Probabilistic and team PFIN-type learning: General properties\note{\copyright{} 2007 Elsevier Inc. All rights reserved.}

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Received 10 November 2004; received in revised form 16 January 2005

Available online 13 June 2007

Abstract

We consider the probability hierarchy for Popperian FINite learning and study the general properties of this hierarchy. We prove that the probability hierarchy is decidable, i.e. there exists an algorithm that receives \( p_1 \) and \( p_2 \) and answers whether PFIN-type learning with the probability of success \( p_1 \) is equivalent to PFIN-type learning with the probability of success \( p_2 \).

To prove our result, we analyze the topological structure of the probability hierarchy. We prove that it is well-ordered in descending ordering and order-equivalent to ordinal \( \epsilon_0 \). This shows that the structure of the hierarchy is very complicated.

Using similar methods, we also prove that, for PFIN-type learning, team learning and probabilistic learning are of the same power.

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Keywords: Inductive inference; Learning in the limit; Finite limits; Ordinals; Decidability

1. Introduction

Inductive inference is a branch of theoretical computer science that studies the process of learning in a recursion-theoretic framework [5,14,22]. Within inductive inference, there has been much work on team learning (see surveys in [2,16,28]).

Probabilistic learning is closely related to team learning. Any team of machines can be simulated by a single probabilistic machine with the same success ratio. The simulation of a probabilistic machine by a team of deterministic machines is often possible as well.

In this paper, we consider finite learning of total recursive functions (abbreviated as FIN). The object to be learned is a total recursive function \( f \). A learning machine reads the values of the function \( f(0), f(1), \ldots \) and produces a program computing \( f \) after having seen a finite initial segment of \( f \). The learning machine is not allowed to change the program later.

FIN is supposed to be one of the simplest learning paradigms. However, if we consider probabilistic and team learning, the situation becomes very complex. Probabilistic FIN-type learning has been studied since 1979 but we are still far from the complete understanding of this area.

\footnote{This research was done while the author was at the University of Latvia and supported by Latvian Science Council Grants No. 93.599 and No. 96.0268 and fellowship “SWH izglītībai, zinānīei un kultūrai”.
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doi:10.1016/j.jcss.2007.06.011
The investigation of probabilistic FINite learning was started by Freivalds [12]. He gave a complete description of the learning capabilities for probabilistic machines with probabilities of success above $\frac{1}{2}$. These results were extended to team learning by Daley et al. [11,29].

The further progress was very difficult. Daley, Kalyanasundaram and Velauthapillai [9] determined the capabilities for probabilistic learners with success probabilities in the interval $[\frac{34}{35}, \frac{1}{2}]$. Later, Daley and Kalyanasundaram [8] extended that to the interval $[\frac{12}{25}, \frac{1}{2}]$. Proofs became more and more complicated. (The full version of [8] is more than 100 pages long.)

PFIN (Popperian FIN)-type learning is a simplified version of FIN-type learning. In a PFIN-type learning, a learning machine is allowed to output only programs computing total recursive functions. Many properties of probabilistic and team PFIN-type learning are similar to FIN-type learning. Yet, PFIN-type learning is simpler and easier to analyze than unrestricted FIN-type learning.

Daley, Kalyanasundaram and Velauthapillai [7,10] determined the capabilities of probabilistic PFIN-type learners in the interval $[\frac{3}{7}, \frac{1}{2}]$. However, even for PFIN-type learning, the situation becomes more and more complicated for smaller probabilities of success.

In this paper, we suggest another approach to PFIN-type and FIN-type learning. Instead of trying to determine the exact points at which the learning capabilities are different (either single points or sequences of points generated by a formula), we investigate global properties of the probability structure.

Our main result is that the probability hierarchy for PFIN-type learning is well-ordered in a decreasing ordering and has a constructive description similar to systems of notations for constructive ordinals. We use this result to construct a decision algorithm for the probability hierarchy. Given two numbers $p_1, p_2 \in [0, 1]$, the decision algorithm answers whether the learning with probability $p_1$ is equivalent to the learning with probability $p_2$. Also, we construct a universal simulation algorithm receiving

- $p_1, p_2 \in [0, 1]$ such that PFIN-learning with these probabilities is equivalent and
- PFIN-learning machine $M$ with the probability of success $p_1$

and transforming $M$ into machine $M'$ with the probability of success $p_2$.

All of these results make heavy use of the well-ordering and the system of notations. To our knowledge, this is the first application of well-orderings to a problem of this character. (They have been used in computational learning theory [1,13], but for entirely different purposes.)

We also determine the exact ordering type of the probability hierarchy. It is order-isomorphic to $\epsilon_0$, a quite large ordinal.  The part of the hierarchy investigated before ($[\frac{3}{7}, 1]$) is order-isomorphic to the ordinal $3\omega$ and is very simple compared to the entire probability hierarchy. Thus, we can conclude that finding a more explicit description for the whole hierarchy is unlikely. (The previous research shows that, even for segments like $[\frac{3}{7}, 1]$ with a simple topological structure, this task is difficult because of irregularities in the hierarchy [7].)

Our results also imply that any probabilistic PFIN-type learning machine can be simulated by a team of deterministic machines with the same success ratio.

2. Technical preliminaries

2.1. Notations

We use the standard recursion theoretic notation [26].

$\mathbb{N}$ denotes $\{0, 1, \ldots\}$, the set of natural numbers. $\mathbb{N}^+$ denotes $\{1, 2, \ldots\}$, the set of positive natural numbers, $\mathbb{Q}$ denotes the set of rational numbers and $\mathbb{R}$ the set of real numbers. $\subseteq$ and $\subset$ denote a subset and a proper subset, respectively.

Let $\varphi$ denote an arbitrary fixed acceptable programming system (a.k.a. Gödel numbering) of all partial recursive functions [21,25,26]. $\varphi_i$ denotes the $i$th program in system $\varphi$.

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1 It is known that $\epsilon_0$ is isomorphic to the set of all expressions possible in first-order arithmetic.
2.2. Finite learning of functions

A learning machine is an algorithmic device which reads values of a recursive function \( f: f(0), f(1), \ldots \). Having seen finitely many values of the function it can output a conjecture. The conjecture is a program in some fixed acceptable programming system. Only one conjecture is allowed, i.e. the learning machine cannot change its conjecture later.

A learning machine \( M \) FIN-learns a function \( f \) if, receiving \( f(0), f(1), \ldots \) as the input, it produces a program computing \( f \). \( M \) FIN-learns a set of functions \( U \) if it FIN-learns any \( f \in U \). A set of functions \( U \) is FIN-learnable if there exists a learning machine that learns \( U \). The collection of all FIN-learnable sets is denoted FIN.

PFIN-learning is a restricted form of FIN-learning. A learning machine \( M \) PFIN-learns \( U \) if it FIN-learns \( U \) and all conjectures (even incorrect ones) of \( M \) on all inputs are programs computing total recursive functions. The collection of all PFIN-learnable sets is denoted PFIN.

2.3. Probabilistic and team learning

Scientific discoveries are rarely done by one person. Usually, a discovery is the result of collective effort. In the area of computational learning theory, this observation has inspired the research on team learning.

A team is just a set of learning machines: \( M = \{M_1, \ldots, M_s\} \). The team \( M \) [\( r, s \)]FIN-learns a set of functions \( U \) if, for every \( f \in U \), at least \( r \) of \( M_1, \ldots, M_s \) FIN-learn \( f \). The collection of all \( [r, s] \)FIN-learnable sets is denoted \( [r, s] \)FIN.

We also consider learning by probabilistic machines. A probabilistic machine has an access to a fair coin and its output depends on both input and the outcomes of coin flips.

Let \( M \) be a probabilistic learning machine. \( M \) FIN\( \langle p \rangle \)-learns (FIN-learns with probability \( p \)) a set of functions \( U \) if, for any function \( f \in U \), the probability that \( M \) outputs a program computing \( f \), given \( f(0), f(1), \ldots \) as the input, is at least \( p \). FIN\( \langle p \rangle \) denotes the collection of all FIN\( \langle p \rangle \)-learnable sets.

Probabilistic and team PFIN-learning is defined by adding a requirement that all conjectures output by the probabilistic machine or any machine in the team must be programs computing total recursive functions.

Definition 1. The probability hierarchy for FIN is the set \( A \subseteq \mathbb{R} \cap [0, 1] \) such that

1. For any two different \( p_1, p_2 \in A \),
   \[ \text{FIN}(p_1) \neq \text{FIN}(p_2) \],
   i.e., learning with probability of success \( p_1 \) is not equivalent to learning with probability of success \( p_2 \).
2. If \( x \in A \), \( x \leq p \) and \( [x, p] \) does not contain any points belonging to \( A \), then \( \text{FIN}(x) = \text{FIN}(p) \).

Essentially, the probability hierarchy is the set of those probabilities at which the learning capabilities of probabilistic machines are different.

The probability hierarchy for PFIN is defined similarly.

2.4. Well-orderings and ordinals

A linear ordering is a well-ordering if it does not contain infinite descending sequences. Ordinals [27] are standard representations of well-orderings.

The ordinal 0 represents the ordering type of the empty set, the ordinal 1 represents the ordering type of any 1 element set, the ordinal 2 represents the ordering type of any 2 element set and so on. The ordinal \( \omega \) represents the ordering type of the set \( \{0, 1, 2, \ldots \} \). The ordinal \( \omega + 1 \) represents the ordering type of \( \{0, 1, 2, \ldots \} \) followed by an element \( \omega \). The ordinal \( 2\omega \) represents the ordering type \( \{0, 1, 2, \ldots \} \) followed by \( \{\omega, \omega + 1, \omega + 2, \ldots \} \). Greater ordinals can be defined similarly [27]. We use arithmetic operations on ordinals defined in two different ways.
Definition 2. (See [20].) Let \( A \) and \( B \) be two disjoint sets, \( \alpha \) be the ordering type of \( A \) and \( \beta \) be the ordering type of \( B \).

1. \( \alpha + \beta \) is the ordering type of \( A \cup B \) ordered so that \( x < y \) for any \( x \in A, y \in B \) and order is the same within \( A \) and \( B \).
2. \( \alpha \beta \) is the ordering type of \( A \times B \) ordered so that \( (x_1, y_1) < (x_2, y_2) \) iff \( x_1 < x_2 \) or \( x_1 = x_2 \) and \( y_1 < y_2 \).

We note that both the sum and the product of ordinals are non-commutative. For example, \( 1 + \omega = \omega \neq \omega + 1 \).

Definition 3. (See [20].) \( \alpha - \beta \) (the difference of \( \alpha \) and \( \beta \)) is an ordinal \( \gamma \) such that \( \alpha = \beta + \gamma \).

\( \alpha - \beta \) always exists and is unique [20]. We also use the natural sum and the natural product of ordinals. These operations use the representation of ordinals as exponential polynomials. In this paper, we consider only ordinals which are less than or equal to \( \epsilon_0 = \lim(\omega, \omega^\omega, \omega^{\omega^\omega}, \omega^{\omega^{\omega^\omega}}, \ldots) \).

Any ordinal \( \alpha < \epsilon_0 \) can be uniquely expressed in the form

\[ \alpha = c_1 \omega^{\alpha_1} + \cdots + c_n \omega^{\alpha_n} \]

where \( \alpha_1 > \alpha_2 > \cdots > \alpha_n \) are smaller ordinals and \( c_1, c_2, \ldots, c_n \in \mathbb{N} \).

Definition 4. (See [20].) Let

\[ \alpha = c_1 \omega^{\alpha_1} + \cdots + c_n \omega^{\alpha_n}, \]
\[ \beta = d_1 \omega^{\beta_1} + \cdots + d_n \omega^{\beta_n}. \]

1. The natural sum of \( \alpha \) and \( \beta \) is
   \[ \alpha(+)\beta = (c_1 + d_1)\omega^{\alpha_1} + \cdots + (c_n + d_n)\omega^{\alpha_n}. \]
2. \( \alpha(\cdot)\beta \), the natural product of \( \alpha \) and \( \beta \) is the product of base \( \omega \) representations as polynomials. \( \omega^{\alpha_1}(\cdot)\omega^{\beta_j} = \omega^{\alpha_1(\cdot)\beta_j} \) and \( \alpha(\cdot)\beta \) is the natural sum of \( c_id_j\omega^{\alpha_1(\cdot)\beta_j} \) for all \( i, j \).

Natural sum and natural product are commutative. They can be used to bound the ordering type of unions.

Theorem 5. Let \( A_1, \ldots, A_s \) be arbitrary subsets of a well-ordered set \( A \), \( \alpha_1, \ldots, \alpha_s \) be the ordering types of \( A_1, \ldots, A_s \) and \( \alpha \) be the ordering type of \( A_1 \cup \cdots \cup A_s \). Then,

\[ \alpha \leq \alpha_1(+)\alpha_2(+)\cdots(+)\alpha_s. \]

The difference between this theorem and Definition 2 is that Definition 2 requires \( x < y \) for all \( x \in A, y \in B \) but Theorem 5 has no such requirement. Next, we give a similar result for the natural product.

Theorem 6. Let \( A_1, \ldots, A_s \) and \( A \) be well-ordered sets with ordering types \( \alpha_1, \ldots, \alpha_s \) and \( \alpha \), respectively. Assume that \( f : A_1 \times A_2 \times \cdots \times A_s \to A \) is a strictly increasing function onto \( A \), i.e.

\[ f(\alpha_1, \ldots, \alpha_{i-1}, \alpha_i, \alpha_{i+1}, \ldots, \alpha_s) < f(\alpha_1, \ldots, \alpha_{i-1}, \alpha'_i, \alpha_{i+1}, \ldots, \alpha_s) \]

for all \( i \in \{1, \ldots, s\} \) and \( \alpha_i < \alpha'_i \). Then

\[ \alpha \leq \alpha_1(\cdot)\alpha_2(\cdot)\cdots(\cdot)\alpha_s. \]

Both Theorems 5 and 6 will be used in Section 4. We will also use the transfinite induction, a generalization of the usual mathematical induction.
Theorem 7. (See [20, Principle of transfinite induction].) Let $A$ be a well-ordered set and $P(x)$ be a predicate. If

1. $P(x)$ is true when $x$ is the smallest element of $A$, and
2. $P(y)$ for all $y \in A$ which are smaller than $x$ implies $P(x)$,

then $P(x)$ for all $x \in A$.

2.5. Systems of notations

In this paper we use subsets of $\mathbb{Q} \cap [0, 1]$ that are well-ordered in decreasing ordering. A subset of $\mathbb{Q}$ is well-ordered in decreasing ordering if it does not contain an infinite monotonically increasing sequence.

Church and Kleene [6,19] introduced systems of notations for constructive ordinals. Intuitively, a system of notations is a way of assigning notations to ordinals which satisfies certain constraints and allows to extract certain information about the ordinal from its notation. Below, we adapt the definition by Church and Kleene [6,19] to well-ordered subsets of $\mathbb{Q}$.

Let $A$ be a subset of $\mathbb{Q}$ which is well-ordered in decreasing ordering. All elements of $A$ can be classified as follows:

1. The greatest element of the set $A$. We call it the maximal element.
2. Elements $x$ which have an immediately preceding element in decreasing ordering (i.e. an element $y$ such that $x < y$ and $[x, y]$ does not contain any points belonging to $A$). Such elements are called successor elements.
3. All other elements $x \in A$. They are called limit elements.

Definition 8. A system of notations for $A$ is a tuple of functions $\langle k_S, p_S, q_S \rangle : \mathbb{Q} \to \mathbb{N}$ such that

1. $k_S(x)$ is equal to
   - (a) 0, if $x$ is the maximal element;
   - (b) 1, if $x$ is a successor element;
   - (c) 2, if $x$ is a limit element;
   - (d) 3, if $x /\in A$;
2. If $k_S(x) = 1$, then $p_S(x)$ is defined and it is the element immediately preceding $x$ in descending ordering.
3. If $k_S(x) = 2$, then $q_S(x)$ is defined and it is a program computing a monotonically decreasing sequence of elements of the set $A$ converging to $x$.

Systems of notations are convenient for manipulating well-ordered sets in our proofs. Possibly, a system of notation is the most appropriate way of describing the probability hierarchy for PFIN. The structure of this hierarchy is quite complicated (Section 4) and it seems unlikely that more explicit descriptions exist.

Below, we give a useful property of systems of notations.

Lemma 9. Let $A \subseteq \mathbb{Q}$ be a set which is well-ordered in descending ordering and has a system of notations $S$. Let $f_1(p)$ be the largest number in $A$ such that $f_1(p) \leq p$ and $f_2(p)$ be the smallest number in $A$ such that $p \leq f_2(p)$. Then $f_1$ and $f_2$ are computable functions.

