Characterization of the minimal and maximal operator ideals associated to the tensor norm defined by a sequence space

J.A. López Molina * and M.J. Rivera

Universidad Politécnica de Valencia, E.T.S. Ingenieros Agrónomos, Camino de Vera, 46072 Valencia, Spain

Received 12 May 2003
Available online 28 May 2004
Submitted by J. Diestel

Abstract

We characterize the minimal and maximal operator ideals associated, in the sense of Defant and Floret, to a wide class of tensor norms derived from a Banach sequence space. Our results are extensions of classical ones about tensor norms of Saphar [Studia Math. 38 (1972) 71–100] and show the key role played by the structure of finite-dimensional subspaces in this kind of problems.

© 2004 Elsevier Inc. All rights reserved.

Keywords: Integral operators; Ultraproducts of spaces and maps

1. Introduction

One of the problems in the concrete theory of tensor products and operator ideals in the class of Banach spaces is to define specific tensor norms and to study its associated operator ideals. For this goal, \( \ell_p \) spaces play a central role in the definition of the classical tensor norms of Grothendieck, Saphar and Lapresté, and it is quite natural to try to replace \( \ell_p \) for a more general Banach sequence space, an idea pointed out many years ago by De Grande-De Kimpe in [2] and by Harksen in his doctoral dissertation [5].
However until now this idea has not gone beyond of a mere, although general, example of tensor norms. Probably this is due to the fact that the study of operators naturally related to classical tensor norms is very dominated by the special properties of $L_p(\mu)$ spaces and thus the crucial key of the problem remains hidden in this setting.

In this paper, we study tensor norms $g^e_\lambda$ defined by a sequence space $\lambda$ in the sense of De Grande-De Kimpe [2] and Harksen [5] and also its associated operator ideals. To be precise, the main goal of this paper is to obtain characterizations of the operators lying in the minimal and maximal operator ideal associated to $g^e_\lambda$ by means of suitable factorization theorems.

Our main idea is to use the so-called “local theory” of Banach spaces, i.e., the study of the involved problems in terms of its finite-dimensional subspaces, a tool which has so much enriched our understanding of Banach spaces in other many aspects. Ultraproducts and finite representability are in the heart of the local theory. The ultraproducts technique allows us to study some operators in terms of its restrictions to finite-dimensional subspaces. In the factorization theorem of classical $\ell_p$-integral operators of Saphar, which is the key of the proof of many metric properties of the related tensor norms and operator ideals, the lattice isomorphism between an ultrapower of $\ell_p$ spaces and some $L_p(\mu)$ space is essential. Unfortunately the structure of ultrapowers of general sequence spaces is not so simple as in the case of $\ell_p$ spaces and they do not have a natural and easy representation. On the other hand, finite representability is not enough for our purposes and we shall need a stronger notion generalizing the concept of $L^p$-spaces of Lindenstrauss and Pelczyński (Section 3).

The notation is standard. All the spaces considered are Banach spaces over the real field in order to use more easily known results in the theory of Banach lattices. If we wish to emphasize in the space $E$ where a norm is defined we shall write $\| \cdot \|_E$. The canonical inclusion map of a Banach space $E$ into the bidual $E''$ will be denoted by $J_E$. The set of finite-dimensional subspaces of a normed space $E$ will be denoted by $\text{FIN}(E)$.

Concerning Banach lattices we refer the reader to [1]. We recall the more relevant definitions and properties for us. A Banach lattice $E$ is order complete or Dedekind complete if every order bounded set in $E$ has a least upper bound in $E$, and it is order continuous if every order convergent filter is norm convergent. Every dual Banach sequence lattice $E'$ is order complete and the reflexive spaces are even order continuous. A linear map $T$ between Banach lattices $E$ and $F$ is said to be positive if $T(x) \geq 0$ in $F$ for every $x \in E$, $x \geq 0$. $T$ is called order bounded if $T(A)$ is order bounded in $F$ for every order bounded set $A$ in $E$.

Let $\omega$ be the vector space of all scalar sequences and $\varphi$ its subspace of the sequences with finitely many nonzero coordinates. A sequence space $\lambda$ is a linear subspace of $\omega$ containing $\varphi$ provided with a topology finer than the topology of coordinatewise convergence. A Banach sequence space $\lambda$ will be a sequence space $\lambda$ provided with a norm which makes it a Banach lattice and an ideal in $\omega$, i.e., such that if $|x| \leq |y|$ with $x \in \omega$ and $y \in \lambda$, then $x \in \lambda$ and $\|x\|_\lambda \leq \|y\|_\lambda$. A sectional subspace $S_k(\lambda)$, $k \in \mathbb{N}$, is the topological subspace of $\lambda$ of those sequences $(\alpha_i)$ such that $\alpha_i = 0$ for every $i \geq k$. Clearly $S_k(\lambda)$ is 1-complemented in $\lambda$.

The Köthe dual (or $\alpha$-dual) $\lambda^\times$ of a sequence space $\lambda$ is defined as the set of scalar sequences $(b_i)$ such that $\sum_{i=1}^{\infty} |a_i b_i|$ converges for every $(a_i) \in \lambda$. In general, if $\lambda$
is a Banach sequence space, the Köthe dual $\lambda^\times$ is a closed subspace of the Banach dual $\lambda'$.

A Banach sequence space $\lambda$ will be called regular whenever the sequence $\{e_i\}_{i=1}^\infty$, where $e_i := (\delta_{ij})$ (Kronecker’s delta) forms a Schauder basis in $\lambda$. Every Banach sequence space $\lambda$ has a solid and regular subspace $\lambda_r := \overline{\phi}\lambda$ such that $\lambda$ is regular if and only if $\lambda = \lambda_r$ (see Lemma 3.3 in [8], for example).

The paper is organized as follows. Section 2 contains the fundamental facts about the tensor norm $g^\!*_\lambda$ derived from a Banach sequence space $\lambda$ and a characterization of the minimal operator ideal associated to the dual tensor norm $(g^\!*_\lambda)'$ (the so-called $\lambda$-nuclear operators) by means of suitable factorizations. Section 3 is technical with respect our principal problems. It contains the main examples of certain class of Banach spaces defined by special properties of its finite-dimensional subspaces, precisely that every one of them be contained in a larger subspace isomorphic to finite-dimensional complemented subspaces of a general regular Banach sequence space with controlled norms of the involved isomorphisms and projections. To get significant examples of these spaces we need to restrict our consideration to the class (quite large) of Banach sequence spaces with the uniform projection property defined by Pelczyński and Rosenthal in [11]. Finally, Section 4 deals with the study of the maximal operator ideal associated to the tensor norm $g^\!*_\lambda$ (the ideal of $\lambda$-integral operators). First we obtain a general sufficient condition for an operator to be $\lambda$-integral and second a necessary condition. In the case $\lambda$ has the uniform projection property we get a complete characterization of $\lambda$-integral operators.

