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Linear Algebra and its Applications 397 (2005) 279–297
LINEAR ALGEBRA AND ITS APPLICATIONS
www.elsevier.com/locate/laa

Additive perturbation results for the Drazin inverse

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Received 11 March 2002; accepted 1 November 2004

Submitted by H.J. Werner

Abstract

In this paper some new additive results for the Drazin inverse are presented. We give a formula for the Drazin inverse of a sum of two matrices under conditions on the matrices less restrictive than those imposed in the corresponding theorem given by Hartwig et al. (Linear Algebra Appl. 322 (2001) 207–217). We consider some aplications of our results to the perturbation of the Drazin inverse and analyze a number of special cases.

AMS classification: Primary 15A09

Keywords: Drazin inverse; Perturbation

1. Introduction

Let $\mathscr{R}(A)$, $\mathscr{N}(A)$ denote the range and null space of $A \in \mathbb{C}^{n\times n}$. The *index* of A, denoted by $\text{ind}(A)$, is the smallest non-negative integer r such that $\mathbb{C}^{n\times n} = \mathscr{R}(A^r) \oplus \mathscr{N}(A^r)$. The *eigenprojection* A^π of A corresponding to the eigenvalue 0 is the uniquely determined idempotent matrix with

$$\mathscr{R}(A^\pi) = \mathscr{N}(A^r) \quad \text{and} \quad \mathscr{N}(A^\pi) = \mathscr{R}(A^r).$$

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doi:10.1016/j.laa.2004.11.001

If $A \in \mathbb{C}^{n\times n}$ is such that $\operatorname{ind}(A) = r$, the Drazin inverse of A is the unique matrix $A^{\mathrm{D}} \in \mathbb{C}^{n\times n}$ satisfying the relations

$$A^{\mathrm{D}}AA^{\mathrm{D}} = A^{\mathrm{D}}, \quad AA^{\mathrm{D}} = A^{\mathrm{D}}A, \quad A^{l+1}A^{\mathrm{D}} = A^{l} \text{ for all } l \geqslant r.$$

By [2–Theorem 7.2.1], for each $A \in \mathbb{C}^{n\times n}$ such that $\operatorname{ind}(A) = r$, there exists a non-singular *core-nilpotent block form*

$$A = P\begin{pmatrix} C & 0 \\ 0 & N \end{pmatrix}P^{-1},$$

where C is non-singular and N is nilpotent of index r. Relative to the above form, the Drazin inverse of A and the eigenprojection A^{π} are given by

$$A^{\mathrm{D}} = P\begin{pmatrix} C^{-1} & 0 \\ 0 & 0 \end{pmatrix}P^{-1}, \quad A^{\pi} = I - AA^{\mathrm{D}} = P\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}P^{-1}.$$

In particular, if A is nilpotent then the block C is empty and $A^{\mathrm{D}} = 0$; if A is non-singular then the block N is empty and $A^{\mathrm{D}} = A^{-1}$. The case when $\operatorname{ind}(A) = 1$, which is equivalent to having $N = 0$ in the above form, is of special interest and the Drazin inverse of A is called the group inverse of A, and is denoted by $A^{\sharp}$. The Drazin inverse of complex square matrices is investigated in the books [1] and [2].

The behaviour of the Drazin inverse with respect to the sum $a + b$ of two Drazin invertible elements of a ring is firstly considered by Drazin in [4]. Herein, it was showed that $(a + b)^{\mathrm{D}} = a^{\mathrm{D}} + b^{\mathrm{D}}$ provided $ab = ba = 0$. In [9–Theorem 2.1] there was constructed, for matrices, a formula for the Drazin inverse $(A + B)^{\mathrm{D}}$ as a function of $A, B, A^{\mathrm{D}}, B^{\mathrm{D}}$ when only the condition $AB = 0$ was assumed. This result was extended in [8] to the generalized Drazin inverse of bounded linear operators in Banach spaces. The aim of this paper is to extend additive Drazin inverse results given in [9] to more general cases, under weaker conditions on the matrices A and B by dropping off the assumption that one of the products of these matrices vanishes. In this paper we apply our results to get a perturbation result that generalizes [9–Corollary 2.2] and admits several special cases.

Next we state one lemma concerning Drazin inverse of a partitioned matrix that will be needed later (see Meyer and Rose [5]).

Lemma 1.1. *Let*

$$M_1 = \begin{pmatrix} A & 0 \\ C & B \end{pmatrix}, \quad M_2 = \begin{pmatrix} B & C \\ 0 & A \end{pmatrix},$$

where A and B are square matrices with $\operatorname{ind}(A) = r$ *and* $\operatorname{ind}(B) = s$. *Then*

$$M_1^{\mathrm{D}} = \begin{pmatrix} A^{\mathrm{D}} & 0 \\ X & B^{\mathrm{D}} \end{pmatrix}, \quad M_2^{\mathrm{D}} = \begin{pmatrix} B^{\mathrm{D}} & X \\ 0 & A^{\mathrm{D}} \end{pmatrix},$$

where

$$X = (B^{\mathrm{D}})^2\left(\sum_{i=0}^{r-1}(B^{\mathrm{D}})^iCA^i\right)A^{\pi} + B^{\pi}\left(\sum_{i=0}^{s-1}B^iC(A^{\mathrm{D}})^i\right)(A^{\mathrm{D}})^2 - B^{\mathrm{D}}CA^{\mathrm{D}}.$$

2. Drazin inverse of a sum of two matrices

First we state one particular case of our main result.

Theorem 2.1. *Let $B \in \mathbb{C}^{n\times n}$, $s = \mathrm{ind}(B)$, let $N \in \mathbb{C}^{n\times n}$ be nilpotent of index r. If $NB^{\mathrm{D}} = 0$ and $B^{\pi}NB = 0$ then*

$$(N+B)^{\mathrm{D}} = B^{\mathrm{D}} + (B^{\mathrm{D}})^2\left(\sum_{i=0}^{r+s-2}(B^{\mathrm{D}})^i NS(i)\right) \tag{2.1}$$

and, for any $i \geqslant 0$,

$$B^{\pi}(N+B)^i = S(i), \tag{2.2}$$

where

$$S(i) = B^{\pi}\left(\sum_{j=0}^{i} B^{i-j}N^j\right).$$

Moreover, if $\max\{r,s\} \leqslant l \leqslant r+s-2$ then for all $i \geqslant l$ we have $S(i) = B^{i-l+1}S(l-1) = S(l-1)N^{i-l+1}$.

