t) & \Sigma_{22}(s,t)\end{pmatrix},$$

where $\{\Sigma_{ij}(s,t)\}$ are known functions and ρ is an unknown real parameter.

We wish to test whether the processes $X(t)$ and $Y(t)$ are independent—that is to test the hypothesis $H_0:\rho=0$, and to estimate ρ.

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In Section 2 we consider the "finite" case where Ω consists of p points

$$\Omega_p = \{t_{ip}, 1 \leqslant i \leqslant p\} \quad \text{in} \quad R^r.$$

Thus we observe $\mathbf{Z}' = (\mathbf{X}', \mathbf{Y}')$ where $X_i = X(t_{ip})$, $Y_i = Y(t_{ip})$. A test of H_0 locally equivalent to the likelihood ratio test is to accept H_0 when $T_p = [\partial \ln f(\mathbf{Z}, \rho)/\partial\rho]_{\rho=0}$ is close to zero, where $f(z, \rho)$ is the density of $\mathbf{Z}$ in R^{2p}. Since $\mathbf{Z} \sim \mathrm{N}_{2p}(0, C(\rho))$ where

$$C(\rho) = \begin{pmatrix} \Sigma_{11p} & \rho\Sigma_{12p} \\ \rho\Sigma_{21p} & \Sigma_{22p} \end{pmatrix}$$

and Σ_{ijp} is the $p \times p$ matrix with (α, β) element $\Sigma_{ij}(t_{\alpha p}, t_{\beta p})$, it follows (Lemma 2.1) that

$$T_p = \mathbf{X}' R_p \mathbf{Y} \quad \text{where} \quad R_p = \Sigma_{11p}^{-1} \Sigma_{12p} \Sigma_{22p}^{-1}. \tag{1.1}$$

In Theorem 2.1 we show that the distribution of T_p is determined by ρ and the canonical correlations of $\mathbf{Z}$: these may be written in he form $\{|\rho|\, \lambda_{ip},\ 1 \leqslant i \leqslant p\}$, where $\{\lambda_{ip}\}$ are the positive roots of $|C(-\lambda^{-1})| = 0$ and $|\cdot|$ denotes the determinant. Equivalently, $\{\lambda_{ip}\}$ are the square roots of the eigenvalues of $A_p A_p'$ or of $A_p' A_p$ where $A_p = \Sigma_{11p}^{-1/2} \Sigma_{12p} \Sigma_{22p}^{-1/2}$. In fact, Theorem 2.1 shows that

$$T_p = \sum_1^p \lambda_{ip}(M_i^2 - N_i^2)/2 + \rho \sum_1^p \lambda_{ip}^2 (M_i^2 + N_i^2)/2 \tag{1.2}$$

where $\{M_i, N_j\}$ are independent $\mathrm{N}(0, 1)$ random variables (r.v.s).

It follows that the null distribution of T_p is symmetric about 0, and that an unbiased estimate of ρ is

$$\hat{\rho}_p = \Lambda_{2,p}^{-1} T_p \quad \text{where} \quad \Lambda_{2,p} = \sum_1^p \lambda_{i,p}^2 = \operatorname{trace} A_p A_p'. \tag{1.3}$$

In Theorem 2.2 we show how to express the distribution of T_p as a power series in ρ, and determine the slope of the power curve near $\rho = 0$ for the one- and two-sided tests of H_0 based on T_p.

A solution for the case when Ω, the domain of observation of Z, has a positive (but bounded) Lebesgue measure may be obtained by choosing $\{t_{ip},\ 1 \leqslant i \leqslant p\}$ to be evenly distributed over Ω and allowing p to increase to ∞. (For example, if $r = 1$ and $\Omega = [0, 1]$ one could choose $t_{ip} = i/p$.) Two distinct situations arise, depending on whether $\Lambda_{2,p}$ remains bounded or not. Section 3 deals with the situation where $\Lambda_{2,p}$ remains bounded and $\lambda_{i,p} \to \lambda_i$ as $p \to \infty$. Theorem 3.1 shows that T_p converges to a r.v.

$$T = \sum_1^\infty \lambda_i (M_i^2 - N_i^2)/2 + \rho \sum_1^\infty \lambda_i^2 (M_i^2 + N_i^2)/2, \tag{1.4}$$

with properties analogous to T_p. In particular the powers of the tests of H_0 are less than 1, and $\hat{\rho}_p$ does not converge to ρ.