2. The tensor norm $g^\!*_\lambda$ associated to a Banach sequence space $\lambda$

and $\lambda$-nuclear operators

We suppose the reader is familiar with the theory of operator ideals and tensor norms. Of course, the fundamental references about these questions are the books [12] and [3] of Pietsch and Defant and Floret, respectively. We set the notation to be used.

Given a pair of Banach spaces $E$ and $F$ and a tensor norm $\alpha$, $E \otimes_\alpha F$ represents the space $E \otimes F$ endowed with the $\alpha$-normed topology. The completion of $E \otimes_\alpha F$ is denoted by $\hat{E} \otimes_\alpha F$, and the norm of $z$ in $\hat{E} \otimes_\alpha F$ by $\alpha(z; E \otimes F)$. If there is no risk of mistake we write $\alpha(z)$ instead of $\alpha(z; E \otimes F)$. For technical requisites of the standard theory of tensor norms (see Criterion 12.2 in [3]), given a Banach sequence space $\lambda$ with the quoted properties in the Introduction, from now on it will be supposed furthermore that $\|e_i\|_\lambda = \|e_i\|_\times = 1$ for every $i \in \mathbb{N}$.

Given a Banach space $E$, a sequence $(x_n)_{n=1}^\infty \in E^\mathbb{N}$ is said to be strongly $\lambda$-summing if $\pi_\lambda((x_i)) := \|((x_n))\|_\lambda < \infty$ and it is said to be weakly $\lambda$-summing if $\varepsilon_\lambda((x_i)) := \sup_{\|x\| \leq 1} \|((x_n, x'))\|_\lambda < \infty$. From now on $\lambda[E]$ (respectively $\lambda(E)$) will denote the space of all strongly (respectively weakly) $\lambda$-summing sequences in $E$ endowed with the norm $\pi_\lambda$ (respectively $\varepsilon_\lambda$).

Let $E$ and $F$ be Banach spaces. For every $z \in E \otimes F$ we define

$$g_\lambda(z) := g_\lambda(z; E \otimes F) := \inf \left\{ \pi_\lambda((x_n))\varepsilon_\lambda((y_n)) \mid z = \sum_{n=1}^m x_n \otimes y_n \right\}.$$
In the case of \( \lambda = \ell_p, \ 1 \leq p < \infty \), \( g_\lambda \) is a norm in \( E \otimes F \), the classical tensor norm \( g_\lambda \) of Saphar (see [13]), but in general \( g_\lambda (\cdot; E \otimes F) \) only is a reasonable quasi-norm in \( E \otimes F \); see [2] and [5]. We denote \( E \hat{\otimes}_{g_\lambda} F \) the corresponding quasi-Banach space.

Let \( g^\lambda_z(z; E \otimes F) \) be the Minkowski functional of the absolutely convex hull of the unit closed ball \( B_{g_\lambda} := \{ z \in E \otimes F \mid g_\lambda(z) \leq 1 \} \) of the quasi-norm \( g_\lambda \) in \( E \otimes F \). It is easy to see that \( g^\lambda_z(z; E \otimes F) \) can be evaluated as

\[
g^\lambda_z(z; E \otimes F) := \inf \left\{ \left( \sum_{i=1}^{n} \pi_\lambda((x_{ij})) \varepsilon_{\lambda^*}((y_{ij})) \right) \mid z = \sum_{i=1}^{n} \sum_{j=1}^{m} x_{ij} \otimes y_{ij} \right\}
\]

and that \( g^\lambda_z \) is a tensor norm in the class of all Banach spaces such that

\[
\forall z \in E \otimes F, \quad g^\lambda_z(z; E \otimes F) \leq g_\lambda(z; E \otimes F). \tag{1}
\]

A suitable representation of the elements of the completed tensor products \( E \hat{\otimes}_{g_\lambda} F \) and \( E \hat{\otimes}_{g^\lambda_z} F \) is a basic tool in the study of the involved operator ideals. It is straightforward to see (see, for instance, [2] and [13]) that if \( z \in E \hat{\otimes}_{g_\lambda} F \), there are \((x_i)_{i=1}^{\infty} \in \lambda_r[E] \) and \((y_i)_{i=1}^{\infty} \in \lambda^*(F) \) such that \( \pi_\lambda((x_i)) \varepsilon_{\lambda^*}((y_i)) < \infty \) and

\[
z = \sum_{i=1}^{\infty} x_i \otimes y_i.
\]

Moreover the quasi-norm of \( z \) in \( E \hat{\otimes}_{g_\lambda} F \) (again denoted by \( g_\lambda(z) \)) is given by

\[
g_\lambda(z) = \inf \pi_\lambda((x_i)) \varepsilon_{\lambda^*}((y_i)),
\]

taking the infimum over all such representations of \( z \). In the same way it can be shown that \( z \in E \hat{\otimes}_{g^\lambda_z} F \) can be represented as

\[
z = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} x_{ij} \otimes y_{ij}, \tag{2}
\]

where \((x_{ij})_{i=1}^{\infty} \mid j \in \mathbb{N} \rangle \subset \lambda_r[E], \ (y_{ij})_{i=1}^{\infty} \mid j \in \mathbb{N} \rangle \subset \lambda^*(F) \) and

\[
\sum_{j=1}^{\infty} \pi_\lambda((x_{ij})) \varepsilon_{\lambda^*}((y_{ij})) < \infty. \tag{3}
\]

Moreover, the norm of \( z \) in \( E \hat{\otimes}_{g^\lambda_z} F \) is the infimum of the numbers in (3) over all representations of type (2).

Following the arguments of [2, Proposition 16], the bilinear onto map

\[
R : \lambda_r[E] \times \lambda^*(F) \to E \hat{\otimes}_{g_\lambda} F
\]

such that \( R((x_i), (y_i)) = \sum_{i=1}^{\infty} x_i \otimes y_i \) is continuous with quasi-norm less than or equal one. Then since \( \lambda_r[E] \) and \( \lambda^*(F) \) are normed spaces by [15] there exists a unique linear and continuous map \( \lambda_r[E] \oplus_{\pi} \lambda^*(F) \to E \hat{\otimes}_{g_\lambda} F \). This map can be extended to a continuous linear and onto map \( \lambda_r[E] \hat{\oplus}_{\pi} \lambda^*(F) \to E \hat{\otimes}_{g_\lambda} F \). By the open mapping theorem, this map is open and hence \( E \hat{\otimes}_{g_\lambda} F \) is isomorphic to a quotient of a Banach space, then it is a Banach space. Then the topology defined by \( g_\lambda \) in \( E \otimes F \) is always normable,
i.e., there is a norm \( w_λ(\cdot; E \otimes F) \) equivalent to the quasi-norm \( g_λ(\cdot; E \otimes F) \). Now it is easy to see that \( w_λ(\cdot; E \otimes F) \), \( g_λ(\cdot; E \otimes F) \) and \( g_3^λ(\cdot; E \otimes F) \) are equivalent with \( g_3^λ(\cdot; E \otimes F) \leq w_λ(\cdot; E \otimes F) \). In view of this equivalence, one is tempted to use the easier \( g_λ \) quasi-norm instead of the norm \( g_3^λ \), but in the last part of the paper the \( g_3^λ \) norm is necessary. In order to simplify, we do the proofs with the \( g_λ \) quasi-norm when the same methods work using \( g_3^λ \) with slight modifications.