Proof. Let P be a non-sigular matrix for which

$$B = P\begin{pmatrix} C_B & 0 \\ 0 & N_B \end{pmatrix}P^{-1},$$

where C_B is non-singular and N_B is nilpotent of index s. From $NB^{\mathrm{D}} = 0$ it follows that N can be written as

$$N = P\begin{pmatrix} 0 & N_1 \\ 0 & N_2 \end{pmatrix}P^{-1},$$

where N_2 is nilpotent of index r. From $B^{\pi}NB = 0$ it follows that $N_2N_B = 0$. Thus, for any $i \geqslant 0$,

$$(N_2+N_B)^i = \sum_{j=0}^{i} N_B^{i-j}N_2^j = \sum_{j=0}^{i} N_B^j N_2^{i-j}.$$

We observe that $N_2 + N_B$ is nilpotent of index $r + s - 1$. We set $t = r + s - 2$. From Lemma 1.1 we get that

$$(N+B)^{\mathrm{D}} = P\begin{pmatrix} C_B & N_1 \\ 0 & N_2+N_B \end{pmatrix}^{\mathrm{D}}P^{-1} = P\begin{pmatrix} C_B^{-1} & X \\ 0 & 0 \end{pmatrix}P^{-1},$$

where

$$X = (C_B^{-1})^2\left(\sum_{i=0}^{t}(C_B^{-1})^i N_1(N_2+N_B)^i\right)$$

$$= (C_B^{-1})^2 \left(\sum_{i=0}^{t} (C_B^{-1})^i N_1 \left(\sum_{j=0}^{i} N_B^{i-j} N_2^j \right) \right).$$

Write $S(i) = B^\pi (\sum_{j=0}^{i} B^{i-j} N^j)$ for all $i \geqslant 0$. Now, we compute, for all $i \geqslant 1$,

$$S(i) = P \left\{ \begin{pmatrix} 0 & 0 \\ 0 & N_B^i \end{pmatrix} + \sum_{j=1}^{i} \begin{pmatrix} 0 & 0 \\ 0 & N_B^{i-j} \end{pmatrix} \begin{pmatrix} 0 & N_1 N_2^{j-1} \\ 0 & N_2^j \end{pmatrix} \right\} P^{-1}$$
$$= P \begin{pmatrix} 0 & 0 \\ 0 & \sum_{j=0}^{i} N_B^{i-j} N_2^j \end{pmatrix} P^{-1}.$$

Hence,

$$B^{\mathrm{D}} + (B^{\mathrm{D}})^2 \left(\sum_{i=0}^{t} (B^{\mathrm{D}})^i N S(i) \right)$$
$$= P \begin{pmatrix} C_B^{-1} & \sum_{i=0}^{t} (C_B^{-1})^{i+2} N_1 \left(\sum_{j=0}^{i} N_B^{i-j} N_2^j \right) \\ 0 & 0 \end{pmatrix} P^{-1}$$
$$= P \begin{pmatrix} C_B^{-1} & X \\ 0 & 0 \end{pmatrix} P^{-1} = (N+B)^{\mathrm{D}}.$$

The equality (2.2) and the second statement of the theorem are easily verified. □

Remark 2.2. Let $B, N \in \mathbb{C}^{n\times n}$ satisfy conditions of Theorem 2.1. Then we have

$$(N+B)^{\mathrm{D}}(N+B) = B^{\mathrm{D}} B + \left(\sum_{i=0}^{r+s-2} (B^{\mathrm{D}})^{i+1} N S(i) \right),$$

where $S(i)$ is defined in (2.2).

Now we can derive some especial cases from Theorem 2.1.

Corollary 2.3. *Let $B \in \mathbb{C}^{n\times n}$, $s = \operatorname{ind}(B)$, and let $N \in \mathbb{C}^{n\times n}$ be nilpotent of index r. If $NB = 0$ then*

$$(B+N)^{\mathrm{D}} = B^{\mathrm{D}} \left(\sum_{i=0}^{r-1} (B^{\mathrm{D}})^i N^i \right).$$

Proof. See [9–Corollary 2.1 (iii)]. □

Corollary 2.4. *Let $B \in \mathbb{C}^{n\times n}$, $s = \operatorname{ind}(B)$, and let $N \in \mathbb{C}^{n\times n}$ be nilpotent of index r. Suppose that $NB^{\mathrm{D}} = 0$ and $B^\pi NB = 0$.*

(i) *If* $N^2 = 0$ *then*

$$(B+N)^{\mathrm{D}} = B^{\mathrm{D}} + (B^{\mathrm{D}})^2 \left(\sum_{i=0}^{s-1} (B^{\mathrm{D}})^i N B^i\right) + (B^{\mathrm{D}})^3 \left(\sum_{i=1}^{s-1} (B^{\mathrm{D}})^i N B^i\right) N.$$

(ii) *If* $NR = 0$, *then*

$$(B+N)^{\mathrm{D}} R = B^{\mathrm{D}} R + (B^{\mathrm{D}})^2 \left(\sum_{i=1}^{r+s-2} (B^{\mathrm{D}})^i N B^i\right) R.$$

(iii) *If* $B^2 = B$, *then*

$$(B+N)^{\mathrm{D}} = B(I-N)^{-1}.$$

Proof. Each of these cases follows directly from Theorem 2.1 and the following simplification.

Write $S(i) = B^{\pi}(\sum_{j=0}^{i} B^{i-j} N^j)$ for all $i \geqslant 0$.

(i) Since $N^2 = 0$, $NS(i) = NB^i + NB^{i-1}N$ for all $i \geqslant 1$.
(ii) Since $NR = 0$, $NS(i)R = NB^i R$.
(iii) Since $B^2 = B$, $B^{\mathrm{D}} = B$ and then the hypothesis $NB^{\mathrm{D}} = 0$ implies $NB = 0$. Then from Corollary 2.3 it follows that $(B+N)^{\mathrm{D}} = B(\sum_{i=0}^{r-1} N^i)$. Now, we use that $(I-N)^{-1} = \sum_{i=0}^{r-1} N^i$. □

Next, we state the main result.

Theorem 2.5. *Let* $A \in \mathbb{C}^{n\times n}$, $r = \operatorname{ind}(A)$, $B \in \mathbb{C}^{n\times n}$, $s = \operatorname{ind}(B)$. *If* $A^{\mathrm{D}} B = 0$, $AB^{\mathrm{D}} = 0$ *and* $B^{\pi} A B A^{\pi} = 0$ *then*

$$\begin{aligned}(A+B)^{\mathrm{D}} = {} & B^{\mathrm{D}} \left(I + \sum_{i=0}^{t} (B^{\mathrm{D}})^{i+1} A Z(i)\right) A^{\pi} \\ & + B^{\pi} \left(I + \sum_{i=0}^{t} Z(i) B (A^{\mathrm{D}})^{i+1}\right) A^{\mathrm{D}} \\ & - (B^{\mathrm{D}})^2 \left(\sum_{i=0}^{t} (B^{\mathrm{D}})^i A Z(i) B\right) A^{\mathrm{D}} \\ & - B^{\mathrm{D}} \left(\sum_{i=0}^{t} A Z(i) B (A^{\mathrm{D}})^i\right) (A^{\mathrm{D}})^2 \\ & - (B^{\mathrm{D}})^2 \left(\sum_{i=0}^{t-1} \sum_{k=0}^{t-1} (B^{\mathrm{D}})^i A Z(i+k+1) B (A^{\mathrm{D}})^k\right) (A^{\mathrm{D}})^2, \end{aligned} \tag{2.3}$$

where $t = r + s - 2$ (*in the case* $r = s = 1$, *we assume that* $\sum_{i=0}^{-1}$ *is an empty sum*)

$$Z(i) = B^{\pi}\left(\sum_{j=0}^{i} B^{i-j}A^{j}\right)A^{\pi}. \tag{2.4}$$

Moreover, if $\max\{r, s\} \leqslant l \leqslant t$ *then we have*

$$Z(i) = B^{i-l+1}Z(l-1) = Z(l-1)A^{i-l+1} \quad \textit{for all } i \geqslant l.$$