However, in the situation where $\Lambda_{2,p} \to \infty$ as $p \to \infty$, dealt with in Section 4, T_p satisfies the Central Limit Theorem (C.L.T), $\hat{\rho}_p$ converges to ρ, and a test of H_0 may be constructed, with power at a given value of ρ arbitrarily close to 1. (In particular this situation occurs when $\Sigma_{ij} \equiv \Sigma_{11}$, as in the case of elliptical Brownian motion.)

Finally Section 5 illustrates an alternative method for obtaining $\{\lambda_i\}$ in the case where Ω is arbitrarily but $\Sigma_{11}(s,t)$, $\Sigma_{22}(s,t)$ have known eigenvalues and eigenfunctions. This approach also enables the construction of $\Sigma_{12}(s,t)$ such that the $\{\lambda_i\}$ have specified properties (such as $\Lambda_{2,\infty}$ being finite or infinite).

2. The Finite Case

Suppose that $Z(t)$ is observed at only the p points in Ω_p.

Lemma 2.1. *T_p is given by* (1.1).

Proof. $-2 \ln f(\mathbf{Z}, \rho) = \ln |C(\rho)| + 2p \ln(2\pi) + \mathbf{Z}'C(\rho)^{-1}\mathbf{Z}$. Let $\Sigma_{i.j} = \Sigma_{iip} - \rho^2 \Sigma_{ijp} \Sigma_{jjp}^{-1} \Sigma_{jip}$, $C^{11} = \Sigma_{1.2}^{-1}$, $C^{22} = \Sigma_{2.1}^{-1}$, and $C^{21} = (C^{12})' = -\rho \Sigma_{22p}^{-1} \Sigma_{21p} \Sigma_{1.2}^{-1}$. Then $|C(\rho)| = |\Sigma_{1.2}|\,|\Sigma_{22p}|$, (see for example De Groot [1, pp. 54–55], so that its derivative vanishes at $\rho = 0$. Also $C(\rho)^{-1} = (C^{ij})$ (e.g., Problem 2.7 of [3]) which has derivative $-C(\rho)^{-1}\, dC(\rho)/d\rho\, C(\rho)^{-1}$ equal to $-\begin{pmatrix} 0 & R_p \\ R_p' & 0 \end{pmatrix}$ when $\rho = 0$. The result follows. ∎

The next result gives the Fourier transform and the Laplace transform of the density T_p.

Theorem 2.1. T_p can be written in the form (1.2). Hence if

$$\operatorname{Re}(\lambda)(\pm\lambda_{ip} + \rho\lambda_{ip}^2) < 2, \qquad 1 \leqslant i \leqslant p, \tag{2.1}$$

then $E \exp(\lambda T_p/2) = D_p(\lambda, \rho)^{-1/2}$ where

$$D_p(\lambda, \rho) = D_{p,\rho}(\lambda)\, D_{p,-\rho}(-\lambda),$$

$$D_{p,\rho}(\lambda) = \prod_{i=1}^{p} \{1 - \lambda(\lambda_{ip} + \rho\lambda_{ip}^2)/2\} = |I - \lambda(K_p + \rho K_p^2)/2|,$$