Every representation of \( z \in E' \hat{\otimes}_{g_3^λ} F \) of the type (2) defines a map \( T_z \in \mathcal{L}(E, F) \) such that \( \forall x \in E \),

\[
T_z(x) := \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} (x'_{ij}, x) y_{ij}.
\]

We remark that all these representations of the same \( z \) define the same map \( T_z \). Let \( \Phi_{EF} : E' \hat{\otimes}_{g_3^λ} F \to \mathcal{L}(E, F) \) be defined by \( \Phi_{EF}(z) := T_z \). We set

**Definition 1.** Let \( E, F \) be Banach spaces. An operator \( T : E \to F \) is said to be \( \lambda \)-nuclear if \( T = \Phi_{EF}(z) \) for some \( z \in E' \hat{\otimes}_{g_3^λ} F \).

\( \mathcal{N}_λ(E, F) \) denotes the space of the \( \lambda \)-nuclear operators \( T : E \to F \) endowed with the topology of the norm

\[
\mathcal{N}_λ(T) := \inf \{ g_3^λ(z) \mid \Phi_{EF}(z) = T \}
\]

or with the equivalent quasi-norm

\[
\mathcal{N}_λ(T) := \inf \{ g_λ(z) \mid \Phi_{EF}(z) = T \}.
\]

For every pair of Banach spaces \( E \) and \( F \), \( (\mathcal{N}_λ(E, F), \mathcal{N}_3^λ) \) is a component of the minimal Banach operator ideal \( (\mathcal{N}_3^λ, \mathcal{N}_3^λ) \) associated to the tensor norm \( g_3^λ \). The space \( \mathcal{N}_λ(E, F) \) coincides with the \( \lambda \)-nuclear operators in the sense of Dubinsky and Ramanujan introduced previously in [4] in order to deal with a different kind of problems.

We have the following characterization of \( \lambda \)-nuclear operators.

**Theorem 2.** Let \( E \) and \( F \) be Banach spaces and let \( T \) be an operator in \( \mathcal{L}(E, F) \). Then the following statements are equivalent:

1. \( T \) is \( \lambda \)-nuclear.
2. \( T \) factors continuously in the following way:

\[
\begin{array}{c}
E \\
A \\
\ell_\infty \\
B \\
\lambda_r \\
\end{array}
\begin{array}{c}
T \\
\mathcal{L}(E, F) \\
C \\
F \\
\end{array}
\]

where \( B \) is a diagonal multiplication operator defined by a positive sequence \( (b_i) \in \lambda_r \).

Furthermore \( \mathcal{N}_λ(T) = \inf \{ ||C|| ||B|| ||A|| \} \), taking it over all such factors.
(3) $T$ factors continuously in the following way:

```
    E  
  ↓   ↓
  A  B  C  
  ↑   ↑
    F

\[ \ell_\infty \{ \ell_\infty \} \quad \ell_1 [\lambda_r] \]
```

where $B$ is a diagonal multiplication operator defined by a positive sequence $(b_i) \in \ell_1 [\lambda_r]$.

Furthermore $N^*_\lambda (T) = \inf\{\|C\|\|B\|\|A\|\}$, taking it over all such factors.

**Proof.** (2) $\Rightarrow$ (1) Assume we have a factorization $T = CBA$ as the given one in the diagram. Consider the transposed map $A' : (\ell_\infty)' \to E'$ and put $x'_i := A'(e_i)$. Then $\langle A(x), e_i \rangle = \langle A'(e_i), x \rangle = \langle x'_i, x \rangle$ for every $i \in \mathbb{N}$ and $A(x) = ((x'_i, x))_{i=1}^\infty$.

Remark that if $u = (u_i)$ such that $u_i = 1$ for every $i \in \mathbb{N}$ and $B(u) = (b_i)_{i=1}^\infty$, then $B$ is the multiplication operator defined by $(b_i)_{i=1}^\infty \in \lambda_r$ and $\|B\| \leq \|b_i\|_{\lambda_r}$.

Let $C(e_i) = y_i$ for every $i \in \mathbb{N}$. Then

\[ \forall (\beta_i) \in \lambda_r, \quad \|C((\beta_i))\|_F = \sup_{1 \leq i' \leq 1, y'} \sum_{i=1}^\infty |\beta_i (y_i, y'_i)| \leq \|C\| \|\beta_i\|_{\lambda_r}. \]

We obtain that for every $y'$ in the unit ball of $F'$ the sequence $((y_i, y'_i))$ lies in $\lambda^\times$ and $\|(y_i, y'_i)\|_{\lambda^\times} \leq \|C\|$. In consequence $(y_i) \in \lambda^\times (F)$ and $\varepsilon_{\lambda^\times} ((y_i)) \leq \|C\|$. Then we have $T = \Phi_{EF}(z)$ with $z = \sum_{i=1}^\infty b_i x'_i \otimes y_i \in E' \otimes_{\varepsilon} F$. Hence $T \in \mathcal{N}(E, F)$ and $\mathcal{N}_\lambda (T) \leq \|A\| \|B\| \|C\|$.

(1) $\Rightarrow$ (2) Assume $T$ is $\lambda$-nuclear. Given $\varepsilon > 0$, there is a representation $T = \sum_{i=1}^\infty x'_i \otimes y_i$ such that $(x'_i)_{i=1}^\infty \in \lambda^\times [E']$, $(y_i)_{i=1}^\infty \in \lambda^\times (F)$ and

\[ N_{\lambda} (T) + \varepsilon \geq \varepsilon_{\lambda^\times} ((y_i)) \leq \varepsilon_{\lambda^\times} ((y'_i)). \]

Indeed we can suppose that $\varepsilon_{\lambda^\times} ((y_i)) = 1$ and $N_{\lambda} (T) + \varepsilon \geq \varepsilon_{\lambda^\times} ((x'_i))$.