Proof. Let P be a non-sigular matrix for which

$$A = P\begin{pmatrix} C_A & 0 \\ 0 & N_A \end{pmatrix}P^{-1},$$

where C_A is non-singular and N_A is nilpotent of index r. From $A^{\mathrm{D}}B = 0$ it follows that B can be written as

$$B = P\begin{pmatrix} 0 & 0 \\ B_1 & B_2 \end{pmatrix}P^{-1}.$$

Thus from the assumptions $AB^{\mathrm{D}} = 0$ and $B^{\pi}ABA^{\pi} = 0$, using Lemma 1.1 to compute B^{D}, we get that $N_AB_2^{\mathrm{D}} = 0$ and $B_2^{\pi}N_AB_2 = 0$. So, we see that B_2 and N_A satisfied conditions of Theorem 2.1.

We set $t = r + s - 2$. From Lemma 1.1 we have that

$$(A+B)^{\mathrm{D}} = P\begin{pmatrix} C_A & 0 \\ B_1 & N_A + B_2 \end{pmatrix}^{\mathrm{D}}P^{-1} = P\begin{pmatrix} C_A^{-1} & 0 \\ X & (N_A + B_2)^{\mathrm{D}} \end{pmatrix}P^{-1},$$

where

$$X = (N_A + B_2)^{\pi}\left(\sum_{k=0}^{t}(N_A + B_2)^{k}B_1(C_A^{-1})^{k}\right)(C_A^{-1})^2 - (N_A + B_2)^{\mathrm{D}}B_1C_A^{-1}.$$

Using Theorem 2.1 we get that

$$(N_A + B_2)^{\pi} = B_2^{\pi} - B_2^{\mathrm{D}}\left(\sum_{i=0}^{t}(B_2^{\mathrm{D}})^{i}N_AS(i)\right),$$

where $S(i) = B_2^{\pi}(\sum_{j=0}^{i} B_2^{j}N_A^{i-j})$ for all $i \geqslant 0$.

Now, expand X as the sum of the following terms X_1, X_2 and X_3.

$$\begin{aligned} X_1 &= B_2^{\pi}\left(\sum_{k=0}^{t}(N_A + B_2)^{k}B_1(C_A^{-1})^{k}\right)(C_A^{-1})^2 \\ &= B_2^{\pi}\left(\sum_{k=0}^{t} S(k)B_1(C_A^{-1})^{k}\right)(C_A^{-1})^2, \end{aligned}$$

where this equality follows by using (2.2) in Theorem 2.1.

$$X_2 = -B_2^{\mathrm{D}} \left(\sum_{i=0}^{t} (B_2^{\mathrm{D}})^i N_A S(i) \right) \left(\sum_{k=0}^{t} (N_A + B_2)^k B_1 (C_A^{-1})^k \right) (C_A^{-1})^2$$
$$= -B_2^{\mathrm{D}} \left(\sum_{k=0}^{t} N_A (N_A + B_2)^k B_1 (C_A^{-1})^k \right) (C_A^{-1})^2$$
$$-(B_2^{\mathrm{D}})^2 \left(\sum_{i=0}^{t-1} \sum_{k=0}^{t-1} (B_2^{\mathrm{D}})^i N_A S(i+k+1) B_1 (C_A^{-1})^k \right) (C_A^{-1})^2,$$

where this equality follows by using (2.2) to obtain that $S(i)(N_A + B_2)^k = B_2^{\pi}(N_A + B_2)^{i+k} = S(i+k)$, after we change $i = i - 1$ in the last sum and we observe that $S(i+t+1) = 0$ for $i = 0, \ldots, t-1$.

$$X_3 = -(N_A + B_2)^{\mathrm{D}} B_1 C_A^{-1}$$
$$= -B_2^{\mathrm{D}} B_1 C_A^{-1} - (B_2^{\mathrm{D}})^2 \left(\sum_{i=0}^{t} (B_2^{\mathrm{D}})^i N_A S(i) B_1 \right) C_A^{-1}.$$

Write $Z(i) = B^{\pi}(\sum_{j=0}^{i} B^{i-j} A^j) A^{\pi}$. By direct computations, for all $i \geqslant 1$ we have,

$$Z(i) = P \begin{pmatrix} I & 0 \\ -B_2^{\mathrm{D}} B_1 & I - B_2 B_2^{\mathrm{D}} \end{pmatrix}$$
$$\times \left\{ \sum_{j=0}^{i-1} \begin{pmatrix} 0 & 0 \\ B_2^{i-j-1} B_1 & B_2^{i-j} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & N_A^j \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & N_A^i \end{pmatrix} \right\} P^{-1}$$
$$= P \begin{pmatrix} 0 & 0 \\ 0 & (I - B_2 B_2^{\mathrm{D}}) \sum_{j=0}^{i} B_2^{i-j} N_A^j \end{pmatrix} P^{-1}$$
$$= P \begin{pmatrix} 0 & 0 \\ 0 & S(i) \end{pmatrix} P^{-1}$$

and

$$A Z(i) B (A^{\mathrm{D}})^q = P \begin{pmatrix} 0 & 0 \\ N_A S(i) B_1 (C_A^{-1})^q & 0 \end{pmatrix} P^{-1} \quad \text{for all } q \geqslant 1.$$

Now, we compute the terms of the expression (2.3) for $(A + B)^{\mathrm{D}}$ using the block descomposition,

$$\Sigma_1 = B^{\mathrm{D}} \left(I + \sum_{i=0}^{t} (B^{\mathrm{D}})^{i+1} A Z(i) \right) A^{\pi}$$

$$
\begin{aligned}
&= P\left\{\begin{pmatrix}0 & 0\\ 0 & B_2^{\mathrm{D}}\end{pmatrix} + \sum_{i=0}^{t}\begin{pmatrix}0 & 0\\ (B_2^{\mathrm{D}})^{i+3}B_1 & (B_2^{\mathrm{D}})^{i+2}\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & N_A S(i)\end{pmatrix}\right\} P^{-1}\\
&= P\begin{pmatrix}0 & 0\\ 0 & B_2^{\mathrm{D}} + \sum_{i=0}^{t}(B_2^{\mathrm{D}})^{i+2}N_A S(i)\end{pmatrix} P^{-1}\\
&= P\begin{pmatrix}0 & 0\\ 0 & (N_A+B_2)^{\mathrm{D}}\end{pmatrix} P^{-1},
\end{aligned}
$$