and

$$K_p = (A_p A_p')^{1/2}.$$

Proof. From the singular value decomposition (e.g., [3, p. 42]), $A_p = U\Lambda V$ where $U'U = V'V = I_p$, $\Lambda = \text{diag}(\nu_1, ..., \nu_p)$, and $\nu_i = \lambda_{ip}$. Set $x = \Sigma_{11p}^{-1/2}\mathbf{X}$, $y = \Sigma_{22p}^{-1/2}\mathbf{Y}$, $u = U'x$, $v = Vy$. Then

$$T_p = x'A_p y = u'\Lambda v = \sum_1^p \nu_i u_i v_i.$$

Also, $\{(u_i, v_i)'\}$ are independently distributed as $\{\text{N}_2(0, L_i)\}$ where $L_i = \begin{pmatrix} 1 & \rho\nu_i \\ \rho\nu_i & 1 \end{pmatrix} = HP_iH$, $H = 2^{-1/2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and $P_i = \begin{pmatrix} 1+\rho\nu_i & 0 \\ 0 & 1-\rho\nu_i \end{pmatrix}$. Hence $\nu_i u_i v_i = a_i M_i^2 + b_i N_i^2$ where $a_i = (\nu_i + \rho\nu_i^2)/2$, $b_i = (-\nu_i + \rho\nu_i^2)/2$ and $(M_i, N_i)' = P_i^{-1/2}H(u_i, v_i)' \sim \text{N}_2(0, I)$. This proves (1.2). That is, T_p has the weighted chi-square form

$$T_p = \sum_1^p (a_i(\rho)\, M_i^2 - a_i(-\rho)\, N_i^2) \tag{2.2}$$

where $\{a_i(\rho) = (\lambda_{ip} + \rho\lambda_{ip}^2)/2\}$ are the eigenvalues of $(K_p + \rho K_p^2)/2$. The rest follows from $E\exp(\lambda N_1^2/2) = (1-\lambda)^{-1/2}$ for $\text{Re}(\lambda) < 1$ as is easily proved by contour integration. ∎

Two approximate methods for finding the distribution of indefinite quadratic forms of normal r.v.'s such as T_p are given in Section 29.7 of Johnson and Kotz [2]. These methods require $\{a_i(\rho)\}$, so that $\{\lambda_{ip}\}$ must be calculated. There are also various methods for inverting the characteristic function of T_p, $D_p(2it, \rho)^{-1/2}$, or its Laplace transform $D_p(-2t, \rho)^{-1/2}$, or of obtaining similarly the distributions of T_p^+, T_p^- and convoluting these, where T_p^+, T_p^- are the positive and negative parts of T_p, that is the components of (2.2) associated with the positive (or negative) values of $\{a_i(\rho), -a_j(-\rho)\}$.

From Theorem 2.1 it is easy to verify

COROLLARY 2.1. *$ET_p = \rho\Lambda_{2,p}$, and $\text{var}(T_p) = \Lambda_{2,p} + \rho^2\Lambda_{4,p}$ where*

$$\Lambda_{n,p} = \sum_1^p \lambda_{ip}^n = \text{trace}(A_pA_p')^{n/2} = \text{trace}(A_p'A_p)^{n/2}.$$

Let z_α denote the $1-\alpha$ quantile of $F(x, 0)$, the null distribution of T_p. The one- and two-sided $1-\alpha$ level tests of H_0 are: "accept $H_0 \Leftrightarrow T_p < z_\alpha$" and "accept $H_0 \Leftrightarrow |T_p| < z_{\alpha/2}$." We now consider the power of these tests near $\rho = 0$.