Let $A : E \to \ell_\infty$ be such that

\[ A(x) := \left( \frac{x'_i, x}{\|x'_i\|} \right)_{i=1}^\infty \]

which is linear and continuous with $\|A\| \leq 1$. Let $B : \ell_\infty \to \lambda_r$ be such that

\[ B((\lambda_i)) := (\lambda_i \|x'_i\|)_{i=1}^\infty. \]

Then

\[ \|B((\lambda_i))\|_1 \leq \|\lambda_i\|_{\ell_\infty} \|\|x'_i\|\|_1 = \|\lambda_i\|_{\ell_\infty} \varepsilon_{\lambda^\times} ((x'_i)) \]

and hence $B$ is linear and continuous with $\|B\| \leq \varepsilon_{\lambda^\times} ((x'_i))$. Finally let $C : \lambda_r \to F$ be such that

\[ C((b_i)) := \sum_{i=1}^\infty b_i y_i. \]
$C$ is linear and continuous with
\[
\|C((\beta_i))\| = \sup_{\|y\| \leq 1} \sum_{i=1}^{\infty} \langle (\beta_i), y_i \rangle \leq \|\beta_i\|_{\lambda} \langle \varepsilon_{\lambda}((y_i)) \rangle = \|\beta_i\|_{\lambda}
\]
and then $\|C\| \leq 1$. Clearly we have $T = CBA$ and
\[N_\lambda(T) + \varepsilon \geq \pi_\lambda((x'_i)) \varepsilon_{\lambda}(\varepsilon_{\lambda}(x'_i))) \geq \|A\|\|B\|\|C\|.
\]
Since $\varepsilon > 0$ is arbitrary the implication is proved.

The proof of (1) $\iff$ (3) is similar using the norm $N_c^\varepsilon$.  

The following proposition provide examples of $\lambda$-nuclear operators.

**Proposition 3.** Let $\lambda$ be a regular order complete Banach sequence space. Then every positive operator $S : L_\infty(\mu) \to \lambda$ and every positive operator $S : L_\infty(\mu) \to \ell_1(\lambda)$ are $\lambda$-nuclear, and there is $g \in \lambda$ ($g \in \ell_1(\lambda)$ in the second case) such that $N_\lambda(S) \leq \|g\|_{\lambda}$ (respectively $N_c^\varepsilon(S) \leq \|g\|_{\lambda}$).

**Proof.** We only will do the proof if $S : L_\infty(\mu) \to \lambda$ is a positive operator. Since $\lambda$ is an order complete Banach lattice there is $g \in \lambda$ such that
\[g = \sup \{S(f) : \|f\|_{L_\infty(\mu)} \leq 1\}.
\]
We define $A : L_\infty(\mu) \to \ell_\infty$ such that
\[\forall h \in L_\infty(\mu), \quad \langle A(h), e_i \rangle := \langle S(h), e_i \rangle \langle g, e_i \rangle\]
if $\langle g, e_i \rangle \neq 0$ and $\langle A(h), e_i \rangle := 0$ otherwise. We define also the diagonal operator $D_g : \ell_\infty \to \lambda$ such that
\[\forall (a_i) \in \ell_\infty, \quad D_g((a_i)) = (a_i \langle g, e_i \rangle).
\]
It is easy to see that $A$ and $D_g$ are linear and continuous with $\|A\| \leq 1$ and $\|D_g\| \leq \|g\|_{\lambda}$, and that $S = D_g A$. By Theorem 2, $S$ is $\lambda$-nuclear with $N_\lambda(S) \leq \|g\|_{\lambda} \|A\| \leq \|g\|_{\lambda}$.  

### 3. Quasi-$\ell^\lambda$-spaces and ultrapowers

To get our most deep results the structure of finite-dimensional subspaces of the involved Banach sequence spaces $\lambda$ and so its behavior under ultraproducts will be crucial.

Concerning to ultraproducts of Banach spaces the standard reference is [7] and we refer to it for concrete definitions. We only set the notation we will use.

Let $D$ be a nonempty index set and $U$ a nontrivial ultrafilter in $D$. Given a family $\{X_d, \ d \in D\}$ of Banach spaces, $(X_d)_U$ denotes the corresponding ultraproduct Banach space. If every $X_d, \ d \in D$, coincides with a fixed Banach space $X$ the corresponding ultraproduct is named an ultrapower of $X$ and is denoted by $(X)_U$. Remark that if every $X_d, \ d \in D$, is a Banach lattice, $(X_d)_U$ has a canonical order which makes it a Banach
lattice. If we have another family of Banach spaces \( \{ Y_d, \ d \in D \} \) and a family of operators \( \{ T_d \in \mathcal{L}(X_d, Y_d), \ d \in D \} \) such that \( \sup_{d \in D} \| T_d \| < \infty \), then \( (T_d)_{|U} \in \mathcal{L}((X_d)_{|U}, (Y_d)_{|U}) \) denotes the canonical ultraproduct operator.

We introduce now two classes of Banach spaces useful for us. The first one is a generalization of \( L^p \) spaces of Lindenstrauss and Pelczyński.

**Definition 4.** Let \( \lambda \) be a Banach sequence space. We say that a Banach space \( X \) is an \( L^\lambda \) space if for every \( F \in \text{FIN}(X) \) and every \( \epsilon > 0 \) there is \( G \in \text{FIN}(X) \) containing \( F \) such that \( d(G, S_{\dim(G)}(\lambda)) \leq 1 + \epsilon \).

An easy example is

**Proposition 5.** Let \( \lambda \) be a regular Banach sequence space. Then \( \lambda \) is an \( L^\lambda \) space.

**Proof.** It is an immediate consequence of Theorem 6 in [9, §16]. \( \square \)

To be an \( L^\lambda \) space turns out to be a very strong condition with bad stability properties under ultraproducts. We need a weaker condition.

**Definition 6.** Let \( \lambda \) be a Banach sequence space. We say that a Banach space \( X \) is a quasi-\( L^\lambda \) space if there are \( a > 0 \) and \( b > 0 \) such that for every \( M \in \text{FIN}(X) \) there are \( M_1 \in \text{FIN}(X) \) containing \( M \) and a \( b \)-complemented subspace \( H \subset X \), such that \( d(M_1, H) \leq a \). Moreover if \( X \) is a Banach lattice we say that it is a quasi-\( L^\lambda \) lattice.

Of course every \( L^\lambda \) space is a quasi-\( L^\lambda \) space. Moreover in many cases quasi-\( L^\lambda \) spaces have better stability properties under ultraproducts than \( L^\lambda \) spaces do. For this we need to consider Banach sequence spaces with the uniform projection property introduced by Pelczyński and Rosenthal [11].

**Definition 7.** A Banach space \( X \) has the uniform projection property if there is a number \( b > 0 \) such that for every \( k \in \mathbb{N} \) there is \( m(k) \in \mathbb{N} \) with the following property: for every \( F \in \text{FIN}(X) \) with dimension \( k \) there is a \( b \)-complement subspace \( G \in \text{FIN}(X) \) containing \( F \) with dimension \( \dim(G) \leq m(k) \).