$$
\begin{aligned}
\Sigma_2 &= B^{\pi}\left(I + \sum_{k=0}^{t} Z(k)B(A^{\mathrm{D}})^{k+1}\right)A^{\mathrm{D}}\\
&= P\begin{pmatrix}C_A^{-1} & 0\\ -B_2^{\mathrm{D}}B_1C_A^{-1} + B_2^{\pi}\left(\sum_{k=0}^{t} S(k)B_1(C_A^{-1})^{k+2}\right) & 0\end{pmatrix} P^{-1},\\
\Sigma_3 &= -(B^{\mathrm{D}})^2\left(\sum_{i=0}^{t}(B^{\mathrm{D}})^i AZ(i)B\right)A^{\mathrm{D}}\\
&= -P\begin{pmatrix}0 & 0\\ \sum_{i=0}^{t}(B_2^{\mathrm{D}})^{i+2}N_A S(i)B_1C_A^{-1} & 0\end{pmatrix} P^{-1},\\
\Sigma_4 &= -B^{\mathrm{D}}\left(\sum_{i=0}^{t} AZ(i)B(A^{\mathrm{D}})^i\right)(A^{\mathrm{D}})^2\\
&= -P\begin{pmatrix}0 & 0\\ \sum_{i=0}^{t} B_2^{D}N_A S(i)B_1(C_A^{-1})^{i+2} & 0\end{pmatrix} P^{-1},\\
\Sigma_5 &= -(B^{\mathrm{D}})^2\left(\sum_{i=0}^{t-1}\sum_{k=0}^{t-1}(B^{\mathrm{D}})^i AZ(i+k+1)B(A^{\mathrm{D}})^k\right)(A^{\mathrm{D}})^2\\
&= -P\begin{pmatrix}0 & 0\\ \sum_{i=0}^{t-1}\sum_{k=0}^{t-1}(B_2^{\mathrm{D}})^{i+2}N_A S(i+k+1)B_1(C_A^{-1})^{k+2} & 0\end{pmatrix} P^{-1}.
\end{aligned}
$$

Thus, $\Sigma_1+\Sigma_2+\Sigma_3+\Sigma_4+\Sigma_5 = P^{-1}\begin{pmatrix}C_A^{-1} & 0\\ X & (N_A+B_2)^{\mathrm{D}}\end{pmatrix}P$, completing the proof of (2.3). The second statement of the theorem is easily verified. □

Remark 2.6. Our conditions in Theorem 2.5, $A^{\mathrm{D}}B = 0$, $AB^{\mathrm{D}} = 0$ and $B^{\pi}ABA^{\pi} = 0$ can be formulated geometrically as

$$\mathscr{R}(B) \subset \mathscr{N}(A^r), \quad \mathscr{R}(B^s) \subset \mathscr{N}(A) \quad \text{and} \quad \mathscr{R}(BA^{\pi}) \subset \mathscr{N}(B^{\pi}A)$$

and we see that when $r = \mathrm{ind}(A) > 1$ and $s = \mathrm{ind}(B) > 1$ these conditions are weaker than condition $AB = 0$, or $\mathscr{R}(B) \subset \mathscr{N}(A)$, assumed in [9–Theorem 2.1].

Remark 2.7. Let $A, B \in \mathbb{C}^{n\times n}$ satisfy conditions of Theorem 2.5. Then for the projection $(A+B)^{\mathrm{D}}(A+B)$ we get, after some computations, the following formula

$$
\begin{aligned}
(A+B)^{\mathrm{D}}(A+B) = B^{\mathrm{D}}B + A^{\mathrm{D}}A + &\left(\sum_{i=0}^{t}(B^{\mathrm{D}})^{i+1}AZ(i)\right)A^{\pi} \\
&+ B^{\pi}\left(\sum_{i=0}^{t}Z(i)B(A^{\mathrm{D}})^{i+1}\right) \\
&- \sum_{i=0}^{t}\sum_{k=0}^{t}(B^{\mathrm{D}})^{i+1}AZ(i+k)B(A^{\mathrm{D}})^{k+1},
\end{aligned}
$$

where $Z(i)$ is defined as in (2.4).

Corollary 2.8. *Let $A, B \in \mathbb{C}^{n\times n}$, $r = \operatorname{ind}(A)$ and $s = \operatorname{ind}(B)$. Suppose that $A^{\mathrm{D}}B = 0$ and $ABA^{\pi} = 0$. Then*

$$
\begin{aligned}
(A+B)^{\mathrm{D}} = &\left(\sum_{i=0}^{r-1}(B^{\mathrm{D}})^{i+1}A^{i}\right)A^{\pi} \\
&+ B^{\pi}\left(\sum_{i=0}^{s-1}B^{i}(A^{\mathrm{D}})^{i+1} + \sum_{i=1}^{r+s-2}\sum_{j=1}^{i}B^{i-j}A^{j}B(A^{\mathrm{D}})^{i+2}\right) \\
&- (B^{\mathrm{D}})^{2}\left(\sum_{i=0}^{r-2}(B^{\mathrm{D}})^{i}A^{i+1}B\right)A^{\mathrm{D}} - B^{\mathrm{D}}\left(\sum_{i=0}^{r-2}A^{i+1}B(A^{\mathrm{D}})^{i}\right)(A^{\mathrm{D}})^{2} \\
&- (B^{\mathrm{D}})^{2}\left(\sum_{i=0}^{r-2}\sum_{k=0}^{r-2-i}(B^{\mathrm{D}})^{i}A^{i+k+1}B(A^{\mathrm{D}})^{k}\right)(A^{\mathrm{D}})^{2}.
\end{aligned}
$$

Proof. From $A^{\mathrm{D}}B = 0$ and $ABA^{\pi} = 0$ it follows that

$$AB^{2} = ABA^{\pi}B + AB(I - A^{\pi})B = ABAA^{\mathrm{D}}B = 0$$

and thus $AB^{\mathrm{D}} = 0$. Then we can apply Theorem 2.5, together with the simplification $AZ(i) = A^{i+1}A^{\pi}$ for all $i \geqslant 0$ and $A^{i}B = A^{i+1}A^{\mathrm{D}}B = 0$ for all $i \geqslant r$, to get the result of this corollary. $\square$

Now, we can derive some special cases from Corollary 2.8.

Corollary 2.9. *Let $A, B \in \mathbb{C}^{n\times n}$, $r = \operatorname{ind}(A)$ and $s = \operatorname{ind}(B)$. Suppose that $A^{\mathrm{D}}B = 0$ and $ABA^{\pi} = 0$.*

(i) *If $B^{2} = B$ then*

$$(A+B)^{\mathrm{D}} = B\left(\sum_{i=0}^{r-1}A^{i}\right)A^{\pi} + (I-B)\left(A^{\mathrm{D}} + \sum_{i=1}^{r-1}A^{i}B(A^{\mathrm{D}})^{i+2}\right)$$

$$-B\left(\sum_{i=0}^{r-2} A^{i+1}B\right)A^{\mathrm{D}} - 2B\left(\sum_{i=0}^{r-2} A^{i+1}B(A^{\mathrm{D}})^i\right)(A^{\mathrm{D}})^2$$
$$-B\left(\sum_{i=0}^{r-3}\sum_{k=0}^{r-2-i} A^{i+k+1}B(A^{\mathrm{D}})^k\right)(A^{\mathrm{D}})^2.$$

(ii) *If B is nilpotent then we get a symmetrical result of Theorem 2.1*,

$$(A+B)^{\mathrm{D}} = A^{\mathrm{D}} + \left(\sum_{i=0}^{r+s-2}\sum_{j=0}^{i} B^{i-j}A^jB(A^{\mathrm{D}})^i\right)(A^{\mathrm{D}})^2.$$

(iii) *In particular*, *if $B^2 = 0$ then*

$$(A+B)^{\mathrm{D}} = A^{\mathrm{D}} + \left(\sum_{i=0}^{r-1} A^iB(A^{\mathrm{D}})^i\right)(A^{\mathrm{D}})^2 + B\left(\sum_{i=0}^{r-1} A^iB(A^{\mathrm{D}})^i\right)(A^{\mathrm{D}})^3.$$

Proof. Each of this cases follows directly from Corollary 2.8 and the following simplification.