THEOREM 2.2. (a) *The density of T_p satisfies*

$$f(x, \rho) = f(x, 0)\{1 + \rho x + \rho^2(x^2 + \Lambda_{2,p})/2\} + O(\rho^3).$$

(b) *The one-sided test has power*

$$\alpha + \rho \int_{z_\alpha} x f(x, 0)\, dx + \rho^2 \left(\alpha \Lambda_{2,p} + \int_{z_\alpha} x^2 f(x, 0)\, dx \right) + O(\rho^3).$$

(c) *The two-sided test has power*

$$\alpha + \rho^2 \left\{ \int_{z_{\alpha/2}} x^2 f(x, 0)\, dx + \alpha \Lambda_{2,p}/2 \right\} + O(\rho^4).$$

Proof. Dropping the subscript p, there exists $\lambda_0 > 0$ such that

$$D_\rho(\lambda)^{-1/2} = D_0(\lambda)^{-1/2} \exp \left\{ \tfrac{1}{2} \sum_1^\infty p_j(\lambda)\, \rho^j / j \right\} \qquad \text{for} \quad |\mathrm{Re}\, \lambda| < \lambda_0,$$

where

$$p_j(\lambda) = \sum_1^p C_i(\lambda)^j \qquad \text{and} \qquad C_i(\lambda) = \lambda \lambda_i^2 (2 - \lambda \lambda_i)^{-1}.$$

Hence $D(\lambda, \rho)^{-1/2} = D(\lambda, 0)^{-1/2} \{1 + \rho L_1(\lambda) + \rho^2 L_2(\lambda) + O(\rho^3)\}$ where $L_1(\lambda) = (p_1(\lambda) - p_1(-\lambda))/2$ and $L_2(\lambda) = (p_2(\lambda) + p_2(-\lambda))/4 + L_1(\lambda)^2/2$. Also, $p_1(\lambda) = -2\partial \ln D_0(\lambda) - \Lambda_1$ and $p_2(\lambda) = 4\partial^2 \ln D_0(\lambda) - 4\lambda^{-1} p_1(\lambda) + \Lambda_2$, where $\partial^i = (\partial/\partial\lambda)^i$. Set $D = D(\lambda, 0)$, $Y = T_p/2$ and let E_0 denote E when $\rho = 0$. Then $D^{-1/2} = E_0 \exp(\lambda Y)$, $D^{-1/2} L_1(\lambda) = 2\partial D^{-1/2} = 2E_0 Y \exp(\lambda Y)$, and $D^{-1/2} L_2(\lambda) = 2\partial^2 D^{-1/2} + \Lambda_2 D^{-1/2}/2 = E_0(2Y^2 + \Lambda_2/2) \exp(\lambda Y)$. Then (a) follows, and hence (b) and (c), using the symmetry of $f(x, 0)$, with ρ^3 for ρ^4 in (c). To replace ρ^3 by ρ^4 in (c), note $T_p = U + \rho V$ where $(U, V) =^{\mathrm{L}} (-U, V)$ so that $|T_p|$ has distribution expandable in powers of ρ^2. ∎

3. The Infinite Case: $\Lambda_{2,p}$ Bounded

Suppose that as $p \to \infty$, $\lambda_{ip} \to \lambda_i$, $i \geqslant 1$ where $\Lambda_2 = \sum_1^\infty \lambda_i^2 < \infty$.

THEOREM 3.1. *As $p \to \infty$, $T_p \to^{\mathrm{L}} T$, given by (1.4). The sum T converges in probability. If also $\Lambda_1 = \sum_1^\infty \lambda_i < \infty$, then $T =^{\mathrm{L}} T^+ + T^- =^{\mathrm{L}} T_+ + T_-$ where*

$$T^+ = \sum a_i^+ M_i^2, \qquad T^- = \sum a_i^- N_i^2,$$

$$T_+ = \sum_1^\infty a_i(\rho)\, M_i^2, \qquad T_- = -\sum_1^\infty a_i(-\rho)\, N_i^2,$$

$\{a_i^+\}$ and $\{a_i^-\}$ are the positive and negative values of $\{a_i(\rho), -a_i(-\rho)\}$ and $a_i(\rho) = (\lambda_i + \rho\lambda_i^2)/2$; these sums converge with probability one.