Proof. $f_1$ and $f_2$ are computed by the algorithm below:

1. Set $x$ equal to an arbitrary element of $A$ smaller than $p$.
2. (a) If $x = p$, output: $f_1(p) = f_2(p) = x$. Stop.
   (b) If $x$ is a successor element and $p_S(x) \geq p$, then output: $f_1(p) = x$ and $f_2(p) = p_S(x)$. Stop.
   (c) If $x$ is a successor element and $p_S(x) \leq p$, set $x = p_S(x)$.
   (d) If $x$ is a limit element and $x \neq p$, take the sequence
      $$\varphi_{q_S(x)}(0), \varphi_{q_S(x)}(1), \ldots.$$  
      Search for the smallest $i$ satisfying $\varphi_{q_S(x)}(i) \leq p$ and set $x = \varphi_{q_S(x)}(i)$. (Such $i$ exists because this sequence is monotonically decreasing and converges to $x$ and $x < p$.)
3. Repeat step (2).
While this algorithm works, $x$ remains less or equal to $p$.

From the definition of the system of notations it follows that the values of $f_1$ and $f_2$ output by the algorithm are correct. It remains to prove that algorithm always outputs $f_1(p)$ and $f_2(p)$.

For a contradiction, assume that the algorithm does not output $f_1(p)$ and $f_2(p)$ for some $p \in Q$. This can happen only if the algorithm goes into eternal loop, i.e. if step (2) is executed infinitely many times.

Each execution of step (2) increases the value of $x$. Let $x_i$ be the value of $x$ after the $i$th repetition of step (2). Then $x_1, x_2, x_3, \ldots$ is an infinite monotonically increasing sequence. This contradicts the set $A$ being well-ordered in decreasing order.

2.6. Three examples

In Fig. 1, we show the known parts of probability hierarchies for three learning criteria:

- **EX** (learning in the limit, Pitt and Smith [23,24]),
- **FIN** (Freivalds [12], Daley, Kalyanasundaram and Velauthapillai [9]), and
- **PFIN** (Daley, Kalyanasundaram and Velauthapillai [7,10]).

We see that these probability hierarchies contain infinite decreasing sequences but none of them contains an infinite increasing sequence. Known parts of these hierarchies are well-ordered in decreasing order.

We will show that, for PFIN-type learning, the entire hierarchy is well-ordered and will use this property to study its properties.

3. Decidability result

The outline for this section is as follows. We start with describing a set $A$ in two equivalent forms in Section 3.1. Then, in Sections 3.2 and 3.3, we show several technical lemmas about the set $A$, including the equivalence of the two descriptions. Then, we show that $A$ is the probability hierarchy for PFIN. The proof of that consists of two parts: diagonalization and simulation. The diagonalization part is shown in Section 3.4. The simulation argument is more complicated. First, in Section 3.5, we show that $A$ is well-ordered and has a system of notations. Finally, in Section 3.6 we use these technical results to construct a universal simulation argument. Our diagonalization theorem uses methods from Kummer’s paper on PFIN-teams [18] but the simulation part uses new techniques and is far more complicated.

3.1. Description of probability hierarchy

Our description has two equivalent forms. First, we describe it as a set of solutions to a particular optimization problem on trees.
Similarly to [18], we define trees as finite non-empty subsets of \( \mathbb{N}^* \) which are closed under initial segments. The root of each tree is the empty string \( \epsilon \). A vertex \( u \) is a child of a vertex \( v \) if \( u = vn \) for some \( n \in \mathbb{N} \). Next, we define labelings of trees by positive reals. The definition below is equivalent to one in [18], with some minor technical modifications.

**Definition 10.** Let \( 0 < p < q \). A \((p, q)\)-labeling of a tree \( T \) is a pair of mappings \( \nu_1, \nu_2 : T \rightarrow \mathbb{R}^+ \) such that

1. \( \nu_1(\epsilon) \geq p \) and \( \nu_2(\epsilon) = 0 \),
2. If \( t_1, \ldots, t_s \) are all direct successors of \( t \), then \( \sum_{i=1}^{s} \nu_2(t_i) \leq \nu_1(t) + \nu_2(t) \) and \( \nu_1(t_i) + \nu_2(t_i) \geq p \) for \( i = 1, \ldots, s \),
3. For each branch the sum of the \( \nu_1 \)-labels of all of its nodes is at most \( q \).

Labelings by natural numbers have an intuitive meaning. \( \nu_1(v) + \nu_2(v) \) is the number of machines that have issued a conjecture consistent with the initial segment \( v \). In particular, \( \nu_2(v) \) is the number of machines that have issued such a conjecture on some prefix of \( v \) and \( \nu_1(v) \) is the number of machines that have output it after seeing the whole segment \( v \).

Then, the requirements of definition have the following interpretation. \( \nu_1(t) + \nu_2(t) \geq p \) means that, for every segment \( t \) in the tree, there must be at least \( p \) machines with conjectures consistent with \( t \).

The second requirement, \( \sum_{i=1}^{s} \nu_2(t_i) \leq \nu_1(t) + \nu_2(t) \) has the following interpretation. \( \nu_2(t_i) \) is the number of machines which have issued a conjecture consistent with \( t_i \) after reading a prefix of \( t_i \). A conjecture consistent with \( t_i \) is also consistent with \( t \). A prefix of \( t_i \) could be either \( t \) or a prefix of \( t \). Since a conjecture can be only consistent with one of segments \( t_i \), \( \sum_{i=1}^{s} \nu_2(t_i) \) must be at most the total number of machines which have issued a conjecture consistent with \( t \) after reading either \( t \) or a prefix of \( t \). The number of such machines is \( \nu_1(t) + \nu_2(t) \).

Finally, the third requirement means that the total number of machines that issue conjectures on any branch is at most \( q \). An example of a labeling is shown in Fig. 2. The first number near node is \( \nu_1(t) \), the second number is \( \nu_2(t) \).

Labelings with reals have a similar interpretation, with \( \nu_1(t) \) and \( \nu_2(t) \) being the probabilities that a probabilistic machine has output a conjecture consistent with \( t \). Let \( p_T \) denote the largest number \( p \) such that there is a \((p, 1)\) labeling of \( T \) (For the tree in Fig. 2, \( p = 6/11 \)). Let \( A = \{ p_T \mid T \text{ is a tree} \} \).

The second description is algebraic, by a recurrence relation. Let set \( A' \) defined by the following rules:

1. \( 1 \in A' \);
2. If \( p_1, p_2, \ldots, p_s \in A' \) and \( p \in [0, 1] \) is a number such that there exist \( q_1, \ldots, q_s \in [0, 1] \) satisfying
   a. \( q_1 + q_2 + \cdots + q_s = p \);
   b. \( \frac{p'}{q_i + 1 - p'} = p_i \) for \( i = 1, \ldots, s \),

\footnote{Definition in [18] incorrectly uses \( \nu_1(t) \) instead of \( \nu_1(t) + \nu_2(t) \) here.}
then \( p \in \mathcal{A} \);

In Section 3.3, we will show that both definitions give the same set \( \mathcal{A} = \mathcal{A}' \). After that, we will prove that \( \mathcal{A} \) is the probability hierarchy for PFIN.

### 3.2. Technical lemmas: Algebraic description

In this subsection, we study the properties of the rule that generates the set \( \mathcal{A}' \). The results of this subsection are used in various parts of Section 3. First, we show that the rule (2) can be described without using variables \( q_i \).

#### Lemma 11. If there exist \( q_1, \ldots, q_s \in [0, 1] \) satisfying \( q_1 + q_2 + \cdots + q_s = p \) and \( \frac{p}{q_i + 1 - p} = p_i \) for \( i = 1, \ldots, s \), then

\[
p = \frac{s}{s} + \sum_{i=1}^{s} \frac{1}{p_i}.
\]

**Proof.** If \( \frac{p}{q_i + 1 - p} = p_i \) is equivalent to \( q_i = \frac{p}{p_i} + p - 1 \). Hence,

\[
p = \sum_{i=1}^{s} q_i = \sum_{i=1}^{s} \left( \frac{p}{p_i} + p - 1 \right) = \left( \sum_{i=1}^{s} \frac{1}{p_i} \right) p + s \cdot p - s,
\]

\[
s = \left( \sum_{i=1}^{s} \frac{1}{p_i} \right) p + (s - 1) p,
\]

\[
p = \frac{s}{(s - 1) + \sum_{i=1}^{s} \frac{1}{p_i}}.
\]

We shall use both forms of the rule (2). The rule with \( q_i \) is more natural in simulation and diagonalization arguments but is less convenient for algebraic manipulations. We also use a version of Lemma 12 where equality is replaced by inequality.

#### Lemma 12. If there exist \( q_1, \ldots, q_s \in [0, 1] \) satisfying \( q_1 + q_2 + \cdots + q_s = p \) and \( \frac{p}{q_i + 1 - p} \leq p_i \) for \( i = 1, \ldots, s \), then

\[
p \leq \frac{s}{s} + \sum_{i=1}^{s} \frac{1}{p_i}.
\]

**Proof.** Similar to the proof of Lemma 11, with \( \leq \) or \( \geq \) instead of \( = \) where necessary.

Lemma 11 suggests that the rule (2) can be considered as a function of \( p_1, \ldots, p_s \). Next lemmas show that this function is monotonic and continuous.

#### Lemma 13. If

1. \( p \in \mathcal{A}' \) follows from \( p_1 \in \mathcal{A}', \ldots, p_s \in \mathcal{A}' \) by rule (2);
2. \( p' \in \mathcal{A}' \) follows from \( p'_1 \in \mathcal{A}', \ldots, p'_s \in \mathcal{A}' \) by rule (2);
3. \( p_i \leq p'_i \), \( i = 1, \ldots, s \),

then \( p \leq p' \). If \( p_i < p'_i \) for at least one \( i \), then \( p < p' \).

**Proof.** By Lemma 11

\[
p = \frac{s}{(s - 1) + \sum_{i=1}^{s} \frac{1}{p_i}} \quad \text{and} \quad p' = \frac{s}{(s - 1) + \sum_{i=1}^{s} \frac{1}{p'_i}}.
\]

From \( p_i \leq p'_i \) it follows that \( \frac{1}{p_i} \geq \frac{1}{p'_i} \) and
(s - 1) + \sum_{i=1}^{s} \frac{1}{p_i} \geq (s - 1) + \frac{s}{\sum_{i=1}^{s} \frac{1}{p'_i}}.

p = \frac{s}{(s - 1) + \sum_{i=1}^{s} \frac{1}{p_i}} \leq \frac{s}{(s - 1) + \sum_{i=1}^{s} \frac{1}{p'_i}} = p'.

If \( p_i < p'_i \) for some \( i \), then \( 1/p_i > 1/p'_i \) and all inequalities are strict. \( \square \)

**Lemma 14.** Let \( p_j = \lim_{i \to \infty} p_{j,i} \) and \( r = \lim_{i \to \infty} r_i \). If, for all \( i \in \mathbb{N} \), \( r_i \in A' \) follows from \( p_{1,i} \in A', \ldots, p_{s,i} \in A' \) by rule (2), then \( r \in A' \) follows from \( p_1 \in A', \ldots, p_s \in A' \) by rule (2).

**Proof.** Assume that Eq. (1) is true for \( p_1 = x_1, \ldots, p_s = x_s \). Then,

\[
\frac{p}{1 + p} = \frac{\sum_{j=1}^{s} \frac{1}{p_j}}{(s - 1) + \sum_{j=1}^{s} \frac{1}{p_j}} = \frac{s}{(s - 1) + \sum_{j=1}^{s} \frac{1}{p_j} + s} = \frac{s}{(s - 1) + \sum_{j=1}^{s} (1 + \frac{1}{x_j})} = \frac{s}{(s - 1) + \sum_{j=1}^{s} \frac{1}{x_j}}.
\]

This is precisely Eq. (1) for \( p_1 = \frac{x_1}{1+x_1}, \ldots, p_s = \frac{x_s}{1+x_s} \).

The opposite direction (Eq. (1) for \( p_1 = \frac{x_1}{1+x_1}, \ldots, p_s = \frac{x_s}{1+x_s} \) implies Eq. (1) is true for \( p_1 = x_1, \ldots, p_s = x_s \)) is similar. \( \square \)

### 3.3. Technical lemmas: Tree description

We start by showing that for a tree \( T \) and its subtrees \( T_i, p_T \) and \( p_{T_i} \) are related similarly to rule (2).

**Lemma 15.** An application of the rule (2) to \( x_1 \in A', \ldots, x_s \in A' \) generates \( p \in A' \) if and only if an application of the rule (2) to \( \frac{x_1}{1+x_1}, \ldots, \frac{x_s}{1+x_s} \in A' \) generates \( \frac{p}{1+p} \in A' \).

**Proof.** We multiply all labels by \( r \) and obtain a \( (p, q) \)-labeling. \( \square \)

**Lemma 17.** Let \( t_1, \ldots, t_s \) be all direct successors of the root in a tree \( T \) and \( T_1, T_2, \ldots, T_s \) be the subtrees with roots \( t_1, t_2, \ldots, t_s \). Assume there are \( q_1, \ldots, q_s \) such that \( \sum_{i=1}^{s} q_i = p \) and

\[
p = p_{T_i}(q_i + 1 - p)
\]

for \( i \in \{1, \ldots, s\} \). Then \( p_T = p \).

**Proof.** First, we construct a \( (p, 1) \)-labeling. Let \( v_1', v_2' \) be a \( (p_{T_i}, 1) \)-labeling for \( T_i \). We define

\[
v_1(t) = \begin{cases} p, & \text{if } t = \epsilon, \\ p - q_i, & \text{if } t = t_i, \\ (1 + q_i - p)v_1'(t), & \text{if } t \text{ is a descendant of } t_i, \end{cases}
\]

\[
v_2(t) = \begin{cases} 0, & \text{if } t = \epsilon, \\ q_i, & \text{if } t = t_i, \\ (1 + q_i - p)v_2'(t), & \text{if } t \text{ is a descendant of } t_i. \end{cases}
\]

Properties (1) and (2) can be checked directly from the definitions of \( v_1 \) and \( v_2 \).
We prove property (3). Let \( u \) be a direct successor of \( t_i \). Then, the sum of \( v_i^j \)-labels on any branch starting at \( u \) is at most \( 1 - pT_i \). (By property (3) of \( v_i^j \), it is at most 1 for any branch starting at \( t_i \) and \( v_i^j(t_i) \geq pT_i \).) Hence, the sum of \( v \)-labels for such a branch is at most \((q_i + 1 - p)(1 - pT_i)\). A branch starting at \( \epsilon \) consists of \( \epsilon, t_i \) and a branch starting at a direct descendant of \( t_i \). Hence, the sum of all its \( v_i \)-labels is at most

\[
p + (p - q_i) + (q_i + 1 - p)(1 - pT_i) = p + 1 - (1 + q_i - p) + (q_i + 1 - p)(1 - pT_i) = p + 1 - (q_i + 1 - p)pT_i = p + 1 - p = 1.
\]

For a contradiction, assume that there is \( p' > p \) and a \((p', 1)\)-labeling \((v'_i, v'_s)\) for \( T \). Let \( q'_i = v'_s(t_i) \). If we restrict ourselves to the subtree \( T_i \) and add \( v'_s(t_i) \) to \( v'_s(t_i) \), we obtain a \((p', 1 - p' + q'_s^2)\)-labeling for \( T_i \). By Lemma 16, there is a \((p'/ (1 - p' + q'_s^2)), 1)\) labeling for \( T_i \). Hence,

\[
\frac{p'}{1 - p' + q'_i} \leq pT_i = \frac{p}{1 - p + q_i} < \frac{p'}{1 - p + q_i},
\]

\((1 - p' + q'_i) > (1 - p + q_i),
\]

\[p' - q'_i < p - q_i.
\]

We consider the sum of these expressions for all \( i \).

\[(s - 1)p' \leq s \cdot p' - \sum_{i=1}^{s} q'_i = \sum_{i=1}^{s} (p' - q'_i) < \sum_{i=1}^{s} (p - q_i) = s \cdot p - \sum_{i=1}^{s} q_i = (s - 1)p
\]

and \( p' < p \). Contradiction, proving the lemma. \(\Box\)

By Lemma 11, the relation between \( p_T \) and \( p_{T_1}, \ldots, p_{T_s} \) is also expressed by Eq. (1). We can now show the equivalence of the two definitions.