(i) Since $B^2 = B$, we have $B^{\mathrm{D}} = B$ and $B^{\pi}B = 0$.
(ii) Since B is nilpotent of index s then $B^s = 0$ and $B^{\mathrm{D}} = 0$.
(iii) Since $B^2 = 0$ then $B^{\mathrm{D}} = 0$. □

Corollary 2.10. *Let $A \in \mathbb{C}^{n\times n}$, $r = \mathrm{ind}(A)$, $B \in \mathbb{C}^{n\times n}$, $s = \mathrm{ind}(B)$. If $AB^{\mathrm{D}} = 0$ and $B^{\pi}AB = 0$ then*

$$(A+B)^{\mathrm{D}} = \left(\sum_{i=0}^{r-1}(B^{\mathrm{D}})^{i+1}A^i + \sum_{i=1}^{r+s-2}\sum_{j=1}^{i}(B^{\mathrm{D}})^{i+2}AB^jA^{i-j}\right)A^{\pi}$$
$$+ B^{\pi}\left(\sum_{i=0}^{s-1} B^i(A^{\mathrm{D}})^{i+1}\right) - (B^{\mathrm{D}})^2\left(\sum_{i=0}^{s-2}(B^{\mathrm{D}})^iAB^{i+1}\right)A^{\mathrm{D}}$$
$$- B^{\mathrm{D}}\left(\sum_{i=0}^{s-2} AB^{i+1}(A^{\mathrm{D}})^i\right)(A^{\mathrm{D}})^2$$
$$- (B^{\mathrm{D}})^2\left(\sum_{i=0}^{s-2}\sum_{k=0}^{s-2-i}(B^{\mathrm{D}})^iAB^{i+k+1}(A^{\mathrm{D}})^k\right)(A^{\mathrm{D}})^2.$$

Proof. From $AB^{\mathrm{D}} = 0$ and $B^{\pi}AB = 0$ it follows that

$$A^2B = AB^{\pi}AB + A(I - B^{\pi})AB = AB^{\mathrm{D}}BAB = 0$$

and thus $A^{\mathrm{D}}B = 0$. Then we can apply Theorem 2.5, together with the simplification $Z(i)B = B^{\pi}B^{i+1}$ for all $i \geqslant 0$ and $AB^i = AB^{\mathrm{D}}B^{i+1} = 0$ for all $i \geqslant s$, to get the result of this corollary. □

Corollary 2.11. *Let* $A \in \mathbb{C}^{n\times n}$, $r = \mathrm{ind}(A)$, $B \in \mathbb{C}^{n\times n}$, $s = \mathrm{ind}(B)$. *Assume* $AB^{\mathrm{D}} = 0$ *and* $B^{\pi}AB = 0$.

(i) *If* $A^2 = A$ *then*

$$(A+B)^{\mathrm{D}} = \left(B^{\mathrm{D}} + \sum_{i=1}^{s-1}(B^{\mathrm{D}})^{i+2}AB^i\right)(I-A) + B^{\pi}\left(\sum_{i=0}^{s-1}B^i\right)A$$
$$-2(B^{\mathrm{D}})^2\left(\sum_{i=0}^{s-2}(B^{\mathrm{D}})^iAB^{i+1}\right)A - B^{\mathrm{D}}\left(\sum_{i=0}^{s-2}AB^{i+1}\right)A$$
$$-(B^{\mathrm{D}})^2\left(\sum_{i=0}^{s-3}\sum_{k=1}^{s-2-i}(B^{\mathrm{D}})^iAB^{i+k+1}\right)A.$$

(ii) *If A is nilpotent then we get Theorem 2.1 as a particular case of Corollary 2.10.*

Proof. We apply Corollary 2.10 and the following simplification.

(i) Since $A^2 = A$, we have $A^{\mathrm{D}} = A$ and $A^jA^{\pi} = 0$ for all $j \geqslant 1$.
(ii) Since B is nilpotent then $B^{\mathrm{D}} = 0$. □

If the stronger condition $AB = 0$ is satisfied then we obtain the Theorem 2.1 given in [9].

Corollary 2.12. *Let* $A, B \in \mathbb{C}^{n\times n}$, $r = \mathrm{ind}(A)$ *and* $s = \mathrm{ind}(B)$. *If* $AB = 0$ *then*

$$(A+B)^{\mathrm{D}} = B^{\mathrm{D}}\left(\sum_{i=0}^{r-1}(B^{\mathrm{D}})^iA^i\right)A^{\pi} + B^{\pi}\left(\sum_{i=0}^{s-1}B^i(A^{\mathrm{D}})^i\right)A^{\mathrm{D}}.$$

Proof. Since $AB = 0$ then it follows that $A^{\mathrm{D}}B = BA^{\mathrm{D}} = 0$. Thus we can apply Corollary 2.8 or Corollary 2.10 to get the above result. □

3. Applications

We can prove a perturbation result concerning the matrix $L - E$, generalizing [9–Corollary 2.2]. Our result recovers all the cases analyzed in [9] and thus the previous perturbation results given in [11,12,15,16]. Continuity properties of the Drazin inverse are investigated in [3] for complex matrices , and in [10,13] for linear operators. Error bounds of the perturbed Drazin inverse with certain restrictions on the perturbing matrices are given in [6,14,15] and in [7] for closed linear operators.

The conditions of the following theorem are satisfied when the idempotent matrix W commutes with L which is the case studied in [9–Corollary 2.2].