The class of Banach spaces with the uniform projection property is quite large. For example, all \( L_p(\mu) \) spaces, \( 1 \leq p \leq \infty \), and all \( L^p(\mu) \) spaces of Lindenstrauss and Pelczyński, \( 1 \leq p \leq \infty \), have this property (see [11]). Moreover all reflexive Orlicz sequence spaces have also the uniform projection property (see remarks to Theorem 4 in [10]). Indeed reflexive modular sequence spaces have the uniform projection property since they can be embedded as complemented subspaces of reflexive Orlicz sequence spaces (same remark in [10]) and we can make a composition of projections. Moreover from [7, Theorem 9.4], if \( E \) has the uniform projection property, then \( L_p(\mu, E) \), \( 1 \leq p \leq \infty \), has the same property with the same projection norm than \( E \). In particular if the Banach sequence space \( \lambda \) satisfies the uniform projection property, \( \ell_1[\lambda] \) does.
Proposition 8. Let $\lambda$ be a Banach sequence space with the uniform projection property. Then every ultrapower $(\lambda)_{\mathcal{U}}$ is a quasi-$L^\infty$ lattice.

Proof. We denote by $D$ the index set of $\mathcal{U}$. Let $M$ be an $n$-dimensional subspace of $(\lambda)_{\mathcal{U}}$. It is known, see, for example, [6, Lemma 4.1 and Proposition 4.2], that $M = (M_d)_{\mathcal{U}}$, where every $M_d$, $d \in D$, is an $n$-dimensional subspace of $\lambda$. From the hypothesis, there are a positive constant $b$, a natural number $m(n)$, a family of $b$-complemented subspaces $(Y_d, \ d \in D)$ in $\lambda$ such that $M_d \subset Y_d$ with respective family of projections denoted $(P_d, \ d \in D)$ such that $\|P_d\| \leq b$ and $\dim(Y_d) \leq m(n)$ for every $d \in D$. Let $P = (P_d)_{\mathcal{U}}$, which is a projection of $(\lambda)_{\mathcal{U}}$ onto a $k$-dimensional subspace $Y$ of $(\lambda)_{\mathcal{U}}$ such that $M \subset Y$, with $\|P\| \leq b$ and $k \leq m(n)$.

For every $d \in D$, let $(\lambda^d_i, i = 1, \ldots, m(n))$ be an Auerbach basis in an $(m(n))$-dimensional subspace $W_d$ of $\lambda$ containing $Y_d$, i.e., $\|\sum_{i=1}^{m(n)} c_i \lambda^d_i\| \geq \max_{i=1,\ldots,m(n)} |c_i|$. Then for every $x = (x_d)_{\mathcal{U}} \in (\lambda)_{\mathcal{U}}$, if

$$P_d(x_d) = \sum_{i=1}^{n} a^d_i x^d_i$$

for some scalars $a^d_i$ such that $|a^d_i| \leq b|x|$. Then

$$P(x) = \left(\sum_{i=1}^{n} a^i x^i\right)_{\mathcal{U}} = \sum_{i=1}^{n} a^i (x^i)_{\mathcal{U}},$$

where $a^i = \lim_d a^d_i$, $i = 1, \ldots, m(n)$. We denote by $W$ the subspace of $(\lambda)_{\mathcal{U}}$ generated for $(\lambda^d_i)_{\mathcal{U}}$, $i = 1, \ldots, m(n))$. It is clear that $W = (W_d)_{\mathcal{U}}$.

Also from [6, Lemma 4.1 and Proposition 4.2], using the basis $(\lambda^d_i)_{\mathcal{U}}$, $i = 1, \ldots, m(n))$ of $W$, given $\varepsilon > 0$ there is $d_0 \in D$ such that for every $x = (x_d)_{\mathcal{U}}$,

$$(1 - \varepsilon)\|x\| \leq \|x_{d_0}\| \leq (1 + \varepsilon)\|x\|.$$ 

We denote $C_{d_0} : W \rightarrow W_{d_0} : C_{d_0}((x_d)_{\mathcal{U}}) = x_{d_0}$ which is an isomorphism satisfying $\|C_{d_0}\| \leq 1 + \varepsilon$ and $\|C_{d_0}^{-1}\| \leq 1/(1 - \varepsilon)$. Then $(C_{d_0})_Y$ is an isomorphism from $Y$ onto $Y_{d_0}$ such that $\|(C_{d_0})_Y\|\|((C_{d_0})_Y)^{-1}\| \leq (1 + \varepsilon)/(1 - \varepsilon).$ \hfill \Box

4. $\lambda$-integral operators

The normed ideal of $\lambda$-integral operators ($I_{\lambda}, I_{\lambda}$) is the maximal operator ideal associated to the tensor norm $g_{\lambda}^\prime$ in the sense of Defant and Floret [3], which coincides with the maximal normed operator ideal associated to the normed ideal of $\lambda$-nuclear operators in the sense of Pietsch [12]. From [3], for every pair of Banach spaces $E$ and $F$, an operator $T : E \rightarrow F$ is $\lambda$-integral if and only if $J_{TF} \in (E \otimes_l g_{\lambda}', F')$.

Given Banach spaces $E, F$ we define the finitely generated tensor norm $g_{\lambda}^\prime$ such that if $M \in \text{FIN}(E)$ and $N \in \text{FIN}(F)$, for every $z \in M \otimes N$,

$$g_{\lambda}^\prime(z; M \otimes N) := \sup \{ \|\langle z, w \rangle\| : g_{\lambda}(w; M' \otimes N') \leq 1 \}. $$
Clearly $g'_\lambda = (g^*_\lambda)'$ since the unit ball in $M' \otimes_{g'_\lambda} N'$ is the convex cover of the unit ball of $M' \otimes_{g_\lambda} N'$. But we remark that $E' \otimes_{g'_\lambda} F'$ (and not $E' \otimes_{g_\lambda} F'$) is an isometric subspace of $(E \otimes_{g_\lambda} F)'$ because $g^*_\lambda$ is finitely generated, see [3, 15.3].

In this case we define $I_\lambda(T)$ to be the norm of $JF T$ considered as an element of the topological dual of the Banach space $E \otimes_{g'_\lambda} F'$. Remark that $I_\lambda(T) = I_\lambda(JF T)$ as a consequence of $F'$ be canonically complemented in $F''$. In the search of a characterization of $\lambda$-integral operators we find the following sufficient conditions which provides also the first nontrivial examples of $\lambda$-integral operators.

**Theorem 9.** Let $(\Omega, \Sigma, \mu)$ be a measure space and let $\lambda$ be a Banach sequence space with a regular predual Banach sequence space $\Delta$. Then every order bounded operator $S : L_\infty(\mu) \to \lambda$ and every order bounded operator $S : L_\infty(\mu) \to \ell_1[\lambda]$ are $\lambda$-integral with $I_\lambda(S) = \|S\|$.

**Proof.** We only will do the proof if $S : L_\infty(\mu) \to \lambda$ is an order bounded operator because the proof of the other case is similar. By hypothesis, the linear span $T$ of the set $\{e_i, i \in \mathbb{N}\}$ is dense in $\Delta$. Then by the representation theorem of maximal operator ideals (see [3, 17.5]) and the density lemma [3, Theorem 13.4] we only have to see that $S \in (L_\infty(\mu) \otimes_{g'_\lambda} T)'$.