**Lemma 18.** \( A = A' \).

**Proof.** By induction. If \( p \in A' \) follows from \( p_1, \ldots, p_s \in A' \) by rule (2) and \( pT_i = p_i \) for trees \( T_i \), we construct a tree \( T \) consisting of the root, \( T_1, \ldots, T_s \) and make the roots of \( T_1, T_2, \ldots, T_s \) children of \( T \)'s root. Then, \( p_T = p \) (by Lemma 17). Hence, for any \( p \in A' \), there is a tree \( T \) with \( p_T = p \). This means \( A' \subseteq A \).

Similarly, we can show that \( p_T \in A \) for any tree \( T \). \(\Box\)

Next, we show that the \((p_T, 1)\)-labeling of Lemma 17 uses only rational numbers and, hence, can be transformed into a labeling that uses only integers.

**Lemma 19.** For any tree \( T \), \( p_T \in \mathbb{Q} \).

**Proof.** By induction over the depth of \( T \). For a tree consisting of root only, \( p = 1 \).

Otherwise, let \( t_1, t_2, \ldots, t_s \) be all direct successors of the root in \( T \) and \( T_1, T_2, \ldots, T_s \) be the subtrees with roots \( t_1, t_2, \ldots, t_s \). The depth of these subtrees is smaller than the depth of \( T \). Hence, all \( pT_i \) are rationals. Equation (1) implies that \( p_T \) is rational, too. \(\Box\)

**Lemma 20.** \((p_T, 1)\)-labeling constructed in the proof of Lemma 17 uses only rational numbers.

**Proof.** By induction over the depth of \( T \). Again, the lemma is evident for the tree with the root only.

For other trees, notice that all \( q_i \) can expressed by \( p \) and \( pT_i \). Hence, \( q_1, \ldots, q_s \) are rationals. Label of the root is the rational number \( p \), labels of \( t_1, \ldots, t_s \) are rationals \( p - q_1, \ldots, p - q_s \) and labels of other nodes are \((1 - p + q_i) v_i^j(t)\).

\((1 - p + q_i) \) is a rational number because \( p \) and \( q_i \) are rationals and \( v_i^j(t) \) is a rational number because \( v_i^j \) is a part of the \((pT_i, 1)\)-labeling for a tree of smaller depth. \(\Box\)

**Corollary 21.** Let \( T \) be a tree. Then there is \( n \in \mathbb{N} \) such that \( T \) has \((p_T n, n)\)-labeling with labels from \( \mathbb{N} \).
Proof. Let $n$ be the least common denominator of all rational numbers in the $(p_T, 1)$-labeling $v_1, v_2$ of Lemma 17. Then, $nv_1(t), nv_2(t)$ is a $(p_T n, n)$ labeling and uses only natural numbers. □

3.4. Universal diagonalization

Let $0^j$ denote a sequence of $j$ zeros and $0^\omega$ denote an infinite sequence of zeros. Let $K$ be the halting set, i.e. the set of all $i$ such that program $\varphi_i$ halts on input $i$. Let $K_i$ be the set of all $i$ such that $\varphi_i$ halts on input $i$ in at most $s$ steps. For a set $S$, let $\chi_S$ be the characteristic function of $S$: $\chi_S(i) = 1$ if $i \in S$ and $\chi_S(i) = 0$ otherwise.

Definition 22. (See [18].) Let $T$ be a tree of depth $d$. $S_T$ is the set of all recursive functions $f$ such that the sequence of values $f(0), f(1), \ldots$ is of the form

$$i_1 \ldots i_d 0^l a_1 0^{l_2} a_2 \ldots 0^l a_1 0^\omega$$

where each $t_i = \min\{t : |\{j : i_j \in K_i\}| \geq h\}$ is finite, $(a_1, \ldots, a_l) \in T$, and either $l = |\{j : i_j \in K\}|$ or $(a_1, \ldots, a_l)$ is a leaf of $T$.

Lemma 23. (See [18].) If $T$ has an $(m, n)$-labeling by integers then

$$S_T \in [m, n]^{\text{PFIN}}.$$  

The next lemma is an extension of Kummer’s results to probabilistic learning. The proof is similar to Theorem 16 in [18]. We give it here for completeness.

Lemma 24. If $S_T \in (p)^{\text{PFIN}}[O]$ and $K$ is not Turing reducible to $O$, then $T$ has a $(p - \epsilon, 1)$ labeling for any $\epsilon > 0$.

Proof. Let $k$ be the depth of $T$. Let $M$ denote an IIM that identifies $S_T$ with the $O$-oracle. For arbitrary $i_1, \ldots, i_k$, we enumerate a set $T_{i_1, \ldots, i_k}$.

Define the event $P(c, s)$ to be true iff $c = |\{j : i_j \in K_i\}|$ and, for each $(a_1, \ldots, a_c) \in T$ and $\sigma_c = i_1 \ldots i_c 0^l a_1 0^{l_2} a_2 \ldots 0^l a_c$ with $\sigma_c 0^\omega$, the probability that $M^O$ outputs a program computing a function with an initial segment $\sigma_c$ while reading $\sigma_c 0^\omega$ is at least $p - \epsilon$.

The procedure for enumerating $T_{i_1, \ldots, i_k}$ is as follows.

Initialization. Let $t = 0, c' = -1, T_{i_1, \ldots, i_k} = \emptyset$.

Step $l$. Search for the smallest $s > i$ satisfying $P(c, s)$ for some $c > c'$. If the search terminates, enumerate $(\chi_{K_1}(i_1), \ldots, \chi_{K_k}(i_k))$ into $T_{i_1, \ldots, i_k}$, set $t = s, c' = c$ and go to Step $l + 1$.

Claim 25. $(\chi_{K_1}(i_1), \ldots, \chi_{K_k}(i_k)) \in T_{i_1, \ldots, i_k}$.

Proof. Let $c = |\{j : i_j \in K_i\}|$. $P(c, s)$ holds for all sufficiently large $s$ because $M^O$ infers all functions $\sigma_c 0^\omega$. After discovering it, $(\chi_{K_1}(i_1), \ldots, \chi_{K_k}(i_k)) = (\chi_{K_1}(i_1), \ldots, \chi_{K_k}(i_k))$ is enumerated into $T_{i_1, \ldots, i_k}$. □

Claim 26. $|T_{i_1, \ldots, i_k}| = k + 1$ for some $i_1, \ldots, i_k$.

Proof. If $(\chi_{K_1}(i_1), \ldots, \chi_{K_k}(i_k)) \in T_{i_1, \ldots, i_k}$ and $|T_{i_1, \ldots, i_k}| \leq k$ for all $i_1, \ldots, i_k$, then, by Fact 6 in [18], $K$ is Turing-reducible to $O$. □

Hence, there exists $i_1, \ldots, i_k$ and $s_1 < \cdots < s_{k+1}$ such that $P(l - 1, s_l)$ for $l = 1, \ldots, k + 1$. Define the label $v_1(\tau)$ of $\tau = (a_1, \ldots, a_{l-1})$ as the probability that:

1. $M$ does not output a program while reading $\sigma_{l-2} 0^{\omega-2}$, where

$$\sigma_c = i_1 \ldots i_k 0^l a_1 0^{l_2} a_2 \ldots 0^l a_c,$$

and

2. $M$ outputs a program computing a function with the initial segment $\sigma_{l-1}$ while reading $\sigma_{l-1} 0^{\omega-1}$. 

For $\tau = \epsilon$, there is no segment $\sigma_{-1}$ and $v_1(\epsilon)$ is just the probability that $M^O$ outputs a program computing a function with the initial segment $0\sigma$ while reading $0^{|\tau|}$.

The label $v_2(\tau)$ is 0 for $\tau = \epsilon$ and the probability that $M^O$ outputs a program computing a function with the initial segment $\sigma_{-1}$ while reading $\sigma_{-1}0^{|\tau|-2}$ for $\tau = (a_1, \ldots, a_{i-1})$.

Next, we verify that all conditions of Definition 10 are satisfied. Property (1) follows from the definitions of $v_1(\epsilon)$, $v_2(\epsilon)$ and $P(0, s)$.

For property (2), notice that $v_1(t) + v_2(t)$ is the total probability that $M^O$ outputs a function consistent with $\sigma_{-1}$ while reading $\sigma_{-1}0^{|\tau|-1}$. $v_1(t)$ are the probabilities that a particular continuation of $\sigma_{-1}$ is an initial segment of the function. These events are mutually exclusive. Hence, $\sum_{i=1}^n v_1(t_i) \leq v_1(t) + v_2(t)$. $v_1(t_i) + v_2(t_i) \geq p - \epsilon$ is true because $M^O$ outputs a program consistent with $\sigma_{i}0^{|\tau|$ with a probability at least $p - \epsilon$ (by the definition of $P(c, s)$).

Property (3) is true because the sum of all $v_1$-labels on any branch is at most the probability that $M^O$ outputs a conjecture while reading $\sigma_{i}0^{|\tau|$ and, hence, is at most 1. □

If there is no oracle $O$, we get

**Corollary 27.** If $ST \in \langle p \rangle PFIN$, then $T$ has a $(p - \epsilon, 1)$ labeling for any $\epsilon > 0$.

**Corollary 28.** For a tree $T$, $ST \in \langle pt \rangle PFIN$ and $ST \notin \langle pt + \epsilon \rangle PFIN$ for any $\epsilon > 0$.

**Proof.** Corollary 21 and Lemma 23 imply that $ST \in [ptn, n]PFIN$ for appropriate $n$. A $[ptn, n]PFIN$ team can be simulated by a $\langle pt \rangle PFIN$ probabilistic machine that chooses one of $n$ machines in the team equiprobably.

If $ST \in \langle pt + \epsilon \rangle PFIN$, then, there is a $(pt + \epsilon/2, 1)$ labeling of $T$ (Corollary 27). This is impossible because $pt$ is the largest number such that there is a $(pt, 1)$ labeling of $T$. □

**Theorem 29.** If $p, q \in A$ and $p \neq q$, then $\langle p \rangle PFIN \neq \langle q \rangle PFIN$.

**Proof.** Follows from Corollary 28 and Lemma 18. □

### 3.5. Well-ordering and system of notations

It remains to prove that, for any probability $p$, $PFIN(p)$-type learning is equivalent to $PFIN$-type learning with some probability belonging to $A$. Our diagonalization technique was similar to [18]. The simulation part is more complicated. Simulation techniques in [18] rely on the fact that each team issues finitely many conjectures and, hence, there are finitely many possible behaviors of these conjectures. A probabilistic machine can issue infinitely many conjectures and these conjectures have infinitely many possible behaviors. This makes simulation far more complicated.

We need an algorithmic structure for manipulating an infinite number of possibilities. We establish it by proving that $A$ is well-ordered and has a system of notations.

**Theorem 30.** The set $A$ is well-ordered in decreasing ordering and has a system of notations.

**Proof.** We construct a system of notations for the set $A$ inductively. First, we construct a system of notations for $A \cap [\frac{1}{2}, 1]$. Then we extend it, obtaining system of notations for $A \cap [\frac{1}{4}, 1]$, $A \cap [\frac{1}{8}, 1]$ and so on.

Freivalds [12] proved

$$A \cap \left[ \frac{1}{2}, 1 \right] = \left\{ \frac{1}{2} \right\} \cup \left\{ \frac{n}{2n - 1} \mid n \in \mathbb{N} \text{ and } n \geq 1 \right\}.$$  

A system of notations for $A \cap [\frac{1}{4}, 1]$ can be easily constructed from this description. Below, we show how to construct a system of notations for $A \cap [\frac{1}{n+1}, 1]$ using a system of notations for $A \cap [\frac{1}{n}, 1]$.

An outline of our construction is as follows:

1. Split the segment $[\frac{1}{n+1}, \frac{1}{n}]$ into smaller segments $[r_{i+1}, r_i]$ so that, if $p \in [r_{i+1}, r_i]$ and $p \in A$ follows from the rule (2), then $p_1 \geq r_i, \ldots, p_s \geq r_i$. (This property allows us to obtain a system of notations for $A \cap [r_{i+1}, r_i]$
from a given system of notations for \( A \cap [r_i, 1] \) without using any knowledge about \( A \cap [\frac{1}{n+1}, 1] \). We give the splitting and prove its properties in Section 3.5.1.

(2) Using transfinite induction over the segments \([r_i+1, r_i]\), extend the system of notations for \( A \cap [\frac{1}{n+1}, 1] \) to larger and larger segments \( A \cap [r_i+1, 1] \), finally obtaining a system of notations for \( A \cap [\frac{1}{n+1}, 1] \). This part is described in Sections 3.5.2–3.5.4 and 3.5.5.

3.5.1. Splitting the segment \([\frac{1}{n+1}, \frac{1}{n}]\)

The splitting consists of two steps.

(1) First, we take \( \frac{p}{1+p} \) for \( p \in A \cap [\frac{1}{n}, \frac{1}{n-1}] \). By Lemma 15, all \( \frac{p}{1+p} \) belong to the set \( A \). These points split \([\frac{1}{n+1}, \frac{1}{n}]\) into segments \([\frac{p}{1+p}, \frac{r}{1+r}]\).

(2) Each segment \([\frac{p}{1+p}, \frac{r}{1+r}]\) is split further by the sequence

\[
\begin{align*}
r_0 &= \frac{r}{1+r}, \\
r_{i+1} &= \frac{2}{1 + \frac{1}{p} + \frac{1}{r}}.
\end{align*}
\]

Let \( r_0, r_1, r_2, \ldots \) be a monotonically decreasing sequence converging to \( \frac{p}{1+p} \). It splits \([\frac{p}{1+p}, \frac{r}{1+r}]\) into segments \([r_1, r_0], [r_2, r_1], \ldots \).

Let \( A_p \) denote the set consisting of all \( \frac{p}{1+p} \) and \( r_0, r_1, \ldots \) for all segments \([\frac{p}{1+p}, \frac{r}{1+r}]\). Next, we prove several properties of the segments \([r_i+1, r_i]\) that will be used further.

**Lemma 31.** Let \([\frac{p}{1+p}, \frac{r}{1+r}]\) be a segment obtained in the first step of the splitting. If \( x \in A \) follows from \( p_1, \ldots, p_s \in A \) by the rule (2) and \( x \in [\frac{p}{1+p}, \frac{r}{1+r}] \) then

\[
p_1 \leq p, \quad p_2 \leq p, \quad \ldots, \quad p_s \leq p.
\]

**Proof.** We have

\[
p_j = \frac{x}{1 - x + q_j} < \frac{x}{1 - x + 0} = \frac{x}{1 - x},
\]

\[
p_j(1 - x) < x,
\]

\[
p_j < x(1 + p_j),
\]

\[
\frac{p_j}{1 + p_j} < x.
\]

Therefore, \( \frac{p}{1+p_j} \leq \frac{p}{1+p} \) and \( p_j \leq p \). \( \square \)

**Lemma 32.** Let \( x \in A \cap [r_i+1, r_i] \). If \( x \in A \) follows from \( p_1, \ldots, p_s \in A \) by the rule (2), then

\[
p_1 \geq r_i, \quad p_2 \geq r_i, \quad \ldots, \quad p_s \geq r_i.
\]

**Proof.** We prove \( p_1 \geq r_i \) only. (\( p_2 \geq r_i, \ldots \) are proved similarly.)

Assume that \([r_i+1, r_i]\) was obtained by splitting \([\frac{p}{1+p}, \frac{r}{1+r}]\). Then, \( p_1 \leq p, \quad p_2 \leq p, \quad \ldots, \quad p_s \leq p \) (Lemma 31). From

\[
\frac{x}{1 - x + q_j} = p_j
\]

it follows that

\[
q_j = \frac{x}{p_j} - 1 + x.
\]

We have \( p_2 \leq p \). Hence,
\[
q_2 \geq \frac{x}{p} - 1 + x,
\]
\[
q_1 \leq x - q_2 \leq 1 - \frac{x}{p},
\]
\[
p_1 = \frac{x}{1 - x + q_1} \geq \frac{x}{2 - x - \frac{x}{p}} = \frac{1}{\frac{x}{2} - 1 - \frac{1}{p}}.
\]

From \(x \in [r_{i+1}, r_i]\) we have that \(x \geq r_{i+1}\) and
\[
p_1 \geq \frac{1}{\frac{x}{2} - 1 - \frac{1}{p}} \geq \frac{1}{\frac{1}{r_i} + 1 + \frac{1}{p}} - \frac{1}{\frac{1}{p}} = r_i. \quad \square
\]

We have proved that all \(x \in A \cap [r_{i+1}, r_i]\) are generated by applications of the rule (2) to \(p_1, \ldots, p_s \in A \cap [r_i, 1]\). The next lemma bounds the number \(s\).