Theorem 3.1. *Consider $L - E$ and let W be an idempotent such that $WE = E$. We set $L_1 = L(I - W)$ and $R = WLW - WEW$. Suppose that*

(i) $L_1 LW = 0$,
(ii) $(I - W)L(LW - E)L_1^{\pi} = 0$,
(iii) $(I - W)LE(I - W)LWL_1^{\mathrm{D}} = 0$.
Then

$$\begin{aligned}(L-E)^{\mathrm{D}} = {} & \left(\sum_{i=0}^{l-1}(R^{\mathrm{D}})^{i+1}L_1^{i} - \sum_{i=1}^{l}(R^{\mathrm{D}})^{i+2}E(I-W)LL_1^{i-1}\right)L_1^{\pi}\\ & + R^{\pi}\left(\sum_{i=0}^{t-1}R^{i}(L_1^{\mathrm{D}})^{i+1} + \sum_{i=0}^{t-1}R^{i}E(I-W)LE(I-W)(L_1^{\mathrm{D}})^{i+4}\right)\\ & - R^{\pi}\left(\sum_{i=0}^{t-1}R^{i}E(I-W)(I+LWL_1^{\mathrm{D}})(L_1^{\mathrm{D}})^{i+2}\right)\\ & + R^{\mathrm{D}}E(I-W)L_1^{\mathrm{D}} + (I+R^{\mathrm{D}}E)(I-W)LW((L_1^{\mathrm{D}})^2\\ & - E(I-W)(L_1^{\mathrm{D}})^3) - (R^{\mathrm{D}})^2E(I-W)\\ & \times (L_1^{\pi} - LWL_1^{\mathrm{D}} - LE(I-W)(L_1^{\mathrm{D}})^2)\\ & - (R^{\mathrm{D}})^3E(I-W)LE(I-W)L_1^{\mathrm{D}},\end{aligned}$$

where $l = \operatorname{ind}(L_1)$ and $t = \operatorname{ind}(R)$.

Proof. We split $L - E$ as

$$L - E = A + B,$$

where $A = L - WLW$ and $B = WLW - E$. In order to compute A^{D} we write A as $A = L_1 + L_2$ where $L_1 = L(I - W)$ and $L_2 = (I - W)LW$. We observe that $L_2^2 = 0$. It follows from assumption (i) that $L_1L_2 = 0$. Thus we may use Corollary 2.12 to obtain

$$A^{\mathrm{D}} = L_1^{\mathrm{D}} + L_2(L_1^{\mathrm{D}})^2 \quad \text{and} \quad A^{\pi} = L_1^{\pi} - L_2L_1^{\mathrm{D}}.$$

Since $WE = E$ then $L_1B = 0$. Hence $L_1^{\mathrm{D}}B = 0$ and $A^{\mathrm{D}}B = (I + L_2L_1^{\mathrm{D}})L_1^{\mathrm{D}}B = 0$.

From (ii) and (iii) it follows that

$$ABA^{\pi} = (I - W)L(LW - E)(L_1^{\pi} - L_2L_1^{\mathrm{D}}) = 0.$$

Then we may apply Corollary 2.8, with the simplification $A^2B = (L_1 + L_2)L_1B = 0$, to give

$$(A+B)^{\mathrm{D}} = \left(\sum_{i=0}^{r-1}(B^{\mathrm{D}})^{i+1}A^i\right)A^{\pi} + B^{\pi}\left(\sum_{i=0}^{s-1}B^i(A^{\mathrm{D}})^{i+1}\right) + B^{\pi}\left(\sum_{i=1}^{s}B^{i-1}AB(A^{\mathrm{D}})^{i+2}\right) - (B^{\mathrm{D}})^2ABA^{\mathrm{D}} - B^{\mathrm{D}}AB(A^{\mathrm{D}})^2, \tag{3.1}$$

where $r = \operatorname{ind}(A) \leqslant \operatorname{ind}(L_1) + 1$ and $s = \operatorname{ind}(B) \leqslant \operatorname{ind}(R) + 1$. In order to compute B^{D} we write B as $B = R - S$ where $S = WE(I - W)$. Since $SR = 0$ and $S^2 = 0$ then we may apply Corollary 2.12 to give

$$B^{\mathrm{D}} = R^{\mathrm{D}} - (R^{\mathrm{D}})^2S \quad \text{and} \quad B^{\pi} = R^{\pi} + R^{\mathrm{D}}S.$$

For all $i \geqslant 1$ we have

$$B^i = (R-S)^i = R^i - R^{i-1}S \quad \text{and} \quad (B^{\mathrm{D}})^i = (R^{\mathrm{D}})^i - (R^{\mathrm{D}})^{i+1}S.$$

On the other hand, for all $i \geqslant 1$ we have

$$A^i = L_1^i + L_2L_1^{i-1} \quad \text{and} \quad (A^{\mathrm{D}})^i = (L_1^{\mathrm{D}})^i + L_2(L_1^{\mathrm{D}})^{i+1}.$$

Now, we compute the first term of $(A+B)^{\mathrm{D}}$:

$$\begin{aligned}
\Sigma_1 &= \left(\sum_{i=0}^{r-1}(B^{\mathrm{D}})^{i+1}A^i\right)A^{\pi} \\
&= -(R^{\mathrm{D}})^2S(L_1^{\pi} - L_2L_1^{\mathrm{D}}) + \left(\sum_{i=0}^{l-1}(R^{\mathrm{D}})^{i+1}L_1^i - \sum_{i=1}^{l}(R^{\mathrm{D}})^{i+2}SLL_1^{i-1}\right)L_1^{\pi} \\
&= -(R^{\mathrm{D}})^2E(I-W)(L_1^{\pi} - LWL_1^{\mathrm{D}}) \\
&\quad + \left(\sum_{i=0}^{l-1}(R^{\mathrm{D}})^{i+1}L_1^i - \sum_{i=1}^{l}(R^{\mathrm{D}})^{i+2}E(I-W)LL_1^{i-1}\right)L_1^{\pi}.
\end{aligned}$$

Let us compute the second and third term of $(A+B)^{\mathrm{D}}$. From assumption (ii) it follows that $L_2R = 0$, then for all $i \geqslant 2$ we have $B^{i-1}AB = R^{i-2}SL_2S$. Thus,

$$\begin{aligned}
\Sigma_2 &= B^{\pi}\left(\sum_{i=0}^{s-1}B^i(A^{\mathrm{D}})^{i+1} + \sum_{i=1}^{s}B^{i-1}AB(A^{\mathrm{D}})^{i+2}\right) \\
&= R^{\pi}\left(\sum_{i=0}^{t-1}R^i(L_1^{\mathrm{D}})^{i+1} + \sum_{i=0}^{t-1}R^iSL_2S(L_1^{\mathrm{D}})^{i+4}\right. \\
&\quad \left. - \sum_{i=0}^{t-1}R^iS(I + L_2L_1^{\mathrm{D}})(L_1^{\mathrm{D}})^{i+2}\right) \\
&\quad + R^{\mathrm{D}}SL_1^{\mathrm{D}} + (I + R^{\mathrm{D}}S)L_2((L_1^{\mathrm{D}})^2 - S(L_1^{\mathrm{D}})^3)
\end{aligned}$$

$$
\begin{aligned}
&= R^{\pi}\left(\sum_{i=0}^{t-1} R^i (L_1^{\mathrm{D}})^{i+1} + \sum_{i=0}^{t-1} R^i E(I-W)LE(I-W)(L_1^{\mathrm{D}})^{i+4}\right)\\
&\quad - R^{\pi}\left(\sum_{i=0}^{t-1} R^i E(I-W)(I+LWL_1^{\mathrm{D}})(L_1^{\mathrm{D}})^{i+2}\right)\\
&\quad + R^{\mathrm{D}}E(I-W)L_1^{\mathrm{D}} + (I+R^{\mathrm{D}}E)(I-W)LW\\
&\quad \times ((L_1^{\mathrm{D}})^2 - E(I-W)(L_1^{\mathrm{D}})^3).
\end{aligned}
$$