Proof. Suppose that λ satisfies (2.1). Then as $p \to \infty$,

$$D_p(\lambda, \rho) \to D_\infty(\lambda, \rho) = \prod_1^\infty (1 - \lambda a_i(\rho))(1 + \lambda a_i(-\rho))$$

which is finite. Convergence in probability follows.

If $\Lambda_1 < \infty$, $T^+ \geqslant 0 \geqslant T^-$ and $\infty > ET^+ \geqslant ET^- > -\infty$, so that T^+, T^- converge with probability 1. The same is true for T_+, T_- since these differ from T^+, T^- by only a fixed finite number of terms. ∎

The analogs of the rest of Theorem 2.1, Corollary 2.1, and Theorem 2.2 also hold. Thus the distribution of T may be determined from $\{\lambda_i\}$. These values depend on the choice of $\{t_{i,p}\}$, but may be obtained independently of them as follows. Let $\mu_p(\cdot)$ be the measure putting weight p^{-1} at $t_{i,p}$, $1 \leqslant i \leqslant p$. Suppose μ_p converges weakly to a measure μ on Ω. (The natural choice for μ is Lebesgue measure on Ω.) Then when the operators $\{\mathrm{S}_{ij}\}$, A, R given by

$$\mathrm{S}_{ij} f(s) = \int \Sigma_{ij}(s, t) f(t)\, d\mu(t),$$

where $\int$ is over Ω,

$$\mathrm{A} = \mathrm{S}_{11}^{-1/2} \mathrm{S}_{12} \mathrm{S}_{22}^{-1/2}, \qquad \mathrm{R} = \mathrm{S}_{11}^{-1} \mathrm{S}_{12} \mathrm{S}_{22}^{-1},$$

are well-defined, clearly $\{\lambda_i^{-2}\}$ are the eigenvalues of AA$'$, $\Lambda_n = \sum_1^\infty \lambda_i^n$ is the trace of the operator $(\mathrm{AA}')^{n/2}$, and $T = \int X(t)\, \mathrm{R} Y(t)\, d\mu(t)$.

It is not actually necessary to obtain $\{\lambda_i\}$ in order to get $E \exp(\lambda T/2) = D_\infty(\lambda, \rho)^{-1/2}$, since when $\Lambda_1 < \infty$, $D_\infty(\lambda, \rho) = D_{\infty,\rho}(\lambda)\, D_{\infty,-\rho}(-\lambda))$ where $D_{\infty,\rho}(\lambda) = \prod_1^\infty \{1 - \lambda(\lambda_i + \rho\lambda_i^2)/2\}$ may be determined from $D_{\infty,\rho}(\lambda) = \exp\{-\int_0^\lambda d\lambda \int B_\rho(t, t, \lambda)\, d\mu(t)\}$, where $B_\rho(s, t, \lambda)$ is the resolvent of the operator $\mathrm{B}_\rho = ((\mathrm{AA}')^{1/2} + \rho \mathrm{AA}')/2$, (see for example, Withers [4]), or from $D_{\infty,\rho}(\lambda) = D(\nu_1)\, D(\nu_2)$ if $D(\lambda) = \prod_1^\infty (1 - \lambda\lambda_i)$ is known and ν_1, ν_2 are the roots of $1 - \lambda(\nu + \rho\nu^2)/2 = 0$.

4. The Infinite Case: $\Lambda_{2,p}$ Unbounded

Theorem 4.1. *Suppose that $\{\lambda_{jp}\}$ are uniformly bounded but $\Lambda_{2,p} \to \infty$ as $p \to \infty$. Then $\hat\rho_p \to^P \rho$, $(T_p - ET_p)(\mathrm{var}\, T_p)^{-1/2} \to^{\mathrm{L}} \mathrm{N}(0, 1)$ as $p \to \infty$, and an asymptotically $1 - \alpha$ level confidence region for ρ is given by*

$$|\rho - \hat\rho_p| \leqslant \Phi^{-1}(1 - \alpha/2)\, \Lambda_{2,p}^{-1} (\Lambda_{2,p} + \hat\rho_p^2 \Lambda_{4,p})^{1/2}.$$

As $p \to \infty$, $\Lambda_{2,p}^{-1/2} T_p \to^{\mathrm{L}} \infty$ if $\rho > 0$, $\mathrm{N}(0, 1)$ if $\rho = 0$, and $-\infty$ if $\rho < 0$.