**Lemma 33.** Let \(x \in A \cap [r_{i+1}, r_i]\), with \([r_{i+1}, r_i]\) being a segment obtained by splitting \([\frac{p}{1+p}, \frac{r_i}{1+r_i}]\). If \(x \in A\) follows from \(p_1, \ldots, p_s \in A\) by the rule (2), then
\[
s \leq \frac{x}{\frac{2}{p} + x - 1}.
\]

**Proof.** From Lemma 31 we have
\[
q_j = \frac{x}{p_j} + x - 1 \geq \frac{x}{p} + x - 1.
\]

Hence,
\[
x = \sum_{j=1}^{s} q_j \geq s \left( \frac{x}{p} + x - 1 \right),
\]
\[
s \leq \frac{x}{\frac{2}{p} + x - 1}. \quad \square
\]

### 3.5.2. Well-ordering

**Lemma 34.** \(A_n\) is well-ordered.

**Proof.** \(A \cap [\frac{1}{n+1}, 1]\) is well-ordered by inductive assumption. Hence, \(A \cap [\frac{1}{n}, \frac{1}{n-1}]\) is well-ordered, too. The set \([\frac{p}{1+p}, \frac{r_i}{1+r_i}]| p \in A \cap [\frac{1}{n}, \frac{1}{n-1}]\) is order-isomorphic to \(A \cap [\frac{1}{n}, \frac{1}{n-1}]\). Hence, it is well-ordered and the set of segments \([\frac{p}{1+p}, \frac{q}{1+q}]\) into which it splits \([\frac{1}{n+1}, \frac{1}{n}]\) is well-ordered, too.

\(A_n\) is obtained by replacing each segment \([\frac{p}{1+p}, \frac{r_i}{1+r_i}]\) with the sequence \(r_0, r_1, \ldots\). Each sequence is well-ordered. Hence, the entire set \(A_n\) is well-ordered. \(\square\)

Hence, we can use transfinite induction over this set.

**Lemma 35.** \(A \cap [\frac{1}{n+1}, \frac{1}{n}]\) is well-ordered in decreasing ordering.

**Proof.** By transfinite induction over \(A_n\).

**Base case.** The set \(A \cap [\frac{1}{n}, 1]\) is well-ordered.

**Inductive case.** Let \(x \in A_n\). We assume that \(A \cap [x', 1]\) is well-ordered for all \(x' > x, x' \in A_n\) and prove that \(A \cap [x, 1]\) is well-ordered, too. There are three cases:

1. \(x = \frac{p}{1+p}\) for \(p \in A \cap [\frac{1}{n+1}, \frac{1}{n}]\) and \(p\) is a limit element.
Let \( p \) be the limit of \( p_1, p_2, \ldots \). Then, \( \frac{p}{1+p} \) is the limit of \( \frac{p_1}{1+p_1}, \frac{p_2}{1+p_2}, \ldots \) because the function \( \frac{x}{1+x} \) is continuous. By inductive assumption, each \( \frac{p_i}{1+p_i}, 1 \) is well-ordered. Hence, their union \( \frac{p}{1+p}, 1 \) is well-ordered.

\[(2) \ x = \frac{p}{1+p} \text{ for } p \in A \cap \left[ \frac{1}{n+1}, \frac{1}{n} \right] \text{ and } p \text{ is not a limit element.}\]

We take the segment \( \frac{p}{1+p}, \frac{r}{1+r} \) obtained in the first step of the splitting and the corresponding sequence \( r_0, r_1, \ldots \).

\[\frac{p}{1+p} \text{ is the limit of } r_0, r_1, \ldots \text{; } \frac{p}{1+p}, 1 \text{ is well-ordered because each } [r_i, 1] \text{ is well-ordered.}\]

\[(3) \ x \neq \frac{p}{1+p} \text{ for any } p \in A \cap \left[ \frac{1}{n+1}, \frac{1}{n} \right]. \text{ Then, } x \neq r_0 \text{ because } r_0 = \frac{r}{1+r} \text{ for } r \in A \cap \left[ \frac{1}{n+1}, \frac{1}{n} \right]. \text{ Hence, } x = r_i + 1 \text{ for some } i \geq 0.\]

\( A \cap [r_i, 1] \) is well-ordered because \( r_i + 1 < r_i \). Hence, it is enough to prove that \( A \cap [r_i, 1] \) is well-ordered.

For a contradiction, assume that \( A \cap [r_i, 1] \) contains an infinite monotonically increasing sequence \( x_1, x_2, \ldots \)

**Claim 36.** Let \( x_1 \in A \cap [r_i, 1], x_2 \in A \cap [r_i, 1], \ldots \) There is an \( s \in \mathbb{N} \) and sequences \( x'_1, x'_2, \ldots \) and \( p_{j,1}, p_{j,2}, \ldots \) for \( j \in \{1, \ldots, s\} \) such that

(a) \( x'_1, x'_2, \ldots \) is a subsequence of \( x_1, x_2, \ldots \),

(b) \( x'_k \in A \) follows from \( p_{1,k}, \ldots, p_{s,k} \in A \) and the rule (2), and

(c) \( p_{j,1} = p_{j,2} = \cdots \) or \( p_{j,1} > p_{j,2} > \cdots \) for all \( j \in \{1, \ldots, s\} \).

**Proof.** Denote

\[s_1 = \left[ \frac{x}{r_{i+1} + x - 1} \right].\]

Consider the applications of the rule (2) that prove \( r_1 \in A, r_2 \in A, \ldots \). By Lemma 33,

\[s \leq \frac{x}{p + x - 1} \leq \frac{x}{r_{i+1} + x - 1} \leq s_1\]

in each of these applications. Hence, there exists an \( s_0 \in \{1, \ldots, s_1\} \) such that infinitely many of \( x_1, \ldots \) are generated by applications of the rule (2) with \( s = s_0 \). We denote this subsequence \( x_1^{(0)}, x_2^{(0)}, \ldots \)

Next, we select \( x_1^{(1)}, x_2^{(1)}, \ldots \), a subsequence of \( x_1^{(0)}, x_2^{(0)}, \ldots \). Then, we select \( x_1^{(2)}, x_2^{(2)}, \ldots \), a subsequence of \( x_1^{(1)}, x_2^{(1)}, \ldots \). We continue so until we obtain \( x_1^{(s_0)}, x_2^{(s_0)}, \ldots \)

The subsequence \( x_1^{(k)}, x_2^{(k)}, \ldots \) is generated from \( x_1^{(k-1)}, x_2^{(k-1)}, \ldots \) as follows:

Let \( p_{1,k}^{(k-1)}, p_{2,k}^{(k-1)}, \ldots, p_{s_0,k}^{(k-1)} \) be the values of \( p_1, \ldots, p_{s_0} \) in the application of the rule (2) that proves \( x_j^{(k-1)} \in A \). We use the infinite version of Dilworth’s lemma.

**Theorem 37.** Let \( y_1, y_2, \ldots \) be a sequence of real numbers. Then \( y_1, y_2, \ldots \) contains

- a subsequence \( y_{n_1}, y_{n_2}, \ldots \) such that \( y_{n_1} = y_{n_2} = \cdots \), or
- an infinite monotonically increasing subsequence, or
- an infinite monotonically decreasing subsequence.

The sequence \( p_{1,k}^{(k-1)}, p_{2,k}^{(k-1)}, \ldots \) does not contain an infinite monotonically increasing subsequence because all elements of this sequence belong to \( A \cap [r_i, 1] \) and \( A \cap [r_i, 1] \) is well-ordered in decreasing ordering. Hence, this sequence contains an infinite subsequence consisting of equal elements or an infinite monotonically decreasing subsequence.

Let this subsequence be \( p_{k,n_1}^{(k-1)}, p_{k,n_2}^{(k-1)}, \ldots \). We choose \( r_{n_1}^{(k-1)}, r_{n_2}^{(k-1)}, \ldots \) as the sequence \( x_1^{(k)}, x_2^{(k)}, \ldots \), \( x_1^{(s_0)}, x_2^{(s_0)}, \ldots \) is the needed sequence \( x_1', x_2', \ldots \). We have

\[p_{1,k} = p_{2,k} = \cdots \text{ or } p_{1,k} > p_{2,k} > \cdots\]

because such property holds for the sequence \( x_1', x_2', \ldots \) and \( x_1^{(s_0)}, x_2^{(s_0)}, \ldots \) is a subsequence of \( x_1^{(k)}, x_2^{(k)}, \ldots \) \qed
We have
\[ p_{1,1} \geq p_{2,1} \geq \cdots, \]
\[ \vdots \]
\[ p_{1,s} \geq p_{2,s} \geq \cdots. \]
By Lemma 13,
\[ x'_1 \geq x'_2 \geq x'_3 \geq \cdots. \]
Hence, \( x_1, x_2, \ldots \) contains an infinite non-increasing subsequence. \( \square \)

This is a contradiction with the assumption that \( x_1, x_2, \ldots \) is monotonically increasing. \( \square \)

Next, we construct a system of notations \( S \) for \( A \cap [\frac{1}{n+1}, \frac{1}{n}] \). We start with technical results necessary for our construction. In Section 3.5.3, we show how to distinguish limit elements from successor elements. In Section 3.5.4, we define \((x, d)\)-minimal sets and show that such sets can be computed algorithmically. Finally, in Section 3.5.5, we use these results to construct a system of notations.

3.5.3. Distinguishing elements of different types

The maximal element of the set \( A \) is 1. It does not belong to \( A \cap [r_{i+1}, r_i] \). Hence, \( A \cap [r_{i+1}, r_i] \) does not contain the maximal element and, constructing a system of notations, we should distinguish numbers \( p \) of three types:

1. \( p \in A \cap [r_{i+1}, r_i] \) and \( p \) is a successor. Then \( k_S(p) = 1 \).
2. \( p \in A \cap [r_{i+1}, r_i] \) and \( p \) is a limit element. Then \( k_S(p) = 2 \).
3. \( p /\in A \cap [r_{i+1}, r_i] \). Then \( k_S(p) = 3 \).

Two lemmas below shows how to distinguish between limit and successor elements.

**Lemma 38.** Let \( x \in A \cap [r_{i+1}, r_i] \). Then \( x \) is a limit element if and only if it can be generated by rule (2) so that at least one of \( p_1, \ldots, p_s \) is limit element.

**Proof.** “if” part. Assume that \( p_j \) is a limit element. Let \( p_{j,1}, p_{j,2}, \ldots \) be a monotonically decreasing sequence converging to \( p_j \) and \( x_k \) be the number generated by the application of the rule (2) to \( p_{j,1}, p_{j,2}, \ldots, p_{j,s} \). Then, \( x_1, x_2, \ldots \) is a monotonically decreasing sequence converging to \( x \). Hence, \( x \) is a limit element.

“only if” part. Let \( x \) be a limit element and \( x_1, x_2, \ldots \) be a monotonically decreasing sequence converging to \( x \). We apply Claim 36 to \( x_1, x_2, \ldots \) and obtain a subsequence \( x'_1, x'_2, \ldots \).

We consider the sequences \( p_{j,1}, p_{j,2}, \ldots \). Let
\[ p'_j = \lim_{k \to \infty} p_{j,k}. \]
By Lemma 14, \( x \) can be generated from \( p'_1, p'_2, \ldots, p'_s \) by an application of rule (2). We have
\[ p_{j,1} = p_{j,2} = \cdots \quad \text{or} \quad p_{j,1} > p_{j,2} > \cdots \]
for any \( j \in \{1, \ldots, m\} \). If \( p_{j,1} = p_{j,2} = \cdots \) for all \( j \), then, \( x'_1 = x'_2 = \cdots \). A contradiction with the assumption that \( x_1, x_2, \ldots \) is monotonically decreasing.

Hence,
\[ p_{j,1} > p_{j,2} \geq \cdots \]
for at least one \( j \) and \( p'_j = \lim_{k \to \infty} p_{j,k} \) is a limit element. \( \square \)

**Lemma 39.** Let \( x \in A_n \). Then \( x \) is a limit element.
Proof. We have three cases.

1. \( x = \frac{p}{1+p} \) for \( p \in \mathcal{A} \cap [\frac{1}{n+1}, \frac{1}{n}] \) and \( p \) is a limit element.

Let \( p \) be the limit of \( p_1, p_2, \ldots \). Then, \( \frac{p}{1+p} \) is the limit of \( \frac{p_1}{1+p_1}, \frac{p_2}{1+p_2}, \ldots \) because the function \( \frac{x}{1+x} \) is continuous.

2. \( x = \frac{p}{1+p} \) for \( p \in \mathcal{A} \cap [\frac{1}{n+1}, \frac{1}{n}] \) and \( p \) is not a limit element.

We take the segment \( [\frac{p}{1+p}, \frac{r}{1+r}] \) obtained in the first step of the splitting and the corresponding sequence \( r_0, r_1, \ldots, r_i \).

3. \( x \neq \frac{p}{1+p} \) for any \( p \in \mathcal{A} \cap [\frac{1}{n+1}, \frac{1}{n}] \).

Then, \( x = r_i \). We prove the lemma by induction over \( i \).

- **Base Case.** If \( i = 0 \), then \( r_i = \frac{r}{1+r} \) and we already know that \( \frac{r}{1+r} \) is a limit element.

- **Inductive Case.** Lemma 11 and the definition of \( r_{i+1} \) imply that \( r_{i+1} \in \mathcal{A} \) follows from \( r_i \in \mathcal{A} \) and \( p \in \mathcal{A} \) by the rule (2). If \( r_i \) is a limit element, then, by Lemma 38, \( r_{i+1} \) is a limit element, too. \( \square \)

3.5.4. \((x,d)\)-minimal sets

In the algorithms of Section 3.5.5, we will often need to compute the largest element of \( \mathcal{A} \cap [r_{i+1}, r_i] \) which is less than some given \( x \). This will be done by checking \( p_1 \in \mathcal{A} \cap [r_i, 1], p_2 \in \mathcal{A} \cap [r_i, 1], \ldots, p_s \in \mathcal{A} \cap [r_i, 1] \) that can generate \( x \in \mathcal{A} \) by rule (2). There are infinitely many possible combinations of \( p_1, \ldots, p_s \). Hence, we need

- to prove that it is enough to check finitely many combinations \( p_1 \in \mathcal{A} \cap [r_i, 1], p_2 \in \mathcal{A} \cap [r_i, 1], \ldots, p_s \in \mathcal{A} \cap [r_i, 1] \), and
- to construct an algorithm finding the list of combinations \( p_1 \in \mathcal{A} \cap [r_i, 1], p_2 \in \mathcal{A} \cap [r_i, 1], \ldots, p_s \in \mathcal{A} \cap [r_i, 1] \) which must be checked when the functions \( k_S, p_S, q_S \) are computed.

We do it below. First, we give formal definitions.

**Definition 40.** A tuple \( \langle p_1, \ldots, p_s \rangle \) is said to be \((x,d)\)-allowed if we have \( p_1 \in \mathcal{A} \cap [r_i, 1], \ldots, p_s \in \mathcal{A} \cap [r_i, 1] \) and \( \sum_{j=1}^{s} (\frac{x}{p_j} + x - 1) \leq d \).

**Definition 41.** A tuple \( \langle p_1, \ldots, p_s \rangle \) is said to be less than or equal to \( \langle p'_1, \ldots, p'_s \rangle \) if \( p_1 \leq p'_1, \ldots, p_s \leq p'_s \).

**Definition 42.** A set of tuples \( P \) is said to be \((x,d)\)-minimal if,

1. It contains only \((x,d)\)-allowed tuples;
2. For each \((x,d)\)-allowed tuple \( \langle p_1, \ldots, p_s \rangle \) there is a tuple belonging to \( P \) which is less than or equal to \( \langle p_1, \ldots, p_s \rangle \).

Next three lemmas show why \((x,d)\)-allowed tuples and \((x,d)\)-minimal sets are important for our construction.

**Lemma 43.** \( \langle p_1, \ldots, p_s \rangle \) is \((x,x)\)-allowed if and only if the application of the rule (2) to \( p_1, \ldots, p_s \) generates a number \( p \) satisfying \( p \geq x \).