Given $z \in L_\infty(\mu) \otimes_{g'_\lambda} T$ and $\varepsilon > 0$, let $X$ and $Y$ be finite-dimensional subspaces of $L_\infty(\mu)$ and $T$, respectively, such that $z \in X \otimes Y$ and

$$g^*_\lambda(z; X \otimes Y) \leq g^*_\lambda(z; L_\infty(\mu) \otimes T) + \varepsilon. \quad (4)$$

Let $(g_s)_{s=1}^m$ be a basis for $Y$ and let $k \in \mathbb{N}$ be such that

$$\forall 1 \leq s \leq m, \quad g_s = \sum_{i=1}^{k} c_{si} e_i.$$

Then

$$\forall f \in X, \forall 1 \leq s \leq m,$$

$$\langle S, f \otimes g_s \rangle = \left\langle f, S'(g_s) \right\rangle = \left\langle f, \left( \sum_{i=1}^{k} c_{si} e_i \right) S'(e_i) \right\rangle = \left\langle f \otimes \sum_{j=1}^{k} c_{sj} e_j, \sum_{i=1}^{k} S'(e_i) \otimes e_i \right\rangle.$$

Then if $U$ denotes the tensor

$$U := \sum_{i=1}^{k} S'(e_i) \otimes e_i \in L_\infty(\mu)' \otimes \lambda,$$

by bilinearity we get

$$\forall z \in X \otimes Y, \quad \langle z, S \rangle = \langle U, z \rangle.$$
Given \( v > 0 \), for every \( 1 \leq i \leq k \), there is \( f_i \in L_\infty(\mu) \) such that \( \| f_i \| \leq 1 \) and \( \| S'(e_i) \| \leq \| \langle S'(e_i), f_i \rangle \| + v \). Then \( f := \sup_{1 \leq i \leq k} f_i \) lies in the closed unit ball of \( L_\infty(\mu) \). On the other hand, \( \lambda \) is a dual lattice and hence it is order complete. By the Riesz–Kantorovich theorem (see Theorem 1.13 in [1], for instance), the modulus \( |S| \) of the operator \( S \) exists in \( L(L_\infty(\mu), \lambda) \). By the lattice properties of \( \lambda \), we have

\[
\pi_\lambda(S'(e_i)) = \left\| \sum_{i=1}^k S'(e_i) e_i \right\| \leq \left\| \sum_{i=1}^k \langle S'(e_i), f_i \rangle e_i \right\| + v \left\| \sum_{i=1}^k e_i \right\|.
\]

Moreover

\[
\varepsilon_{\lambda_\Delta}(e_i)^k = \sup_{\|\beta_j\|_\Delta \leq 1} \left\| \sum_{i=1}^k (e_i, \beta_j) e_i \right\| = \left\| \sum_{i=1}^k \beta_i e_i \right\| \leq 1.
\]

Hence, denoting by \( I_X \) and \( I_Y \) the corresponding inclusion maps into \( L_\infty(\mu) \) and \( \lambda \), respectively, we have

\[
\langle S, z \rangle = \| U, (I_X)' \otimes (I_Y)' \| (z) \| U \| (I_X)' \otimes (I_Y)' \| (z) \| U \| (I_X)' \otimes (I_Y)' \| (z)
\]

\[
\leq g_\lambda^*(U; X \otimes Y)g_\lambda^* \left( (I_X)' \otimes (I_Y)' \right)(z) \| X' \otimes Y' \|
\]

\[
\leq g_\lambda(U; X \otimes Y)g_\lambda \left( (I_X)' \otimes (I_Y)' \right)(z) \| X' \otimes Y' \|
\]

\[
\leq g_{\lambda_\Delta}^*(z; L_\infty(\mu) \otimes \lambda') (\pi_\lambda(S'(e_i)) \varepsilon_{\lambda_\Delta}(e_i)) + \varepsilon
\]

\[
\leq g_{\lambda_\Delta}^*(z; L_\infty(\mu) \otimes \lambda') \left( \| S \| + v \left\| \sum_{i=1}^k e_i \right\| + \varepsilon \right)
\]

and \( v \) being arbitrary

\[
\langle S, z \rangle \leq g_{\lambda_\Delta}^*(z; L_\infty(\mu) \otimes \lambda')(\| S \| + \varepsilon).
\]
Finally, by the arbitrariness of \( \varepsilon \) we get
\[
|\langle S, z \rangle| \leq g'_{\lambda}(z; L_\infty(\mu) \otimes \lambda') \|S\|.
\]
But from [1, Theorem 1.10],
\[
|S(\chi_\Omega)| = \sup \left\{ |S(f)|, |f| \leq \chi_\Omega \right\}
\]
and as \( \lambda \) is order continuous
\[
\|S\| = \|S(\chi_\Omega)\| = \sup \left\{ \|S(f)\|, \|f\| \leq 1 \right\} = \|S\|.
\]
Then \( S \) is \( \lambda \)-integral with \( I_\lambda(S) \leq \|S\| \). But as \( (I_\lambda, I_\lambda) \) is a Banach operators ideal, \( \|S\| \leq I_\lambda(S) = \|S\| \). \( \square \)

**Corollary 10.** Let \((\Omega, \Sigma, \mu)\) be a measure space and \(n, k \in \mathbb{N}\). Then every operator \(T : L_\infty(\mu) \to S_k(\lambda)\) and every operator \(T : L_\infty(\mu) \to S_\ell(\ell_1)[S_k(\lambda)]\) satisfy that \(I_\lambda(T) = \|T\|\).

**Proof.** The results follows easily from Theorem 9, because every operator \(T : L_\infty(\mu) \to S_k(\lambda)(T : L_\infty(\mu) \to S_\ell(\ell_1)[S_k(\lambda)])\) in the other case is order bounded and \(S_k(\lambda)\) (respectively \(S_\ell(\ell_1)[S_k(\lambda)]\)) is reflexive hence order continuous. \( \square \)

For our next theorem we need a very deep technical result of Lindenstrauss and Tzafriri [10] which gives us a kind of “uniform approximation” of finite-dimensional subspaces by finite-dimensional sublattices in Banach lattices.

**Lemma 11.** Let \( \varepsilon > 0 \) and \( n \in \mathbb{N} \) be fixed. There is a natural number \( h(n, \varepsilon) \) such that for every Banach lattice \( X \) and every subspace \( F \subset X \) of dimension \( \dim(F) = n \) there are \( h(n, \varepsilon) \) disjoints elements \( \{z_i, 1 \leq i \leq h(n, \varepsilon)\} \) and an operator \( A \) from \( F \) into the linear span \( G \) of \( \{z_i, 1 \leq i \leq h(n, \varepsilon)\} \) such that
\[
\forall x \in F, \quad \|A(x) - x\| \leq \varepsilon \|x\|.
\]

**Theorem 12.** Let \( \lambda \) be a regular Banach sequence space, \( G \) an abstract \( M \)-space, and \( X \) a quasi-\( L^\infty \)-space or a quasi-\( L^{\ell,1}(\ell_1) \)-space. Then every operator \(T : G \to X\) is \( \lambda \)-integral and there is a constant \( K > 0 \) such that \(I_\lambda(T) \leq K \|T\|\).