Proof. By Theorem 2.1, Corollary 2.1, and Lyapounov's Theorem, the C.L.T. holds for T_p since

$$\sum_1^p (a_i(\rho)^4 + a_{ip}(-\rho)^4)/(\Lambda_{2,p} + \rho^2 \Lambda_{4,p})^2 \to 0 \qquad \text{as} \quad p \to \infty.$$

Hence

$$\Lambda_{2,p}(\hat{\rho}_p - \rho)(\Lambda_{2,p} + \rho^2 \Lambda_{4,p})^{-1/2} \xrightarrow{\text{L}} \text{N}(0, 1),$$

and

$$\Lambda_{2,p}^{-1/2} T_p = \rho \Lambda_{2,p}^{1/2} + 0_p(1). \quad \blacksquare$$

Thus by taking p suitably large one may decide—with negligible error—whether $\rho = 0$, and if it is not, obtain its value. For example, if p is chosen so that $(10^{-3}/3.291)^2 \geqslant (\Lambda_{2,p} + \hat{\rho}_p^2 \Lambda_{4,p}) \Lambda_{2,p}^{-2}$ then with probability greater or equal to $.999 + o(1)$, ρ must lie between $\hat{\rho}_p - .001$ and $\hat{\rho}_p + .001$. (Here $o(1)$ is a term which tends to zero as $p \to \infty$. Typically for each n, $\Lambda_{n,p}/p$ is a power series in p^{-1} so that this term can be improved to $O(p^{-\zeta})$ for any desired $\zeta > 0$, by the method of Withers [5].)

EXAMPLE 4.1. $\Sigma_{ij}(s, t) \equiv \sigma_{ij} C(s, t)$ where $\sigma_{12} C \not\equiv 0$. In this case $\sigma_{21} = \sigma_{12}$ and $A_p = \sigma I_p$ where $\sigma = \sigma_{12}(\sigma_{11}\sigma_{22})^{-1/2}$, and $\Lambda_{n,p} = \sigma^n p$. Hence with probability $1 - \alpha + O(p^{-1/2})$,

$$|\rho - \hat{\rho}_p| \leqslant \Phi^{-1}(1 - \alpha/2)\, \sigma^{-1} p^{-1/2} (1 + \hat{\rho}_p^2 \sigma^2)^{1/2}$$

where

$$\hat{\rho}_p = \sigma^{-2} p^{-1} T_p, \qquad T_p = \sigma_0 \mathbf{X}' C_p^{-1} \mathbf{Y},$$

$$\sigma_0 = (\sigma_{11}\sigma_{22})^{-1/2}\, \sigma, \qquad \text{and} \qquad (C_p)_{\alpha,\beta} = C(t_{\alpha p}, t_{\beta p}).$$

EXAMPLE 4.1. Suppose $r = 1$, $\Omega = [a, b]$, $C(s, t) = \min(s, t)$. Then

$$\mathbf{X}' C_p^{-1} \mathbf{Y} = c^{-1} X_1 Y_1 + d^{-1} \sum_1^p X_i D_i$$

where $c = a + d$, $d = (b - a)/p$, $D_1 = Y_1 - Y_2$, $D_i = -Y_{i-1} + 2Y_i - Y_{i+1}$ for $1 < i < p$, $D_p = -Y_{p-1} + Y_p$.

For $Z(\cdot)$ non-Gaussian, a Central Limit Theorem for T_p may still be proved under suitable conditions (such as when T_p is strong-mixing); however the confidence interval in Theorem 4.1 is no longer consistent, and a consistent estimate for var T_p is more difficult to obtain.