**Proof.** Let \( d = \sum_{j=1}^{s} x + \frac{x}{p_j} - 1 \). \( \langle p_1, \ldots, p_s \rangle \) is \((x,x)\)-allowed if and only if \( d \leq x \). Hence, it is enough to prove that \( d \leq x \) if and only if \( x \leq p \).

\[
\begin{align*}
d &= \sum_{j=1}^{s} \left( x + \frac{x}{p_j} - 1 \right) = \sum_{j=1}^{s} \left( x + \frac{x}{p_j} - 1 \right) - p + p = \sum_{j=1}^{s} \left( x + \frac{x}{p_j} - 1 \right) - \sum_{j=1}^{s} \left( \frac{p + p}{p_j} - 1 \right) + p \\
&= \left( \sum_{j=1}^{s} \left( 1 + \frac{1}{p_j} \right) \right) (x - p) + p.
\end{align*}
\]
We have
\[ \sum_{j=1}^{s} \left( 1 + \frac{1}{p_j} \right) \geq 1 + \frac{1}{p_j} > 1. \]

Hence, if \( x > p \), then \( (x - p) > 0 \) and \( d > (x - p) + p = x \). If \( x \leq p \), then \( (x - p) \leq 0 \) and \( d \leq (x - p) + p = x \). \( \square \)

**Lemma 44.** Let \( P \) be a \((x, x)\)-minimal set. Then, for any \( p_1, \ldots, p_s \) that generates \( p \geq x \) by an application of the rule (2), there exists a tuple \( \langle p_1', \ldots, p_s' \rangle \in S \) such that \( p_1' \leq p_1, \ldots, p_s' \leq p_s \).

**Proof.** By Lemma 43, \( \langle p_1, \ldots, p_s \rangle \) is \((x, x)\)-allowed. By the definition of \((x, x)\)-minimal set, \( P \) contains a tuple \( \langle p_1', \ldots, p_s' \rangle \) such that \( p_1' \leq p_1, \ldots, p_s' \leq p_s \). \( \square \)

**Lemma 45.** Let \( P \) be a \((x, x)\)-minimal set, \( p_1 \in A \cap [r_i, 1] \), \( \ldots, p_s \in A \cap [r_i, 1] \). If \( x \in A \) follows from \( p_1, \ldots, p_s \in A \) and the rule (2), then \( \langle p_1, \ldots, p_s \rangle \in P \).

**Proof.** By Lemma 43, \( \langle p_1, \ldots, p_s \rangle \) is \((x, x)\)-allowed. Hence, by Lemma 44, there exists \((x, x)\)-allowed \( \langle p_1', \ldots, p_s' \rangle \in P \) such that \( p_1' \leq p_1, \ldots, p_s' \leq p_s \).

Let \( x' \) be the number generated by an application of the rule (2) to \( p_1' \in A \), \( \ldots, p_s' \in A \). If \( p_j' < p_j \) for some \( i \), then \( x' < x \) (Lemma 13) and \( \langle p_1', \ldots, p_s' \rangle \) is not \((x, x)\)-allowed (Lemma 43).

However, \((x, x)\)-allowed set contains only \((x, x)\)-allowed tuples. Hence, \( p_1 = p_1', \ldots, p_s = p_s' \), i.e. \( \langle p_1, \ldots, p_s \rangle \in P \). \( \square \)

Next lemma shows that \((x, d)\)-minimal sets can be computed algorithmically. Its proof also shows that a finite \((x, d)\)-minimal set always exists.

**Lemma 46.** Assume that a system of notations for \( A \cap [r_i, 1] \) is given. There is an algorithm \text{findsmallest}\((x, d)\) which receives \( x \in A \cap [r_{i+1}, 1] \) and \( d \in [0, x] \) and returns a \((x, d)\)-minimal set.

**Proof.** We use an auxiliary procedure \text{findsmallest}\((P, x, d)\). It receives numbers \( x, d \) and an \((x, d)\)-minimal set \( P \) and returns the smallest \( d' \) such that \( d' > d \) and \( \sum_{j=1}^{s} (\frac{x}{p_j} + x - 1) = d' \) for some \( p_1, \ldots, p_s \in A \).

Both \text{findsmallest} and \text{xminimal} use a constant \( p_0 \). \( p_0 \) is defined as the largest number in \( A \cap [r_i, 1] \) such that \( x + \frac{x}{p_0} - 1 > 0 \). Equivalently, \( p_0 \) is the number in \( A \cap [r_i, 1] \) with the smallest \( x + \frac{x}{p_0} - 1 \) such that \( x + \frac{x}{p_0} - 1 > 0 \). \( \Delta \) denotes \( \frac{x}{p_0} + x - 1 \).

**Algorithm** \text{findsmallest}\((P, x, d)\):

\begin{enumerate}
\item Let \( d' = 1 \);
\item For each \( \langle p_1, \ldots, p_s \rangle \in P \) do:
\begin{enumerate}
\item For each \( j \in \{1, \ldots, s\} \):
\begin{enumerate}
\item Find \( p_j' = \max\{p \mid p \in A \cap [r_i, 1] \text{ and } p < p_j\} \), using the given system of notations for \( A \cap [r_i, 1] \).
\item \( d_1 = \sum_{k=1}^{j-1} (\frac{x}{p_k} + x - 1) + (\frac{x}{p_j'} + x - 1) + \sum_{k=j+1}^{s} (\frac{x}{p_k} + x - 1) \). If \( d_1 > d \), then \( d' = \min(d', d_1) \).
\end{enumerate}
\item \( d_2 = \sum_{j=1}^{s} (\frac{x}{p_j} + x - 1) + (\frac{x}{p_0} + x - 1) \); If \( d_2 > d \), then \( d' = \min(d', d_2) \).
\end{enumerate}
\end{enumerate}
\item Return \( d' \) as the result.

**Algorithm** \text{xminimal}\((x, d)\):

\begin{enumerate}
\item Let \( P = \emptyset \);
\item If \( d < \Delta \), return the empty set as the result;
\item Let \( y \) be the smallest number in \( A \cap [r_i, 1] \) such that \( \frac{x}{y} + x - 1 < d \);
\item While \( (\frac{x}{y} + x - 1 > 0) \) do:
\end{enumerate}
(a) \( d' = d - \left( \frac{x}{y} + 1 \right) \);
(b) \( P_1 = \text{xdmin}(x, d') \);
(c) If \( P_1 = \emptyset \), add \((y)\) to \( P \). Otherwise, for each \((p_1, \ldots, p_s) \in P_1\), add \((y, p_1, \ldots, p_s)\) to \( P \);
(d) Replace \( y \) by a greater element of \( A \cap [r_l, 1] \):
   (i) If \( y \) is a successor element, replace \( y \) by \( p_{S}(y) \), using the given system of notations for \( A \cap [r_l, 1] \);
   (ii) If \( y \) is a limit element, replace \( y \) by \( y' \) where \( y' \) is the smallest element of \( A \cap [r_l, 1] \) such that
   \[
   \frac{x}{y'} + 1 \leq d - \text{findsmallest}(P_1, x, d').
   \]
(5) Return \( P \).

**Proof of correctness for \( \text{xdmin} \).** We prove the correctness by induction over \( \floor{\frac{d}{\Delta}} \).

**Base Case.** \( d \in [0, \Delta] \).

Then, \( \frac{x}{y} + 1 \geq \Delta \) for any \( y \). Hence, \( \sum_{j=1}^{i} \left( \frac{x}{p_j} + 1 \right) \geq \Delta \) for any \((p_1, \ldots, p_s)\) and there are no \((x, d)\)-allowed tuples. In this case, the algorithm returns the empty set. Hence, it works correctly.

**Inductive Case.** We assume that the lemma holds for \( d \in [0, k \Delta] \) and prove it for \( d \in [k \Delta, (k + 1) \Delta] \). We use

**Claim 47.** If \( \text{xdmin}(x, d) \) calls \( \text{xdmin}(x, d') \), then \( d' \leq d - \Delta \)

**Proof.** From the description of \( \text{xdmin} \) we have \( d' = d - \left( \frac{x}{y} + 1 \right) \). By definition of \( p_0 \) and \( \Delta \), \( \frac{x}{y} + 1 \geq \Delta \) and \( d' \leq d - \Delta \). \( \square \)

Hence, \( \text{xdmin}(x, d) \) calls only \( \text{xdmin}(x, d') \) with \( d' < (k + 1) \Delta - \Delta = k \Delta \). The correctness \( \text{xdmin}(x, d') \) for such values follows from the inductive assumption.

First, we prove that the computation of \( \text{xdmin}(x, d) \) always terminates. Each \( \text{xdmin}(x, d') \) called by \( \text{xdmin}(x, d) \) terminates because of the correctness of \( \text{xdmin}(x, d') \), by the inductive assumption. Hence, each while loop terminates and, if \( \text{xdmin}(x, d) \) does not stop then while loop is executed infinitely many times.

Let \( y_j \) be the value of \( y \) during the \( j \)th execution of while loop. \( y \) is increased at the end of each while loop. Hence, \( y_1 < y_2 < \cdots \)

\( y_1 \in A \cap [r_l, 1], y_2 \in A \cap [r_l, 1], \ldots \). If while loop is executed infinitely many times, then \( y_1, y_2 \ldots \) is an infinite monotonically increasing sequence. However, \( A \cap [r_l, 1] \) does not contain such sequences because it is well-ordered.

Hence, while loop is executed finitely many times and \( \text{xdmin}(x, d) \) terminates. Let \( P = \text{xdmin}(x, d) \). Next, we prove that \( P \) is a \((x, d)\)-minimal set.

For a contradiction, assume that it is not. Then, there exists an \((x, d)\)-allowed tuple \((p_1, \ldots, p_s)\) such that \( P \) does not contain any tuple that is less than or equal to \((p_1, \ldots, p_s)\).

We assume that \((p'_1, p_2, \ldots, p_s)\) is not \((x, d)\)-allowed for any \( p'_1 \in A \cap [r_l, 1] \) satisfying \( p'_1 < p_1 \). (Otherwise, we can replace \( p_1 \) by the smallest \( p'_1 \in A \cap [r_l, 1] \) such that \((p'_1, p_2, \ldots, p_s)\) is \((x, d)\)-allowed.)

Consider two cases:

(1) In \( \text{xdmin}(x, d) \), while loop is executed with \( y = p_1 \).
Denote \( d'' = d - \left( \frac{x}{p_1} + 1 \right) \). The tuple \((p_2, \ldots, p_s)\) is \((x, d'')\)-allowed.
\( \text{xdmin}(x, d) \) calls \( \text{xdmin}(x, d'') \) which \( \text{xdmin}(x, d') \) works correctly, i.e. returns an \((x, d'')\)-minimal set \( P_1 \). Hence, \( P_1 \) contains a tuple \((p'_2, \ldots, p'_s)\) that is less than or equal to \((p_2, \ldots, p_s)\).
\( \text{xdmin}(x, d) \) adds \((p_1, p'_2, \ldots, p'_s)\) to \( P \) because \((p'_2, \ldots, p'_s)\) belongs to the set returned by \( \text{xdmin}(x, d') \). Hence, \( P \) contains the tuple \((p_1, p'_2, \ldots, p'_s)\) that is less than or equal to \((p_1, p_2, \ldots, p_s)\). A contradiction.

(2) While loop is not executed with \( y = p_1 \).
Let \( y_1 \) be the greatest number such that \( y_1 < p_1 \) and while loop is executed with \( y = y_1 \). Let \( y_2 \) be the number by which \( y_2 \) is replaced in the end of while loop.
\( y_1 < y_2 \) because \( y \) is always replaced by a greater number. By definition of \( y_1, y_2 > p_1 \). (Otherwise \( y_2 \) would have been instead of \( y_1 \).)
(a) $y_1$ is a successor element.

In this case, $y_1$, $y_2$, $p_1$ all belong to $A$ and $y_1 < p_1 < y_2$. When $x_{\text{minimal}}(x, d)$ replaces $y_1$ by a greater element of $A$, it chooses the smallest element of $A$ that is greater than $y_1$. It can be $p_1$ or some number between $y_1$ and $p_1$ but not $y_2$. A contradiction.

(b) $y_1$ is a limit element.

We assumed that $\langle p_1', p_2, \ldots, p_s \rangle$ is not $(x, d)$-allowed for any $p_1' \in A \cap [r_i, 1]$ satisfying $p_1' < p_1$. Hence, $\langle y_1, p_2, \ldots, p_s \rangle$ is not $(x, d)$-allowed i.e.

$$\left(\frac{x}{y_1} + x - 1\right) + \sum_{j=2}^{s} \left(\frac{x}{p_j} + x - 1\right) > d,$$

$$\sum_{j=2}^{s} \left(\frac{x}{p_j} + x - 1\right) > d - \left(\frac{x}{y_1} + x - 1\right) = d'.$$

Hence,

$$\sum_{j=2}^{s} \left(\frac{x}{p_j} + x - 1\right) \geq \text{findsmallest}(P_1, x, d')$$

where $P_1$ is the $(x, d')$-minimal set obtained by $x_{\text{minimal}}(x, d')$. However,

$$\sum_{j=1}^{s} \left(\frac{x}{p_j} + x - 1\right) \leq d$$

because $\langle p_1, p_2, \ldots, p_s \rangle$ is $(x, d)$-allowed. Hence,

$$\frac{x}{p_1} + x - 1 \leq d - \sum_{j=2}^{s} \left(\frac{x}{p_j} + x - 1\right) \leq d - \text{findsmallest}(P_1, x, d').$$

By the definition, $y_2$ is the smallest number such that

$$\frac{x}{y_2} + x - 1 \leq d - \text{findsmallest}(P_1, x, d').$$

This implies $y_2 \leq p_1$. A contradiction with $y_2 > p_1$. □

3.5.5. System of notations

We show how to extend a system of notations $S$ from $A \cap \left[\frac{1}{n}, 1\right]$ to $A \cap \left[\frac{1}{n+1}, 1\right]$. Below, we give the algorithms computing $k_S(x)$, $p_S(x)$ and $q_S(x)$ for $x \in \left[\frac{1}{n+1}, \frac{1}{n}\right]$. These algorithms use the procedure $x_{\text{minimal}}(x, d)$ defined in the previous subsection. They also use the system $S$ for $A \cap \left[\frac{1}{n}, 1\right]$.

Function $k_S(x)$.

1. Use the system for $A \cap \left[\frac{1}{n}, \frac{1}{n+1}\right]$ to find whether $x = \frac{p}{1+q}$ for some $p \in A \cap \left[\frac{1}{n}, \frac{1}{n+1}\right]$. If yes, then $k_S(x) = 2$.
2. Otherwise, find the segments $\left[\frac{p}{1+q}, \frac{r_i}{1+q}\right]$ and $\left[r_{i+1}, r_i\right]$ containing $x$. If $x = r_{i+1}$ or $x = r_i$, then $k_S(x) = 2$.
3. Otherwise, find an $(x, x)$-minimal set $P$ using $x_{\text{minimal}}(x, x)$.
4. If there exists $\langle p_1, \ldots, p_s \rangle \in P$ such that $x$ is generated by an application of the rule (2) to $p_1, \ldots, p_s$ and at least one of $p_1, \ldots, p_s$ is a limit element, then $k_S(x) = 2$.
5. Otherwise, if there exists $\langle p_1, \ldots, p_s \rangle \in P$ such that $x$ is generated by an application of the rule (2) to $p_1, \ldots, p_s$, then $k_S(x) = 1$.
6. Otherwise, $k_S(x) = 3$.

Function $p_S(x)$.

1. Find the interval $[r_{i+1}, r_i]$ containing $x$. Execute $x_{\text{minimal}}(x, x)$ and find a $(x, x)$-minimal set.
Lemma 48. Let $P_1$ be the set consisting of all tuples $\langle p_1, \ldots, p_s \rangle$ such that

(a) $\langle p_1, \ldots, p_s \rangle \in P$ or
(b) $\langle p_1, \ldots, p_{j-1}, p_j', p_{j+1}, \ldots, p_s \rangle \in P$ and $p_j = qS(p_j')$ for some $j \in \{1, \ldots, s\}$ or
(c) $\langle p_1, \ldots, p_{j-1}, p', p_{j+1}, \ldots, p_s \rangle \in P$ for some $j \in \{1, \ldots, s\}$ and $p' \in A \cap [r_i, 1]$.

(3) For each tuple $\langle p_1, \ldots, p_s \rangle \in P_1$ find the number $p \in A$ generated by an application of the rule (2) to $p_1, \ldots, p_s$. $pS(x)$ is the smallest of those $p$ which are greater than $x$.

Function $qS(x)$.

(1) If $x = \frac{p}{1+p}$, $p \in A \cap \left[\frac{1}{n}, \frac{1}{n-1}\right]$ and $p$ is a limit element, $qS(x)$ is a program computing $\frac{\varphi_{qS(p)}(0)}{1+\varphi_{qS(p)}(0)}$, $\frac{\varphi_{qS(p)}(1)}{1+\varphi_{qS(p)}(1)}$, $\ldots$.

(2) If $x = \frac{p}{1+p}$, $p \in A \cap \left[\frac{1}{n}, \frac{1}{n-1}\right]$ and $p$ is a successor element, find $r = pS(p)$. $qS(x)$ is a program computing the sequence $r_0, r_1, \ldots$ corresponding to $\left[\frac{p}{1+p}, \frac{r}{1+r}\right]$.