**Proof.** We will prove the case where \( X \) is a quasi-\( L^\infty \)-space. The same technique works “almost” word by word if \( X \) is a quasi-\( L^{\ell,1}(\ell_1) \) space. By the representation theorem of maximal operator ideals (see 17.5 in [3]), we only need to show that \(J_X T \in (G \otimes_G X')'\).

Given \( z \in G \otimes X' \) and \( \varepsilon > 0 \), let \( M \subset G \) and \( N \subset X' \) be finite-dimensional subspaces and let \( z = \sum_{i=1}^n f_i \otimes x_i' \) be a fixed representation of \( z \) with \( f_i \in M \) and \( x_i' \in N \), \( i = 1, 2, \ldots, n \), such that
\[
g^1_i(z; G \otimes X') \leq g^1_i(z; M \otimes N) \leq g^1_i(z; G \otimes X') + \varepsilon.
\]
By Lemma 11 we find a finite-dimensional sublattice \( M_1 \) of \( G \) and an operator \( A : M \to M_1 \) so that
\[
\forall f \in M, \quad \|A(f) - f\| \leq \varepsilon \|f\|.
\]
Then, if id\(_G\) denotes the identity map on \(G\) we have

\[
\left| \left\langle J_X T, z \right\rangle \right| = \sum_{i=1}^{n} \left| \left\langle T(f_i), x'_i \right\rangle \right| \leq \sum_{i=1}^{n} \left| \left\langle T(\text{id}_G - A)(f_i), x'_i \right\rangle \right| + \sum_{i=1}^{n} \left| \left\langle T(A(f_i), x'_i \right) \right| \\
\leq \varepsilon \| T \| \sum_{i=1}^{n} \| f_i \| \| x'_i \| + \sum_{i=1}^{n} \left| \left\langle T(A(f_i), x'_i \right) \right| \\
\leq \varepsilon \| T \| \sum_{i=1}^{n} \| f_i \| \| x'_i \| + \sum_{i=1}^{n} \left| \left\langle T(A(f_i), x'_i \right) \right| .
\]

Put \(X_1 := T(M_1)\). By hypothesis \(X\) is a quasi-\(L^\lambda\) space and hence, there are a finite-dimensional subspace \(X_2\) of \(X\) containing \(X_1\), some complemented finite-dimensional subspace \(H\) of \(\lambda\) with projection \(P_H : \lambda \to H\) such that \(\| P_H \| \leq b\) for some \(b > 0\), and an isomorphism \(V : X_2 \to H\) such that \(X_1 \subset X_2\) and \(\| V \| \| V^{-1} \| \leq a\) for some positive real constant \(a\). Let \(I_{X_1} : X_1 \to X_2\) be the inclusion map. To simplify notation we denote \(R : M_1 \to H\) such that \(R := V I_{X_1} T\). Let \(K_2 : X_2'' \to X_2' = X_2''/X_2\) be the canonical quotient map. Then

\[
\sum_{i=1}^{n} \left| \left\langle T(A(f_i)), x'_i \right\rangle \right| = \sum_{i=1}^{n} \left| \left\langle I_{X_1} T(A(f_i)), K_2(x'_i) \right\rangle \right| \\
\text{(it should be noted that the later equality is the key of the method and what makes necessary the strong notion of quasi-\(L^\lambda\) space)}
\]

\[
= \sum_{i=1}^{n} \left| \left\langle V^{-1} V I_{X_1} T(A(f_i)), K_2(x'_i) \right\rangle \right| \\
= \sum_{i=1}^{n} \left| \left\langle R A(f_i), (V^{-1})' K_2(x'_i) \right\rangle \right| \\
= \left\langle R, \sum_{i=1}^{n} A(f_i) \otimes (V^{-1})' K_2(x'_i) \right\rangle
\]

with \(\sum_{i=1}^{n} A(f_i) \otimes (V^{-1})' K_2(x'_i) \in M_1 \otimes H'\).

As \(\lambda\) is an \(L^\lambda\) space, \(H\) is contained in some \(r\)-dimensional subspace \(W\) of \(\lambda\) such that \(d(W, S_\varepsilon(\lambda)) < 1 + \varepsilon\). We denote \(I_W\) the inclusion of \(H\) in \(W\) and by \(C : W \to S_\varepsilon(\lambda)\) an isomorphism such that \(\| C \| \| C^{-1} \| \leq 1 + \varepsilon\).

Since \(M_1\) is a reflexive abstract \(M\)-space it is lattice isometric to some \(L_\infty(\mu)\) space. As the map

\[
CI_W R : M_1 \to S_\varepsilon(\lambda)
\]

is order bounded, from Corollary 10 this map is \(\lambda\)-integral with \(I_\varepsilon(CI_W R) \leq \| C \| \| R \| \leq \| C \| \| V \| \| T \|\). But

\[
R = (P_H) W C^{-1} CI_W R,
\]

hence \(R\) is again \(\lambda\)-integral with

\[
I_\varepsilon(R) \leq \| P_H \| \| C^{-1} \| \| C \| \| V \| \| T \| \leq (1 + \varepsilon) b \| V \| \| T \|.
\]
Then
\[
\left| \sum_{i=1}^{n} T(A(f_i), x_i) \right| = \left| \left\langle R, \sum_{i=1}^{n} A(f_i) \otimes (V^{-1})' K_2(x_i) \right\rangle \right| \\
\leq I_\lambda(R) g'_\lambda \left( \sum_{i=1}^{n} A(f_i) \otimes (V^{-1})' K_2(x_i); M_1 \otimes H' \right) \\
\leq (1 + \varepsilon) b \| V \| \| g'_\lambda (z; M \otimes N) \| \\
\leq (1 + \varepsilon)^2 ab \| T \| g'_\lambda (z; G \otimes X') + \varepsilon
\]
and by the arbitrariness of \( \varepsilon > 0 \) we obtain
\[
| \langle J_T, z \rangle | \leq ab \| T \| g'_\lambda (z; G \otimes X'). \quad \square
\]

Corollary 13. Let \( \lambda \) be a regular Banach sequence space, \( G \) an abstract \( M \)-space, and \( X \) a complemented subspace of a quasi-\( L^\lambda \) space or a quasi-\( L_{1\lambda} \) space. Then every operator \( T : G \to X \) is \( \lambda \)-integral and there is a constant \( K > 0 \) such that \( I_\lambda(T) \leq K \| T \| \).