5. Fourier Methods

In some applications it may be desirable to try a variety of $\Sigma_{12}(s,t)$ or $\{\Sigma_{ij}(s,t)\}$. Fourier expansions provide a way to construct a covariance for $Z(t)$ with given values of $\{\lambda_i\}$, and hence to make Λ_1 or Λ_2 finite or infinite as desired.

Alternatively, when $\{\Sigma_{ij}(s,t)\}$ are given and the eigenvalues $\{\theta_{1i}, \theta_{2i},...\}$ and eigenfunctions $\{\phi_{1i}(t), \phi_{2i}(t),...\}$ of $\Sigma_{ii}(s,t)$ (w.r.t. the measure μ on Ω) are known for $i=1$ and 2, (including the solutions of $S_{ii}\phi = \theta^{-1}\phi$ with $\theta = \infty$), then the Fourier expansions provide an alternative way of calculating $\{\lambda_i\}$.

We illustrate this with the following example—the case when the marginals $X(t)$, $Y(t)$ are Brownian motion.

Example 5.1. Take $r=1$, $\Omega=[0,b]$, $\Sigma_{ii}(s,t) = \sigma_{ii}\min(s,t)$, $i=1,2$, and μ Lebesgue measure. Then the eigenfunctions and eigenvalues of $\Sigma_{ii}(s,t)$ are $\{\phi_j(t), \theta_j/\sigma_{ii}, j \geqslant 1\}$ where $\theta_j = (j-\frac{1}{2})^2\pi^2/b^2$ and $\phi_j(t) = (2/b)^{1/2}\sin\theta_j^{1/2}t$. Choose $\{q_i\}$ such that $\sum_1^\infty q_i^2 < \infty$ and set

$$\Sigma_{12}(s,t) = \sum_1^\infty \phi_i(s)\,\phi_i(t)\,q_i.$$

Then $\Sigma(s,t,\rho)$ will be a covariance $\Leftrightarrow \max_i \rho^2 q_i^2 \leqslant \sigma_{11}\sigma_{22}$, since $\Sigma(s,t,\rho) = \sum_1^\infty \phi_i(s)\,\phi_i(t)\,\tau_i$ where $\tau_i = \begin{pmatrix} \sigma_{11} & \rho q_i \\ \rho q_i & \sigma_{22} \end{pmatrix} = EX(s)\,X(t)'$ where $X(s) = \sum_1^\infty \phi_i(s)\,\mathbf{X}_i$ and $\{\mathbf{X}_i\}$ are independent r.v.'s with means $\mathbf{0}$ and covariance $\{\tau_i\}$.

Also $\{\lambda_i\} = \{|q_i|\,\theta_i(\sigma_{11}\sigma_{22})^{-1/2}\}$ and so $\Lambda_2 < \infty \Leftrightarrow \sum_1^\infty q_i^2 i^4 < \infty$. This clearly fails for Example 4.1.1 with $a=0$, since that example corresponds to the choice $q_i \equiv \sigma_{12}/\theta_i$.

However for a choice such as $q_i \equiv \sigma_{12}/\theta_i^2$, Λ_2 is finite and so is Λ_1; this choice yields

$$\Sigma_{12}(s,t) = \sigma_{12}\int_0^b \min(s,u)\min(u,t)\,du = \sigma_{12}(stb - st^2/2 - s^3/6) \qquad \text{for} \quad s \leqslant t;$$

also $\{\lambda_i\} = \{\alpha\theta_i^{-1}\}$ where $\alpha = |\sigma_{12}|\,(\sigma_{11}\sigma_{22})^{-1/2}$, so that the Laplace transform/characteristic function of T is given by Section 3 in terms of $D(\lambda) = \prod_1^\infty (1-\lambda\lambda_i) = \operatorname{cosin}\{(\alpha\lambda)^{1/2}\,b\}$.

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