(3) Otherwise, search the set $P$ returned by $x_{\text{minimal}}(x, x)$ and find $p_1 \in A \cap [r_i, 1], \ldots, p_s \in A \cap [r_i, 1]$ such that $x$ is generated by an application of the rule (2) to $p_1, \ldots, p_s$ and $p_j$ is a limit element.

Lemma 48. $S$ is a system of notations for $A \cap \left[\frac{1}{n+1}, 1\right]$.

Proof. By transfinite induction over $A_n$.

**Base Case.** $S$ is a correct system of notations for $A \cap \left[\frac{1}{n}, 1\right]$.

**Inductive Case.** Let $y \in A_n$. We assume that $S$ is correct for all $A \cap [y', 1]$ with $y' < y$ and prove that it is correct for $A \cap [y, 1]$. We consider two cases:

(1) $y = \frac{p}{1+p}$ and $p \in A \cap \left[\frac{1}{n}, \frac{1}{n-1}\right]$.

Similarly to the proof of Lemma 39, $y$ is a limit of a sequence consisting of elements of $A_n$. Hence, if $x > y$, then $x > y'$ where $y'$ is some element of this sequence. The functions $kS(x)$, $pS(x)$, $qS(x)$ are correct because $S$ is correct for $A \cap [y', 1]$ (by inductive assumption). It remains to prove the correctness of $kS(x)$, $pS(x)$, $qS(x)$ for $x = y$.

$kS(y) = 2$. This is correct because, by Lemma 39, $y$ is a limit element. The function $pS(x)$ is defined only for successor elements. Hence, we do not need to check its correctness for the limit element $y$. The correctness of the sequence computed by $qS(y)$ is proved in the proof of Lemma 39.

(2) $y = r_{i+1}$ for $i \geq 0$. In this case, we assume that $S$ is correct for $A \cap [r_i, 1]$ and prove the correctness for $A \cap [r_{i+1}, r_i]$.

By Lemma 46, $x_{\text{minimal}}(x, d)$ returns an $(x, d)$-minimal set if it has access to a system of notations for $A \cap [r_i, 1]$. We know that $S$ is correct for $A \cap [r_i, 1]$. Hence, the set $P$ returned by $x_{\text{minimal}}(x, x)$ is $(x, x)$-minimal.

2.1. Proof of correctness for $kS$.

If $x \in A \cap [r_{i+1}, r_i]$ then $x \in A$ follows from $p_1 \in A$, $\ldots$, $p_s \in A$ and the rule (2), for some $p_1, \ldots, p_s$. By Lemma 32, $p_1 \in A \cap [r_i, 1], \ldots, p_s \in A \cap [r_i, 1]$. By Lemma 45, $\langle p_1, \ldots, p_s \rangle$ belongs to $P$.

Correctness of $x_{\text{minimal}}(x, x)$ implies that, if $x \in A$, the algorithm computing $kS$ finds $p_1, \ldots, p_s$ such that $p \in A$ follows from $p_1 \in A, \ldots, p_s \in A$ and the rule (2).

Hence, it distinguishes $x \in A$ and $x \notin A$ correctly. By Lemma 38, it distinguishes limit and successor elements correctly.

2.2. Proof of correctness for $pS$.

We prove that $pS(x)$ returns the element of $A \cap [r_{i+1}, r_i]$ immediately preceding $x$, i.e. $(\forall z \in A \cap [r_{i+1}, r_i]) (x < z \Rightarrow pS(x) \leq z)$.

Let $z \in A \cap [r_{i+1}, r_i]$ and $x < z$. Consider $p_1, \ldots, p_s$ that generate $z \in A$ by rule (2).

$P$ contains a tuple $\langle p'_1, \ldots, p'_s \rangle$ such that $p'_1 \leq p_1, \ldots, p'_s \leq p_s$ (Lemma 44). An application of the rule (2) to $p'_1, \ldots, p'_s$ generates $p \in A$ with $p \geq x$ (Lemma 43). Consider two cases:
(a) \( p > x \).

The algorithm computing \( p_S \) adds \( \langle p'_1, \ldots, p'_s \rangle \) to the set \( P_1 \). Later, it sets \( p_S(x) \) equal to a number that is less or equal to \( p \). (This is true because \( \langle p'_1, \ldots, p'_s \rangle \in P_1 \) and \( p'_1, \ldots, p'_s \) generates \( p > x \). The algorithm selects \( p_S(x) \) as the smallest of all \( p \) satisfying these conditions.) By Lemma 13, \( p \leq z \). Hence, \( p_S(x) \leq p \leq z \).

(b) \( p = x \).

If \( p_1 = p'_1, \ldots, p_s = p'_s \) then \( p = z \). However, \( p < z \). Hence, \( p_j < p'_j \) for some \( i \). Let \( p''_j = p_S(p'_j) \).

We have \( p''_j \leq p_j \) because \( p_S(p'_j) \) is the smallest element of \( A \) that is greater than \( p'_j \). Let \( p \) denote the number generated by the rule (2) from \( p'_1, \ldots, p'_{j-1}, p''_j, p'_{j+1}, \ldots, p'_s \).

By Lemma 13, \( x < p \). Hence, the algorithm for \( p_S(x) \) adds \( \langle p'_1, \ldots, p'_j, p''_j, p'_{j+1}, \ldots, p'_s \rangle \) to the set \( P_1 \) and, then, checking tuples in \( P_1 \), sets \( p_S(x) \) equal to a number which is greater than or equal to \( p \).

This implies \( p_S(x) \leq p \).

From \( p'_1 \leq p_1, \ldots, p'_{j-1} \leq p_{j-1}, p''_j \leq p_j, p'_{j+1} \leq p_{j+1}, \ldots, p'_s \leq p_s \) it follows that \( p \leq z \) (Lemma 13). Hence, \( p_S(x) \leq p \leq z \).

So, in both cases \( p_S(x) \) is less than or equal to any \( z \in A \) satisfying \( x < z \). On the other hand, \( p_S(x) \in A \) and \( x < p_S(x) \). (It can be seen from the algorithm computing \( p_S \).)

Hence, \( p_S(x) \) is the smallest element of \( A \) satisfying \( x < p_S(x) \), i.e. the algorithm computes \( p_S \) correctly.

2.3. Proof of correctness for \( q_S \).

We already proved that, if there exist \( p_1, \ldots, p_s \) such that \( x \in A \) follows from \( p_1 \in A, \ldots, p_s \in A \), then such combination is found by \( \text{admind}(x, x) \) (see proof of correctness for \( k_S \)). If there exists such a combination with one of \( p_1, \ldots, p_s \) being limit element, it is found. The algorithm computing \( q_S \) generates a program computing required sequence from such combination correctly.

The correctness of \( S \) for \( A \cap [\frac{1}{n}, 1) \) follows by transfinite induction. \( \square \)

By Lemmas 35 and 48, \( A \cap [\frac{1}{n}, 1) \) is well-ordered and has a system of notations for any \( n \). Hence, \( A \) is well-ordered and has a system of notations. This completes the proof of Theorem 30. \( \square \)

3.6. Universal simulation

**Theorem 49.** For any \( p \in A \) there exists \( k \) such that \( \text{PFIN}(x) \subseteq [pk, k] \text{PFIN} \) for all \( x \) which are greater than any \( p' \in A \cap [0, p] \). There exists an algorithm which receives a probabilistic machine \( M \) and a probability \( x \) and outputs a team \( L_1, \ldots, L_k \) which identifies the same set of functions.

**Proof.** By transfinite induction.

**Base Case.** For \( p > \frac{1}{2} \), the theorem follows from the results of [12].

**Inductive Case.** We assume that the theorem is true for all \( p \in A \) such that \( p > p_0 \) and prove it for \( p = p_0 \).

Let \( p'_0 \) be the largest element of \( A \) for which \( \frac{p_0}{p'_0} + x - 1 > 0 \). \( p'_0 \) is always a successor element. (If it was a limit element, let \( q_1, q_2, \ldots \) be a decreasing sequence that converges to \( p'_0 \). For some element \( q_i \) in this sequence, \( \frac{p_0}{q_i} + x - 1 > 0 \), implying that \( p'_0 \) is not the largest element with this property.) Let \( p''_0 \) be the predecessor of \( p'_0 \).

Let \( P \) be a \((x, x)\)-minimal set (see Section 3.5.4). Let \( P' \) be the set of all \( p \) that appear in some tuple in the set \( P \).

We define two functions \( g(r) \) and \( g'(r) \), for \( r \in [0, x] \). To define \( g(r) \), let \( y \) be the smallest element of \( A \) which is at least \( \frac{x}{1 + x - y} \). If \( y = p''_0 \), we define \( g(r) = 0 \). Otherwise, let \( y' \) be the largest element of \( P' \) satisfying \( y' \leq y \). Let \( g(r) \) be the solution to \( y' = \frac{p_0}{1 - p''_0 + g(r)} \). (Equivalently, \( g(r) = p_0 + \frac{p_0}{1 - p''_0 + g(r)} - 1 \).) To define \( g'(r) \), let \( S(r) \) be the set of all tuples \( \langle r_1, r_2, \ldots, r_m \rangle \) such that \( r_1 + \cdots + r_m \leq r, r_1 > 0, \ldots, r_m > 0 \). Then,

\[
g'(r) = \sup_{\langle r_1, r_2, \ldots, r_m \rangle \in S(r)} g(r_1) + g(r_2) + \cdots + g(r_m).
\]

In the simulation algorithm for \( p = p_0 \), we use several simulation algorithms for \( p > p_0 \) as subroutines. Namely, we use:
(1) A simulation algorithm for \( p = p_0^* \).
(2) Simulation algorithms for all \( p \in P' \).

The existence of these simulation algorithms is implied by the assumption that Theorem 49 holds for \( p > p_0 \).

A \([ pk, k]PFIN\)-team \( L = \{ L_1, \ldots, L_k \} \) simulates a probabilistic PFIN\((x)\)-machine \( M \) as follows:

(1) \( L_1, \ldots, L_k \) read \( f(0), f(1), \ldots, \) simulate \( M \) and wait until the probability that \( M \) has issued a conjecture reaches \( x \). Then \( pk \) machines \( \{ L_1, \ldots, L_{pk} \} \) issue conjectures \( h_1, \ldots, h_{pk} \).

(2) The first values of the functions computed by \( h_1, \ldots, h_{pk} \) are identical to the values of \( f \), i.e.

\[
\varphi_{h_1}(i) = \cdots = \varphi_{h_{pk}}(i) = f(i)
\]

for \( i \leq m \) where \( f(m) \) is the last value of \( f \) read by \( L \) before issuing conjectures. The next values of these functions are computed as follows:

Let \( n = m + 1 \). Let \( T = \{ (f(0), f(1), \ldots, f(m)) \} \). We repeat the following sequence of operations. For each segment \( \rho = (f(0), \ldots, f(n - 1)) \) in \( T \):

(a) Find all conjectures of \( M \) (among ones issued until the probability reached \( x \)) that output \( f(0), \ldots, f(n - 1) \). Run each of those conjectures on input \( n \). Let \( d_1, \ldots, d_i \) be the values that are output by at least one of conjectures. For \( i \in \{ 1, \ldots, s \} \), let \( r_i \) be the total probability of \( M \)'s conjectures outputting \( f(n) = d_i \). The programs \( h_1, \ldots, h_{pk} \) output the next value as follows. Out of those programs, which have output the segment \( (f(0), \ldots, f(n - 1)) \), \( g'(r_1)k \) output \( f(n) = d_1 \), \( g'(r_2)k \) output \( f(n) = d_2 \) and so on. If the number of programs that have output the segment \( (f(0), \ldots, f(n - 1)) \) is larger than \( \sum g'(r_i)k \), the remaining programs are not necessary for the further steps. Make them output \( f(n) = f(n + 1) = \cdots = 0 \), to ensure that every program computes a total function.

(b) The programs which have output \( f(n) = d_i \) then simulate the machine \( M \) on input \( f(0), \ldots, f(n) \). If the total probability of \( M \) giving a conjecture consistent with \( f(0), \ldots, f(n) \) reaches \( x \), invoke the simulation algorithm for simulating a probabilistic machine with the success probability \( p = \frac{x}{1-x+r_i} \) by a team of \( (1 - p_0 + g(r_i))k \) machines, \( p_0k \) of which have to be successful. Let \( g(r_i)k \) of \( g'(r_i)k \) programs which have output \( f(0), \ldots, f(n) \) simulate the first \( g(r_i)k \) machines in this simulation. If \( g'(r_i)k > g(r_i)k \), make the remaining \( g'(r_i)k - g(r_i)k \) programs output \( f(n + 1) = f(n + 2) = \cdots = 0 \).

(c) Otherwise (if the probability of \( M \) giving a conjecture consistent with \( f(0), \ldots, f(n) \) does not reach \( x \)), add \( (f(0), \ldots, f(n)) \) to the set \( T \).

After steps (2a)–(2c) have been done for every \( (f(0), \ldots, f(n - 1)) \in T \), increase \( n \) by 1 and repeat.

(3) After \( L_1, \ldots, L_{pk} \) have issued conjectures, all remaining machines in the team \( L \) read the next values of the input function and simulate the conjectures issued by the probabilistic machine \( M \) before conjectures of \( L_1, \ldots, L_{pk} \). They wait until the step (2b) happens, for a segment \( f(0), \ldots, f(n) \) consistent with the input. Then, \( L_{pk+1}, \ldots, L_k \) (i.e. all machines which have not issued conjectures yet) participate in one of two simulations:

(a) If \( g(r_i) > 0 \), they, together with \( g(r_i)k \) of programs \( h_1, \ldots, h_{pk} \), form an \([ p_0k, (1 - p_0 + g(r_i))k ] \) team. This team simulates a probabilistic machine \( M' \) according to the algorithm for \( p = \frac{p_0}{1-p_0+g(r_i)} \). (Note that, by the definition of \( g, p \in P' \).)

The machine \( M' \) is defined as follows. It reads \( f(0), \ldots, f(n) \) and then simulates \( M \). If, while reading \( f(0), \ldots, f(n), M \) outputs a conjecture inconsistent with the segment \( f(0), \ldots, f(n), M' \) restarts the simulation of \( M \). If \( M \) outputs a conjecture consistent with \( f(0), \ldots, f(n), M' \) outputs this conjecture as well. If \( M \) outputs no conjecture while reading \( f(0), \ldots, f(n), M' \) proceeds to read the next values of \( f \) and keeps simulating \( M \).

(b) If \( g(r_i) = 0 \), they form an \([ p_0k, (1 - p_0)k ] \) team and simulate the probabilistic machine \( M' \), defined as above, according to the algorithm for \( p = p_0^* \).

**Proof of correctness.** We need to show two statements.

- When we use an \([ p_0k, (1 - p_0 + g(r_i))k ] \) or an \([ p_0k, (1 - p_0)k ] \) team to simulate a probabilistic machine, the team is able to perform the simulation.
• For a segment \( \langle f(0), f(1), \ldots, f(n-1) \rangle \in T \), the sum of numbers \( g'(r_i) \) of programs \( h_1, \ldots, h_{pk} \) asked to output various extensions \( \langle f(0), f(1), \ldots, f(n-1), f(n) \rangle \) of this segment is never more than the number of programs which have output the segment \( \langle f(0), f(1), \ldots, f(n-1) \rangle \).

We start with the first statement. We consider two cases.

(1) Case 3a.
Here, we use \( g(r_i)k \) programs and \( (1 - p_0)k \) machines \( L_{p_0k+1}, \ldots, L_k \) to simulate \( M \) on functions with the given \( f(0), \ldots, f(n) \) according to the algorithm for \( p = p'_0 \).

The success probability for \( M' \) is at least \( \frac{x}{1-x+r_i} \), since \( M \) succeeds with probability at least \( x \) and the probability that \( M \) outputs a conjecture inconsistent with \( f(0), \ldots, f(n) \) is \( x - r_i \). Let \( y \) and \( y' \) be as in the definition of \( g(r_i) \). Then, \( \frac{x}{1-x+r_i} \) is greater than any \( p' \in A \cap [0, y] \). Since \( y > y' \), \( \frac{x}{1-x+r_i} \) is greater than any \( p' \in A \cap [0, y'] \).

Therefore, by inductive assumption, it can be simulated by an \([y'k', k']\) team, for some \( k' \). Since \( y' = \frac{p_0}{1-p_0 + g(r_i)} \), a \([p_0k, (1 - p_0 + g(r_i)k]\) team can do this task, as long as \( k \) is appropriately chosen.