Concerning necessary conditions for an operator be \( \lambda \)-integral we find

Theorem 14. For every pair of Banach spaces \( E, F \) and a Banach sequence space \( \lambda \), if \( T \in I_\lambda(E, F) \) then \( J_F T \) factors in the following way:

\[
\begin{array}{c}
\begin{array}{c}
E \\
A \\
L_\infty(\mu)
\end{array} \quad J_F T \\
F'' \\
C \\
B \\
X
\end{array}
\]

where \( X \) is some ultrapower \( (\ell_1[\lambda_r])_{U_1} \) of \( \ell_1[\lambda_r] \) and \( B \) is a lattice homomorphism. Moreover \( I_\lambda(T) \geq \inf \| C \| \| B \| \| A \| \) taking it over all such factors.

Proof. We define the set
\[
D := \left\{ (M, N) : M \in FIN(E), \ N \in FIN(F') \right\},
\]
where \( FIN(Y) \) is the set of finite-dimensional subspaces of a Banach space \( Y \), endowed with the natural inclusion order
\[
(M_1, N_1) \leq (M_2, N_2) \iff M_1 \subset M_2, \ N_1 \subset N_2.
\]
For every \( (M_0, N_0) \in D \), put \( R(M_0, N_0) := \{ (M, N) \in D : (M_0, N_0) \subset (M, N) \} \) and \( \mathcal{R} = \{ R(M, N), \ (M, N) \in D \} \). \( \mathcal{R} \) is filter basis in \( D \), and according to Zorn’s lemma, let \( D \)
be an ultrafilter on $D$ containing $\mathcal{R}$. If $d \in D$, $M_d$ and $N_d$ denote the finite-dimensional subspaces of $E$ and $F'$, respectively, so that $d = (M_d, N_d)$. For every $d \in D$, if $z \in M_d \otimes N_d$, $J_F (M_d \otimes N_d)^{\prime} = M_d' \otimes \gamma_{d'} N_d' = \mathcal{N}_{\lambda}(M_d, N_d')$. Then from Theorem 2 of characterization of $\lambda$-nuclear operators, $J_F T_{M_d \otimes N_d}$ factors as $\ell_\infty[\ell_\infty]$.

\[
\begin{array}{cccc}
M_d & \rightarrow & \mathcal{J}_F (M_d \otimes N_d) & \rightarrow \ N_d' \\
A_d & \downarrow & B_d & \downarrow C_d \\
\ell_\infty[\ell_\infty] & \rightarrow & \ell_1[\lambda_r] & \rightarrow \ \\
\end{array}
\]

where $B_d$ is a positive diagonal operator and $\|A_d\|\|B_d\|\|C_d\| \leq \mathcal{N}_{\lambda}(T_{M_d \otimes N_d}) + \varepsilon = I_{\lambda}(T_{M_d \otimes N_d}) + \varepsilon$. Then

\[
\|A_d\|\|B_d\|\|C_d\| \leq I_{\lambda}(T_{M_d \otimes N_d}) + \varepsilon \leq I_{\lambda}(T) + \varepsilon.
\]

Without loss of generality we can suppose that $\|A_d\| = \|C_d\| = 1$. We define $W_E : E \rightarrow (M_d)_D$ such that $W_E(x) = (x_d)_D$ so that $x_d = x$ if $x \in M_d$ and $x_d = 0$ if $x \notin M_d$. In the same way we define $W_{F'} : F' \rightarrow (N'_d)_D$ such that $W_{F'}(a) = (a_d)_D$ so that $a_d = a$ if $a \in N_d$ and $a_d = 0$ if $a \notin N_d$. Then we have the following commutative diagram:

\[
\begin{array}{cccc}
E & \xrightarrow{W_E} & (M_d)_D & \xrightarrow{(J_F T)_{M_d \otimes N_d}} (N'_d)_D & \xrightarrow{I} ((N_d)_D)' \\
& & (A_d)_D & \xrightarrow{(J_F T)_{M_d \otimes N_d}} (N'_d)_D & \xrightarrow{(C_d)_D} \\
& & \ell_\infty[\ell_\infty]_D & \xrightarrow{(B_d)_D} & (\ell_1[\lambda_r])_D \\
\end{array}
\]

where $I$ is the canonical inclusion map. As from [10] $((\ell_1[\lambda_r])_D)'$ is a 1-complemented subspace of some ultrapower $((\ell_1[\lambda_r])_D)_U$ which from [14] is another ultrapower $(\ell_1[\lambda_r])_{U \varepsilon}$ with projection $Q$, the result follows with $A = (A_d)_D$, $B = ((B_d)_D)''$ which is a lattice homomorphism, $C = P_{F''} (W_{F'} I(C_d)_D)' Q$, where $P_{F''}$ is the projection of $F''$ in $F''$, and $X = (\ell_1[\lambda_r])_{U \varepsilon}$, having in mind that as $(\ell_\infty[\ell_\infty])_D$ is an abstract $M$-space, there is a measure space such that $L_\infty(\mu) = ((\ell_\infty[\ell_\infty])_D)'$, where the equality means that the spaces are lattice isometric. □

**Theorem 15.** Let $\lambda$ be a Banach sequence space with the uniform projection property and let $E$ and $F$ be Banach spaces. The following statements are equivalent:

1. $T \in \mathcal{I}_\lambda(E, F)$. 
(2) $JFT$ factors continuously in the following way:

$$
\begin{array}{c}
E \\
\downarrow \\
L_\infty(\mu) \\
\downarrow \\
A \\
\downarrow \\
B \\
\downarrow \\
X
\end{array}
\xrightarrow{JFT}
\begin{array}{c}
E'' \\
\downarrow \\
C
\end{array}
$$

where $X$ is a quasi-$\mathcal{L}_{\ell_1[\lambda_r]}$ space. Furthermore the norm $I_X(T)$ is equivalent to $\inf\{\|D\|\|B\|\|A\|\}$, taking it over all such factors.

(3) $JFT$ factors continuously in the following way:

$$
\begin{array}{c}
E \\
\downarrow \\
L_\infty(\mu) \\
\downarrow \\
A \\
\downarrow \\
B \\
\downarrow \\
X
\end{array}
\xrightarrow{JFT}
\begin{array}{c}
E'' \\
\downarrow \\
C
\end{array}
$$

where $X$ is a quasi-$\mathcal{L}_{\ell_1[\lambda_r]}$ lattice and $B$ is a lattice homomorphism. Furthermore $I_X(T)$ is equivalent to $\inf\{\|D\|\|B\|\|A\|\}$, taking it over all such factors.

**Proof.** It follows easily from Theorems 14 and 12 and Proposition 8. □

**Acknowledgment**

The authors are very indebted to Prof. A. Pelczyński for drawing their attention to papers [11] and [10] and for his helpful suggestions.

**References**