On the other hand, for the other terms of $(A+B)^{\mathrm{D}}$ in (3.1) we get

$$
\begin{aligned}
\Sigma_3 &= -(B^{\mathrm{D}})^2 ABA^{\mathrm{D}} - B^{\mathrm{D}}AB(A^{\mathrm{D}})^2,\\
&= -(R^{\mathrm{D}})^3 SL_2SL_1^{\mathrm{D}} - (R^{\mathrm{D}})^2 SL_2S(L_1^{\mathrm{D}})^2,\\
&= -(R^{\mathrm{D}})^3 E(I-W)LE(I-W)L_1^{\mathrm{D}} - (R^{\mathrm{D}})^2 E(I-W)LE(I-W)(L_1^{\mathrm{D}})^2.
\end{aligned}
$$

Finally, the result follows by adding $\Sigma_1 + \Sigma_2 + \Sigma_3$ to get $(L-E)^{\mathrm{D}}$. □

Here we discus some special interesting cases of Theorem 3.1. The following lemma is needed for the cases to follow.

Lemma 3.2. *Let $L \in \mathbb{C}^{n\times n}$ and let $W \in \mathbb{C}^{n\times n}$ be an idempotent. Assume that $L^2W = WL^2W = (LW)^2$. If we set $L_1 = L(I-W)$ and $L_3 = WLW$ then*

$$
L^{\mathrm{D}} = \left(\sum_{i=0}^{l-1} (L_3^{\mathrm{D}})^{i+1} L_1^i\right) L_1^{\pi} + L_3^{\pi}\left(\sum_{i=0}^{r-1} L_3^i (L_1^{\mathrm{D}})^{i+1}\right) + (I-W)LW(L_1^{\mathrm{D}})^2,
$$

$l_1 = \operatorname{ind}(L_1)$, $r = \operatorname{ind}(L_3)$, *and*

$$
(I-W)L^{\pi} = (I-W)(L_1^{\pi} - LWL_1^{\mathrm{D}}).
$$

Moreover, for all $i \geqslant 1$ we have

$$
(I-W)(L^{\mathrm{D}})^i = (I-W)((L_1^{\mathrm{D}})^i + LW(L_1^{\mathrm{D}})^{i+1})
$$

and

$$
(I-W)L^{i+1} = (I-W)LL_1^i.
$$

Proof. We split L as $L = L_1 + L_2 + L_3$, where $L_1 = L(I-W)$, $L_2 = (I-W)LW$ and $L_3 = WLW$. We observe that $L_2^2 = 0$. Condition $L^2W = (LW)^2$ implies $(L_1 + L_3)L_2 = 0$ and then

$$
L^{\mathrm{D}} = (L_1 + L_3)^{\mathrm{D}} + L_2((L_1 + L_3)^{\mathrm{D}})^2
$$

and since $L_1L_3 = 0$ we can apply Corollary 2.12 to get

$$
(L_1 + L_3)^{\mathrm{D}} = \left(\sum_{i=0}^{l-1} (L_3^{\mathrm{D}})^{i+1} L_1^i\right) L_1^{\pi} + L_3^{\pi}\left(\sum_{i=0}^{r-1} L_3^i (L_1^{\mathrm{D}})^{i+1}\right),
$$

where $l = \text{ind}(L_1)$ and $r = \text{ind}(L_3)$. Condition $L^2W = WL^2W$ implies $L_2L_3 = 0$ then $L_2L_3^{\mathrm{D}} = 0$, then

$$L_2((L_1 + L_3)^{\mathrm{D}})^2 = L_2(L_1^{\mathrm{D}})^2.$$

Then the first part of the lemma is proved. Now, from the formula for L^{D} we get that for all $i \geqslant 1$

$$(I - W)(L^{\mathrm{D}})^i = (I - W)((L_1^{\mathrm{D}})^i + LW(L_1^{\mathrm{D}})^{i+1})$$

and

$$(I - W)L^{\pi} = (I - W)(L_1^{\pi} - LWL_1^{\mathrm{D}}).$$

By other way, we can easily prove that for all $i \geqslant 1$,

$$(I - W)L^{i+1} = (I - W)LL_1^i. \qquad \square$$

Case (1) We assume (i) $WE = E$ and $EW = 0$. (ii) $L^2W = WL^2W = (LW)^2$. (iii) $(I - W)LEL^{\pi} = 0$.

Then we can apply Lemma 3.2 and Theorem 3.1, with $R = L_3 = WLW$ which implies $R^i = WL^iW$ and $(R^{\mathrm{D}})^i = (L^{\mathrm{D}})^iW$, to get

$$\begin{aligned}(L - E)^{\mathrm{D}} = {} & WL^{\mathrm{D}} - \left(\sum_{i=0}^{l-1}(L^{\mathrm{D}})^{i+1}EL^i\right)(I - W)L^{\pi} \\ & + (I - W + L^{\mathrm{D}}E)(L^{\mathrm{D}} + LW(L^{\mathrm{D}})^2) \\ & + L^{\pi}W\left(\sum_{i=0}^{t-1}L^iELE(L^{\mathrm{D}})^{i+4} - \sum_{i=0}^{t-1}L^iE(L^{\mathrm{D}})^{i+2}\right) \\ & - (I - W + L^{\mathrm{D}}E)LE(L^{\mathrm{D}})^3 - (L^{\mathrm{D}})^3ELEL^{\mathrm{D}} \\ & - (L^{\mathrm{D}})^2ELE(L^{\mathrm{D}})^2.\end{aligned}$$

Case (1a) We assume (i) and (ii) as in Case (1) and $LE = WLE$. Then

$$\begin{aligned}(L - E)^{\mathrm{D}} = {} & WL^{\mathrm{D}} - \left(\sum_{i=0}^{l-1}(L^{\mathrm{D}})^{i+1}EL^i\right)(I - W)L^{\pi} \\ & + (I - W + L^{\mathrm{D}}E)(L^{\mathrm{D}} + LW(L^{\mathrm{D}})^2) \\ & - L^{\pi}W\left(\sum_{i=0}^{t-1}L^iE(L^{\mathrm{D}})^{i+2}\right).\end{aligned}$$

Case (1b) We assume (i) as in Case (1) and $LW = WLW$. Then

$$(L - E)^{\mathrm{D}} = WL^{\mathrm{D}} - \left(\sum_{i=0}^{l-1}(L^{\mathrm{D}})^{i+1}EL^i\right)(I - W)L^{\pi}$$

$$+ (I - W + L^{\mathrm{D}}E)L^{\mathrm{D}} - L^{\pi}W\left(\sum_{i=0}^{t-1} L^{i}E(L^{\mathrm{D}})^{i+2}\right).$$

Case (1c) We assume (i) as in Case (1) and $LW = WL$. Then

$$(L - E)^{\mathrm{D}} = L^{\mathrm{D}}W - \left(\sum_{i=0}^{l-1}(L^{\mathrm{D}})^{i+1}EL^{i}\right)L^{\pi} + (I - W + L^{\mathrm{D}}E)L^{\mathrm{D}} - \left(\sum_{i=0}^{t-1} L^{i}L^{\pi}E(L^{\mathrm{D}})^{i+2}\right).$$

Example 3.3. We set

$$L = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad E = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}.$$

Consider

$$W = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then

$$LW = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad WL = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

We have $LW = WLW$ and $WE = E$, so we can apply Case (1b). However we see $LW \neq WL$.