(2) Case 3b.
Here, we use \((1 - p_0)k\) machines \( L_{p_0k+1}, \ldots, L_k \) to simulate \( M \) on functions with the given \( f(0), \ldots, f(n) \) according to the algorithm for \( p = p'_0 \).

The success probability for \( M' \) is at least \( \frac{x}{1-x+r_i} \), by the same argument as before. Since \( g(r_i) = 0 \), this is more than any \( p' \in A \cap [0, p'_0] \). By inductive assumption, this means \( M' \) can be simulated by an \([p''_0k', k']\) team, for some \( k' \). We will choose \( k \) so that \( k' = (1 - p_0)k \). Then, \( M' \) can be simulated by an \([p''_0(1 - p_0)k, (1 - p_0)k]\) team. It remains to prove that this simulation yields at least \( pk \) correct programs. This is equivalent to \( p''_0(1 - p_0)k \geq pk \).

If \( p''_0 < \frac{p_0}{1-p_0} \), then \( p''_0(1 - p_0)k \) would belong to the interval \([x, p_0]\), contradicting the assumption that this interval does not contain any elements of \( A \). Therefore, \( p''_0(1 - p_0) \geq p_0 \) and \( p''_0(1 - p_0)k \geq pk \).

Next, we show that the programs output by \( L_1, \ldots, L_{pk} \) are sufficient to conduct the necessary simulations. Let \( \langle f(0), \ldots, f(n-1) \rangle \in T \) be an initial segment, output by \( M \) with probability \( r \) and let \( r_1, \ldots, r_s \) be the probabilities of its possible extensions \( \langle f(0), \ldots, f(n) \rangle \). Then, the number of programs \( h_1, \ldots, h_{pk} \) outputting the segment \( \langle f(0), \ldots, f(n-1) \rangle \) is \( pk \) if \( n = m + 1 \) and \( g'(r)k \) if \( n > m + 1 \). The number of programs outputting its extensions \( \langle f(0), \ldots, f(n) \rangle \) is \( g'(r_1)k, \ldots, g'(r_s)k \).

For the \( n > m + 1 \) case, it suffices to show that \( \sum_{i=1}^{t} g'(r_i) \leq g'(r) \), whenever \( \sum_{i=1}^{t} r_i \leq r \). The \( n = m + 1 \) case follows from the \( n > m + 1 \) case, once we prove \( g'(x) \leq p_0 \). We now proceed to show those two results.

**Lemma 50.** If \( \sum_{i=1}^{t} r_i \leq r \), then \( \sum_{i=1}^{t} g'(r_i) \leq g'(r) \).

**Proof.** Immediate from the definition of \( g' \). \( \square \)

**Lemma 51.** \( g'(x) \leq p_0 \).

**Proof.** We need to prove that \( g(r_1) + \cdots + g(r_m) \leq p_0 \), whenever \( r_1 + \cdots + r_m \leq x \) and \( r_1, \ldots, r_m \geq 0 \). Let \( y'_i \) be the value of \( y' \) in the calculation of \( g(r_i) \).

We claim that the tuple \( \langle y'_1, y'_2, \ldots, y'_m \rangle \) is \((x, x)\)-allowed. To prove that, we need to show \( \sum_{i=1}^{m} x + \frac{x}{y'_i} - 1 \leq x \).

This is true because, \( y'_i \geq \frac{x}{1-x+r_i} \) and, therefore

\[
x + \frac{x}{y'_i} - 1 \leq x + \frac{x}{x/(1-x+r_i)} - 1 = r_i,
\]

implying \( \sum_{i=1}^{m} x + \frac{x}{y'_i} - 1 \leq \sum_{i=1}^{m} r_i \leq x \). Since the tuple is \((x, x)\) allowed, applying rule (2) to it generates \( p \geq x \). Since \( p_0 \) is the smallest element of \( A \) satisfying \( p_0 \geq x \), this also means \( p \geq p_0 \).
Consider the application of rule (2) to $y'_1, y'_2, \ldots, y'_m$. Consider the values of $q_1, \ldots, q_m$ in this application. We have 

$$
\frac{p}{q_i + 1 - \frac{p}{y'_i} - 1} = y'_i
$$

which is equivalent to 

$$
q_i = p + \frac{p}{y'_i} - 1 = (p - p_0) + g(r_i).
$$

Summing over all $i$ gives 

$$
\sum_{i=1}^{m} q_i = p_0 - m(p - p_0) = m(p_0 - p) = p_0.
$$

This proves the lemma. □

The size of $L$. We show how to select the size of the team $L$ so that be it will able to perform all described simulations. Two conditions must be satisfied:

1. When the machines of the team split, the amount of machines saying that $f(m) = d_i$ must be integer for any $d_i$, i.e., $g(r)k$ must be integer in all cases.
2. When the simulation algorithm for the success ratio $p_0$ uses another simulation algorithm (with the ratio of successful machines $p' > p_0$), a certain team size $k'$ is required for simulation with $[p'k', k']$PFIN-team. The amount of machines participating in this simulation (when it is used as the subroutine of the simulation for the ratio $p_0$) must be multiple of $k'$.

For the first condition, notice that $g'(r)k$ is, by definition, a sum of $g(r)k$ for smaller $r$. Therefore, it suffices to choose $k$ so that $g(r)k$ is an integer. By definition, $g(r) = p_0 + \frac{p_0}{y'_i} - 1$ where $y'$ is belongs to a finite set $P'$. Since $P' \subseteq A$ and $A$ is the subset of rational numbers (Section 3.4), this means that $k$ must be chosen so that $g(r)k$ is an integer for finitely many rationals $g(r)$. Each of those requirements is equivalent to requiring that the denominator of $g(r)$ divides $k$.

The second condition is equivalent to:

1. For all $p' \in P'$, the team size $(1 - p_0 + g(r))k = \frac{p_0}{p}k$ must be a multiple of $k_i$ where $k_i$ is the size of the team with the success ratio $p'$.
2. $(1 - p_0)k$ must be a multiple of $k_0$, the size of the simulation team with the success ratio $p'_0$.

Overall, we have finitely many requirements. Each of them requires that the team size is a multiple of some finite number of integers $k_1, \ldots, k_m$. If we select the size $k$ so, the simulation algorithm will be able to perform all required simulations. □

Theorem 49 implies

**Corollary 52.** Let $x, y \in [0, 1]$ and $x < y$. If there is no $p \in A$ satisfying $x \leq p < y$, then

$$
PFIN(x) = PFIN(y).
$$

**Proof.** Any machine which succeeds with probability $y$, succeeds with probability $x < y$, too. Hence, it suffices to prove that any machine with the probability of success $x$ can be simulated by a machine with the probability of success $y$, i.e.

$$
PFIN(x) \subseteq PFIN(y).
$$

Let $p$ be the smallest element of $A$ which is greater than $x$. Theorem 49 implies

$$
PFIN(x) \subseteq [pk, k]PFIN \subseteq PFIN(p).
$$

We have $y \leq p$ and, hence,
Lemma 56. 
\[ \text{PFIN}(p) \subseteq \text{PFIN}(y), \]
\[ \text{PFIN}(x) \subseteq \text{PFIN}(y). \]  
\[ \square \]

So, if Theorem 29 does not prove that the power of learning machines with probabilities \( x \) and \( y \) is different, then these probabilities are equivalent. Hence,

**Theorem 53.** \( A \) is the probability hierarchy for probabilistic PFIN-type learning in the range \([0, 1]\).

**Proof.** Follows from Theorem 29 and Corollary 52.  
\[ \square \]

Theorem 53 has a following important corollary.

**Theorem 54.** Probabilistic PFIN-type learning probability structure is decidable, i.e. there is an algorithm that receives as input two probabilities \( p_1 \) and \( p_2 \) and computes whether \( \text{PFIN}(p_1) = \text{PFIN}(p_2) \).

**Proof.** Use the algorithm of Lemma 9 to find the intervals \([f_1(p_1), f_2(p_1)]\) and \([f_1(p_2), f_2(p_2)]\). If these two intervals are equal, \( \text{PFIN}(p_1) = \text{PFIN}(p_2) \). Otherwise, \( \text{PFIN}(p_1) \neq \text{PFIN}(p_2) \).  
\[ \square \]

4. Relative complexity

From Theorem 30 we know that PFIN-type probability hierarchy is well-ordered. A question appears: what is the ordering type of this hierarchy? To what particular ordinal is it order-isomorphic? We analyze the proof of Theorem 30 step by step.

Let \( \alpha(x) \) denote the ordering type of \( A \cap \{ x \} \) for \( x \leq \frac{1}{2} \) and the ordering type of \( A \cap [x, 1) \) for \( x > \frac{1}{2} \). If \( x \geq y \), then \( \alpha(x) \leq \alpha(y) \) because \( A \cap \{ x \} \subseteq A \cap \{ y \} \). We will often use this inequality.

**Lemma 55.** \( \alpha(\frac{1}{2}) = \omega \).

**Proof.** \( A \cap \{ \frac{1}{2} \} \) consists of a single sequence \( 1,2/3,3/5,\ldots \) [12].  
\[ \square \]

First, we prove lower bounds on the ordering type of \( A \). \( l(p) \) is the largest ordinal \( \alpha \) such that there is an \( \omega^\alpha \)-sequence in \( A \cap \{ p \} \) which converges to \( p \). We define \( l(p) = 0 \) if there is no such sequence for any \( \alpha \).

It is easy to see that \( \alpha(p) \geq \omega(l(p)) \). However, there may be a large gap between these two ordinals. For example, if \( A \cap \{ p \} \) has the ordering type \( \omega^\omega + 1 \), there is no infinite monotonic sequence converging to \( p \) and \( l(p) = 0 \). We use the function \( l \) to prove lower bounds.

**Lemma 56.** \( l(\frac{p}{1+p}) \geq \alpha(p) \).

**Proof.** Transfinite induction over \( p \in A \).

**Base Case.** Let \( p = 1 \). The ordering type of \( A \cap \{ 1 \} = \{ 1 \} \) is 1. The ordering type of \( A \cap [1/2, 1] \) is \( \omega \) and \( l(1/2) = 1 \).

**Inductive Case.** Consider two cases:

1. \( p \) is a successor element.

Let \( p \in [\frac{1}{n+1}, \frac{1}{n-1}] \). Let \( r \) denote the element immediately preceding \( p \). We have \( \alpha(p) = \alpha(r) + 1 \) because \( p \) is the only element of \( A \cap \{ p \} \) which does not belong to \( A \cap [r, 1] \). By inductive assumption, \( l(\frac{r}{1+r}) \geq \alpha(r) \).

Consider the splitting of \( [\frac{1}{n+1}, \frac{1}{n-1}] \) in the proof of Theorem 30 (Section 3.5.1). In the first step, one of segments is \( [\frac{p}{1+p}, \frac{r}{1+r}] \) because \( [p, r] \) does not contain other elements of \( A \). We consider the sequence \( r_0, r_1, \ldots \) corresponding to \( [\frac{r}{1+r}, \frac{r}{1+r}] \).

**Claim 57.** \( l(r_i) \geq \alpha(r) \).
Proof. By induction.

Base Case. Let \( i = 0 \). Then, \( r_0 = \frac{r}{1+r} \) and \( l(r_0) = l\left(\frac{r}{1+r}\right) \geq \alpha(r) \).

Inductive Case. We prove \( l(r_{i+1}) \geq l(r_i) \). Then \( l(r_{i+1}) \geq \alpha(r) \) follows from \( l(r_i) \geq \alpha(r) \). We use

Claim 58. If a set is obtained from \( \omega^\alpha \) by removing a proper initial segment, it still has ordering type \( \omega^\alpha \).

Proof. If we remove a segment with ordering type \( \beta \), we obtain the set with ordering type \( \omega^\alpha - \beta \) (Definition 3). We have \( \omega^\alpha - \beta = \omega^\alpha \) for all \( \beta < \omega^\alpha \).

Let \( f(x) = \frac{2}{1 + \frac{1}{x} + \frac{1}{p}}. \)

\( f(x) \) maps \( x \in A \) to the number generated from \( x \) and \( p \) by rule (2) (Lemma 11). Let \( x_0 \) be such that \( f(x_0) = r_i \). The function \( f \) maps \( (x_0, r_i) \) to \( (r_i, r_{i+1}) \). Then \( l(r_i) \geq \alpha(r) \).

We take a \( \omega^{\alpha(r)} \) sequence converging to \( r_i \) and remove all \( x < x_0 \) from it. The ordering type of the remaining sequence is still \( \omega^{\alpha(r)} \) (Claim 58). \( f \) maps it to a sequence converging to \( r_{i+1} \) and preserves the ordering. Hence, \( l(r_{i+1}) \geq \alpha(r) \).

We take the union of \( \omega^{\alpha(r)} \) sequences converging to \( r_0, r_1, \ldots \) and obtain a \( \omega^{\alpha(r)+1} \) sequence converging to

\[ \lim_{i \to \infty} r_i = \frac{p}{1+p}. \]

Hence, \( l\left(\frac{p}{1+p}\right) \geq \alpha(r) + 1 = \alpha(p) \).

(2) \( p \) is a limit element.

Let \( p_0, p_1, \ldots \) be a decreasing sequence converging to \( p \). Then, \( \alpha(p) = \lim_{i \to \infty} \alpha(p_i) \).

We take the union of \( \omega^{\alpha(p_i)} \) sequences converging to \( \frac{p_i}{1+p_i} \). It has the ordering type

\[ \lim_{i \to \infty} \alpha^{\alpha(p_i)} = \omega^{\lim_{i \to \infty} \alpha(p_i)} = \omega^{\alpha(p)} \]

and converges to \( \frac{p}{1+p} \). Hence, \( l\left(\frac{p}{1+p}\right) \geq \alpha(p) \).

Lemma 59. \( \alpha\left(\frac{p}{1+p}\right) \geq \omega^{\alpha(p)} \).

Proof. Follows from Lemma 56 and \( \alpha\left(\frac{p}{1+p}\right) \geq \omega^{l\left(\frac{p}{1+p}\right)} \).

The upper bound proof is more complicated. We prove a counterpart of Lemma 59.

Lemma 60. \( \alpha\left(\frac{p}{1+p}\right) \leq \omega^{\alpha(p)} \).

Proof. Transfinite induction over \( p \in A \).

Base Case. Let \( p = 1 \). The ordering type of \( A \cap [1, 1] \) is 1 and the ordering type of \( A \cap [1/2, 1] \) is \( \omega \).

Inductive Case. Consider two cases:

1. \( p \) is a successor element.
2. \( p \) is a limit element.

Let \( r \) be the element immediately preceding \( p \). We split the interval \( \left[\frac{p}{1+p}, \frac{r}{1+r}\right] \) into subintervals \( [r_1, r_0], [r_2, r_1], \ldots \), as in Section 3.5.1.

Claim 61. \( \alpha(r_j) \leq c_i \omega^{\alpha(r)} \) for some \( c_i \in \mathbb{N} \).

Proof. By induction.

Base Case. If \( i = 0 \), \( r_0 = \frac{r}{1+r} \) and \( \alpha\left(\frac{r}{1+r}\right) \leq \omega^{\alpha(r)} \) by inductive assumption.
Inductive Case. Let $P$ be a $(r_{i+1}, r_{i+1})$-minimal set (Section 3.5.4). Let $A(p_1, \ldots, p_s)$ denote the set of all $x \in A \cap [r_{i+1}, r_i]$ generated by applications of the rule (2) to $p'_1 \in A, \ldots, p'_s \in A$ such that $p_1 \leq p'_1, \ldots, p_s \leq p'_s$. Let $\alpha'(p_1, \ldots, p_s)$ denote the ordering type of $A(p_1, \ldots, p_s)$.

Claim 62. $\alpha(r_{i+1}) \leq \alpha(r_i)(+) \sum_{(p_1, \ldots, p_s) \in P} \alpha'(p_1, \ldots, p_s)$.

Proof. We have

$$A \cap [r_{i+1}, 1] = (A \cap [r_i, 1]) \cup \bigcup_{(p_1, \ldots, p_s) \in P} A(p_1, \ldots, p_s).$$

By Lemma 5, the ordering type of $A \cap [r_{i+1}, 1]$ is less than or equal to the natural sum of the ordering types of $A \cap [r_i, 1]$ and all $A(p_1, \ldots, p_s)$. □

Next, we bound each $\alpha'(p_1, \ldots, p_s)$. We start with an auxiliary lemma.

Claim 63. If $p \in A$ follows from an application of the rule (2) to $p_1 \in A, \ldots, p_s \in A$, then

$$\alpha(p) \geq \alpha(p_1)(+) \cdots (+) \alpha(p_s).$$

Proof. Without loss of generality, we assume that $p_1 \leq p_2 \leq \cdots \leq p_s$. Then, $\alpha(p_1) \geq \alpha(p_2) \geq \cdots \geq \alpha(p_s)$. We prove the lemma by transfinite induction over $p_1$.