Example 3.4. We set

$$L = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix}, \text{ then } \quad L^{\pi} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix}$$

and consider

$$W = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}, \quad E_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad E_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 \end{pmatrix}.$$

Then

$$L^2W = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{pmatrix}, \quad LE_2 = \begin{pmatrix} -1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \end{pmatrix}, \quad LE_2L^{\pi} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

We see that $L^2W = WL^2W = (LW)^2$. Moreover, for the matrix E_1 we have $WE_1 = E_1$, $E_1W = 0$ and $LE_1 = WLE_1$. Thus we can apply Case(1a) to the matrix $L - E_1$. For the matrix E_2 we have $WE_2 = E_2$, $E_2W = 0$ and $(I - W)LE_2L^{\pi} = 0$, however $(I - W)LE_2 \neq 0$. Thus, we can apply Case (1) to the matrix $L - E_2$.

Case (2) We assume (i) $WE = E$. (ii) $L^2W = WL^2W = (LW)^2$. (iii) $ELW = EWLW$, $(I - W)LEW = 0$ and $(I - W)LEL^{\pi} = 0$.

We apply Lemma 3.2 and Theorem 3.1 having in count that condition $ELW = EWLW$ implies that, for all $i \geqslant 1$,

$$W(L - E)^iW = (WLW - EW)^i = R^i.$$

Thus,

$$\begin{aligned}(L - E)^{\mathrm{D}} = {} & \left(\sum_{i=0}^{l-1}(R^{\mathrm{D}})^{i+1}L(I - W)L^{i-1} - \sum_{i=0}^{l-1}(R^{\mathrm{D}})^{i+2}E(I - W)L^i\right) \\ & \times (I - W)L^{\pi} + R^{\pi}W\left(\sum_{i=0}^{t-1}(L - E)^iWL(I - W)(L^{\mathrm{D}})^{i+2}\right. \\ & \left. + \sum_{i=0}^{t-1}(L - E)^iE(I - W)LE(I - W)(L^{\mathrm{D}})^{i+4}\right) \\ & - R^{\pi}W\left(\sum_{i=0}^{t-1}(L - E)^iE(I - W)(L^{\mathrm{D}})^{i+2}\right) + R^{\mathrm{D}}E(I - W)L^{\mathrm{D}} \\ & + (I - W)LW(L^{\mathrm{D}})^2 - (I + R^{\mathrm{D}}E))(I - W)LE(I - W)(L^{\mathrm{D}})^3 \\ & - (R^{\mathrm{D}})^3E(I - W)LE(I - W)L^{\mathrm{D}} \\ & - (R^{\mathrm{D}})^2E(I - W)LE(I - W)(L^{\mathrm{D}})^2).\end{aligned}$$

Case (2a) Assume (i) and (ii) as in Case (2) and $E(I - W)LW = (I - W)LE = 0$. Then

$$\begin{aligned}(L - E)^{\mathrm{D}} = {} & \left(\sum_{i=0}^{l-1}(R^{\mathrm{D}})^{i+1}L(I - W)L^{i-1} - \sum_{i=0}^{l-1}(R^{\mathrm{D}})^{i+2}E(I - W)L^i\right) \\ & \times (I - W)L^{\pi} + R^{\pi}W\left(\sum_{i=0}^{t-1}(L - E)^iWL(I - W)(L^{\mathrm{D}})^{i+2}\right.\end{aligned}$$

$$\left. - \sum_{i=0}^{t-1}(L-E)^i E(I-W)(L^{\mathrm{D}})^{i+2}\right) + R^{\mathrm{D}}E(I-W)L^{\mathrm{D}}$$
$$+ (I-W)LW(L^{\mathrm{D}})^2.$$

Case (2b) Assume (i) as in Case (2) and $(I-W)LW=0$. Then

$$(L-E)^{\mathrm{D}} = \left(\sum_{i=0}^{l-1}(R^{\mathrm{D}})^{i+1}L(I-W)L^{i-1} - \sum_{i=0}^{l-1}(R^{\mathrm{D}})^{i+2}E(I-W)L^i\right)$$
$$\times (I-W)L^{\pi} + R^{\pi}W\left(\sum_{i=0}^{t-1}(L-E)^i WL(I-W)(L^{\mathrm{D}})^{i+2}\right.$$
$$\left. - \sum_{i=0}^{t-1}(L-E)^i E(I-W)(L^{\mathrm{D}})^{i+2}\right) + R^{\mathrm{D}}E(I-W)L^{\mathrm{D}}.$$

Example 3.5. We consider the matrix L and the idempotent W of Example 3.4 and we set

$$E_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & -1 & 2 & 3 \end{pmatrix}.$$

Then

$$E_3LW = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 0 & -3 & 3 \end{pmatrix}, \quad LE_3W = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 0 & 2 & 3 \end{pmatrix}.$$

We see that $E_3LW = E_3WLW$, $(I-W)LE_3W = 0$ and $(I-W)LE_3L^{\pi} = 0$ thus we can apply Case (2) to the matrix $L - E_3$.

Case (3) Assume $WE = E$, $LW = 0$. In this case $LE = 0$ and we get Corollary 2.12 as a particular case of our perturbation result.

$$(L-E)^{\mathrm{D}} = \left(\sum_{i=0}^{l-1}(-E^{\mathrm{D}})^{i+1}WL^i\right)L^{\pi} + E^{\pi}\left(\sum_{i=0}^{t-1}(-E)^i W(L^{\mathrm{D}})^{i+1}\right).$$

Case (4) Assume $WE = E$, $EW = WLW$, $L^2W = WL^2W = (LW)^2$ and $(I-W)LE(I-W)L^{\pi} = 0$. Then we can apply Lemma 3.2 and Theorem 3.1 with $R = 0$ to get

$$(L-E)^{\mathrm{D}} = (I-W)L^{\mathrm{D}} - WL(I-W)(L^{\mathrm{D}})^2 - E(I-W)(L^{\mathrm{D}})^2 + E(I-W)LE(I-W)(L^{\mathrm{D}})^4 - (I-W)LE(I-W)(L^{\mathrm{D}})^3.$$

Case (4a) Assume $WE = E$, $EW = WLW$, $L^2W = WL^2W = (LW)^2$ and $LE = WLE$. Then

$$(L-E)^{\mathrm{D}} = (I-W)L^{\mathrm{D}} - WL(I-W)(L^{\mathrm{D}})^2 - E(I-W)(L^{\mathrm{D}})^2.$$

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