Base Case. $p_1$ is the maximum element, i.e. $p_1 = 1$.

Then, $p_1 = \cdots = p_s = 1$. An application of the rule (2) to $p_1, \ldots, p_s$ generates $p = s/(2s - 1)$.

$$A \cap \left[ \frac{s}{2s - 1}, 1 \right] = \left\{ \frac{s}{2s - 1}, \frac{s - 1}{2s - 3}, \ldots, \frac{2}{3}, 1 \right\}.$$

The ordering type of this set is $s$, i.e. $\alpha(p) = s$. On the other hand, $\alpha(p_1) = \cdots = \alpha(p_s) = 1$ and $\alpha(p_1)(+) \cdots (+) \alpha(p_s) = s$.

Inductive Case. We have two possibilities:

(a) $p_1$ is a successor element.

$\lambda$ denotes the maximum number such that $p_1 = \cdots = p_j$. Let

$$p'_i = \begin{cases} \text{predecessor of } p_1, & \text{if } i \leq j, \\ p_i, & \text{if } i > j. \end{cases}$$

We have $\alpha(p_i) = \alpha(p'_i) + 1$ for $i \leq j$ and $\alpha(p_i) = \alpha(p'_i)$ for $i > j$. Hence,

$$\alpha(p_1)(+) \cdots (+) \alpha(p_s) = (\alpha(p'_1) + 1)(+) \cdots (+)(\alpha(p'_j) + 1)(+) \cdots (+) \alpha(p'_s) \alpha(p'_j) + j.$$

Let $x_0$ be the number generated by an application of the rule (2) to $p'_1, \ldots, p'_s$ and $x_i$, for $i \in \{1, \ldots, j\}$, be the number generated by an application of the rule (2) to $p_1, \ldots, p_i, p'_{i+1}, \ldots, p'_s$. By inductive assumption,

$$\alpha(x_0) \geq \alpha(p'_1)(+) \cdots (+) \alpha(p'_s).$$

We have $p_i < p'_i$ for $i \leq j$. By Lemma 13, $x_i < x_{i-1}$. Hence,

$$\alpha(x_i) \geq \alpha(x_{i-1}) + 1.$$

We have $p_i = p'_i$ for $i > j$. Hence, $x_j = p$ and

$$\alpha(p) = \alpha(x_j) \geq \alpha(x_0) + j \geq \alpha(p'_1)(+) \cdots (+) \alpha(p'_s) + j = \alpha(p_1)(+) \cdots (+) \alpha(p_s).$$
(b) \( p_1 \) is a limit element.

Again, \( j \) is the maximum number such that \( p_1 = \cdots = p_j \). Let \( p'_1, p'_2, \ldots \) be a monotonically decreasing sequence converging to \( p_1 \). Without loss of generality, we can assume that all elements of \( p'_1, p'_2, \ldots \) are less than or equal to \( p_{j+1} \). (Otherwise, just remove the elements that are larger than \( p_{j+1} \) and use the sequence consisting of remaining elements.)

Let \( x_i \) be the number generated by an application of the rule (2) to \( p'_1, \ldots, p'_j, p_{j+1}, \ldots, p_k \). By inductive assumption,

\[
\alpha(x_i) \geq \alpha(p'_1)(+)[j] \cdots (+)\alpha(p'_j)(+)[j+1] \cdots (+)\alpha(p_k).
\]

We have \( p_1 = \cdots = p_j = \lim_i x_i \). By Lemma 14, \( p = \lim_i x_i \). Hence, if we take \( i \to \infty \) in (3) and apply the fact that \(+\) is continuous, we get

\[
\alpha(p) \geq \alpha(p'_1)(+) \cdots (+)\alpha(p_k). \quad \square
\]

**Claim 64.** Let \( \langle p_1, \ldots, p_s \rangle \in P \). Then

\[
\alpha'(p_1, \ldots, p_s) \leq \text{const} \omega^{\alpha(r)}.
\]

**Proof.** Lemma 6 implies that \( \alpha'(p_1, \ldots, p_s) \) is at most the natural product of \( \alpha(p_1), \ldots, \alpha(p_s) \). Let \( \alpha_j \) be the largest ordinal such that \( \alpha(p_j) \geq \omega^{\alpha_j} \). Then, \( \alpha(p_j) \leq \text{const} \omega^{\alpha_j} \). (If there is no such \( \alpha_j \), then \( \alpha(p_j) \leq \lim_i c_i \omega^{\alpha_j} = \omega^{\alpha_j+1} \) and \( \alpha_j \) is not the largest ordinal with this property.) Hence,

\[
\alpha'(p_1, \ldots, p_s) \leq c_1 \omega^{\alpha_1} c_2 \omega^{\alpha_2} \cdots c_s \omega^{\alpha_s} = (c_1c_2 \cdots c_s) \omega^{\alpha_1(+)\alpha_2(+) \cdots (+)\alpha_s}.
\]

Let \( p'_j \) be such that \( p'_j \in A \) and \( \alpha(p'_j) = \alpha_j \). We have \( \alpha(p'_j) = \alpha_j \leq \alpha(r) \) because

\[
\omega^{\alpha_j} \leq \alpha(p_j) \leq \alpha(\{r\}) \leq \text{const} \omega^{\alpha(r)} < \omega^{\alpha(r+1)},
\]

where \( \alpha(p_j) \leq \alpha(\{r\}) \) follows from \( p_j \geq r \). \( \alpha(p'_j) = \alpha_j \leq \alpha(r) \) implies \( p'_j \geq r \). Therefore, both Lemmas 56 and 60 are true for \( p = p'_j \). This means that \( \alpha(p'_j) = \alpha(r) \). Hence, \( p'_j \geq p \) because \( \alpha(p_j) \geq \omega^{\alpha_1} \).

Let \( p' \) be the number generated by an application of the rule (2) to \( p'_1, \ldots, p'_s \). By Lemma 15, \( p'_1 + p'_2 + \cdots + p'_s \) is greater than or equal to the number generated by an application of rule (2) to \( p_1, \ldots, p_s \) because \( p'_j \geq p_j \). This number is at least \( r_{i+1} \) because the tuple \( \langle p_1, \ldots, p_s \rangle \) belongs to the \( (r_{i+1}, r_{i+2}) \)-allowed set \( P \). Hence, \( \frac{p'_1 + p'_2 + \cdots + p'_s}{p_1 + p_2 + \cdots + p_s} \geq r_{i+1} \). We have \( \frac{p'_1 + p'_2}{p_1 + p_2} \geq \frac{r_{i+1}}{r_{i+2}} \) because \( \frac{p'_1 + p'_2}{p_1 + p_2} \) does not contain any points of type \( \frac{p'_1 + p'_2}{p_1 + p_2} \) with \( p' \in A \). This implies \( p' \geq r \).

By Claim 63,

\[
\alpha(p') \geq \alpha(p_1)(+) \cdots (+)\alpha(p_s).
\]

This implies

\[
\alpha'(p_1, \ldots, p_s) \leq (c_1 \cdots c_s) \omega^{\alpha(p_1)(+) \cdots (+)\alpha(p_s)} \leq (c_1 \cdots c_s) \omega^{\alpha(p')} \leq (c_1 \cdots c_s) \omega^{\alpha(r)} \quad \square
\]

Now, we are ready to finish the proof of Claim 61. By Claim 62, \( \alpha(r_{i+1}) \) is less than or equal to the natural sum of \( \alpha(\{r\}) \) and \( \alpha'(p_1, \ldots, p_s) \). We have \( \alpha(\{r\}) \leq \text{const} \omega^{\alpha(r)} \) by inductive assumption and

\[
\alpha'(p_1, \ldots, p_s) \leq \text{const} \omega^{\alpha(r)}
\]

by Claim 64. Hence, the natural sum of these ordinals is at most \( \text{const} \omega^{\alpha(r)} \), too. \( \square \)

\[
\alpha\left(\frac{p}{1+p}\right) = \lim_{i \to \infty} \alpha(\{r\}) \leq \lim_{i \to \infty} c_i \omega^{\alpha(r)} \leq \lim_{i \to \infty} \omega \cdot \omega^{\alpha(r)} = \omega^{\alpha(r)+1} = \omega^{\alpha(p)}.
\]
(2) \( p \) is a limit element.

Let \( p_0, p_1, \ldots \) be a decreasing sequence converging to \( p \). By inductive assumption, \( \alpha\left(\frac{p_i}{1+p_i}\right) \leq \omega\alpha(p_i) \). We have

\[
\alpha\left(\frac{p}{1+p}\right) = \lim_{i \to \infty} \alpha\left(\frac{p_i}{1+p_i}\right) \leq \lim_{i \to \infty} \omega\alpha(p_i) = \omega^{\lim_{i \to \infty} \alpha(p_i)} = \omega^{\alpha(p)}.
\]

**Lemma 65.** \( \alpha\left(\frac{p}{1+p}\right) = \omega^{\alpha(p)} \).

**Proof.** Follows from Lemmas 59 and 60. \( \square \)

**Theorem 66.** The ordering type of \( A \) is at least \( \epsilon_0 \).

**Proof.** The ordering type of \( A \cap \left(\frac{1}{2}, 1\right] \) is \( \omega \) (Lemma 55). The ordering type of \( A \cap \left(\frac{1}{4}, 1\right] \) is \( \omega^\omega \) (Lemma 65 with \( p = 1/2 \)), the ordering type of \( A \cap \left(\frac{1}{3}, 1\right] \) \( \omega^{\omega^\omega} \) and so on.

The ordering type of \( A \) is the limit of this sequence, i.e.

\[
\epsilon_0 = \lim (\omega, \omega^\omega, \omega^{\omega^\omega}, \ldots).
\]

It is known that the ordinal \( \epsilon_0 \) expresses the set of all expressions possible in first-order arithmetic. We see that PFIN, a very simple learning criterion, generates a very complex probability hierarchy.

The table below shows how the complexity of the hierarchy increases. All results in this table can be obtained using Lemma 65.

<table>
<thead>
<tr>
<th>Interval</th>
<th>Ordering type of the probability hierarchy</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \left[\frac{1}{2}, 1\right] )</td>
<td>( \omega )</td>
</tr>
<tr>
<td>( \left[\frac{1}{4}, 1\right] )</td>
<td>( 2\omega )</td>
</tr>
<tr>
<td>( \left[\frac{3}{7}, 1\right] )</td>
<td>( 3\omega )</td>
</tr>
<tr>
<td>( \left[\frac{2}{5}, 1\right] )</td>
<td>( \omega^2 )</td>
</tr>
<tr>
<td>( \left[\frac{3}{8}, 1\right] )</td>
<td>( \omega^3 )</td>
</tr>
<tr>
<td>( \left[\frac{1}{3}, 1\right] )</td>
<td>( \omega^\omega )</td>
</tr>
<tr>
<td>( \left[\frac{1}{4}, 1\right] )</td>
<td>( \omega^{\omega^\omega} )</td>
</tr>
<tr>
<td>( [0, 1] )</td>
<td>( \epsilon_0 )</td>
</tr>
</tbody>
</table>

It shows that the known part of hierarchy (\( [\frac{1}{2}, 1] \)) is very simple compared to the entire hierarchy.

**Notes.** The points of the probability hierarchy in the intervals \( [\frac{1}{2}, 1], [\frac{3}{8}, 1] \) and \( [\frac{1}{4}, 1] \) were explicitly described in [12], [10] and [7], respectively.

In [7], an \( \omega^2 \) sequence of points converging to \( \frac{3}{5} \) was presented and it was conjectured that this sequence forms the backbone of the learning capabilities in the interval \( [\frac{2}{5}, 1] \).

5. **Probabilistic versus team learning**

For EX-identification, there is a precise correspondence between probabilistic and team learners (Pitt’s connection [23]). Any probabilistic learner can be simulated by any team with the ratio of successful machines equal to the probability of success for the probabilistic learner.

However, the situation is more complicated for finite learning (FIN and PFIN). Here, the learning power of a team depends not only on the ratio of successful machines. Team size is also important.

**Theorem 67.** (See [17,29].) \( [1, 2]\) PFIN \( \subset [2, 4]\) PFIN.
So, a team of 4 learning machines where 2 machines are required to be successful has more learning power than team of 2 learning machines where 1 must succeed. However, in both teams the ratio of successful machines to all machines is the same (2/3).

This phenomena is called redundancy. Various redundancy types have been discovered for various ratios of successful machines [7,11,17]. The theorem below is the example of infinite redundancy [7,11].

**Theorem 68.** (See [7].) If \( k \mod 3 \neq 0 \), then

\[
[2k, 5k]_{\text{PFIN}} \subset [8k, 20k]_{\text{PFIN}}.
\]

In particular,

\[
[2, 5]_{\text{PFIN}} \subset [8, 20]_{\text{PFIN}} \subset [32, 80]_{\text{PFIN}} \subset \cdots
\]

So, for the ratio of successful machines 2/5 there are infinitely many different team sizes with different learning power.

However, even for PFIN, any probabilistic machine can be simulated by a team with the same ratio of success, if we choose the team size carefully. A simple corollary of Theorem 49 is

**Corollary 69.** If \( p, q \in \mathbb{N^+} \), then there exists \( k \) such that

\[
\text{PFIN} \left( \frac{p}{q} \right) = [pk, qk]_{\text{PFIN}}.
\]

This shows that probabilistic PFIN-learning and team PFIN-learning are of the same power.

**Corollary 70.** If \( p, q \in \mathbb{N^+} \), then there exists \( k \) such that

\[
[p, q]_{\text{PFIN}} \subseteq [pk, qk]_{\text{PFIN}}
\]

for any \( l \in \mathbb{N^+} \).

**Proof.** The team of \( ql \) machines can be simulated by single probabilistic machine which equiprobably chooses one of machines in team and simulates it. Hence, Corollary 69 implies that

\[
[p, q]_{\text{PFIN}} \subseteq \text{PFIN} \left( \frac{p}{q} \right) = [pk, qk]_{\text{PFIN}}.
\]

So, we see that redundancy structures can be very complicated but always there is the “best” team size such that team of this size can simulate any other team with the same ratio of successful machines. It exists even if there are infinitely many team sizes with different learning power (like for ratio 2/5, Theorem 68).

6. Conclusion

We have investigated the structure of probability hierarchy for PFIN-type learning. Instead of trying to determine the exact points at which the learning capabilities change, we focused on the structural properties of the hierarchy.

We have developed a universal diagonalization algorithm (Theorem 29) and a universal simulation algorithm (Theorem 49). These algorithms are very general forms of diagonalization and simulation arguments used for probabilistic PFIN [7,10].

Universal diagonalization theorem gives the method that can be used to obtain any possible diagonalization for probabilistic PFIN. Universal simulation algorithm can be used for any possible simulation.

These two results together give us a recursive description of the set of points \( \mathcal{A} \) at which the learning capabilities are different.

This set is well-ordered in decreasing ordering. (This property is essential to the proof of Theorem 49.) Its structure is quite complicated. Namely, its ordering type is \( \epsilon_0 \), the ordering-type of the set of all expressions possible in first-order arithmetic.
It shows the huge complexity of the probabilistic PFIN-hierarchy and explains why it is so difficult to find the points at which the learning capabilities are different. A simple corollary of our results is that the probabilistic and team PFIN-type learning is of the same power, i.e. any probabilistic learning machine can be simulated by a team with the same success ratio. Several open problems remain:

(1) Unrestricted finite learning (FIN).

The major open problem is the generalization of our results for other learning paradigms such as (non-Popperian) FIN-type learning and language learning in the limit. Theorem 29 can be proved for (non-Popperian) FIN-type learning, too. Hence, if

\[ \text{PFIN}(p_1) \neq \text{PFIN}(p_2), \]

then

\[ \text{FIN}(p_1) \neq \text{FIN}(p_2). \]

So, the probability hierarchy of FIN is at least as complicated as the probability hierarchy of PFIN. It is even more complicated because it is known [9,10] that

\[ \text{FIN}(24/49) \subset \text{FIN}(1/2) \]

but

\[ \text{PFIN}(24/49) = \text{PFIN}(1/2). \]

The simulation techniques for FIN are much more complicated than simulation techniques for PFIN. However, we hope that some combination of our methods and other ideas (e.g. [8,9]) can help to identify the set of all possible diagonalization methods for FIN and to prove that no other diagonalization methods exists (i.e. to construct universal simulation for FIN).

A step in that direction was made in [3] by proving that FIN-hierarchy is well-ordered and recursively enumerable. It still remains open whether it is decidable. The proof technique in [3,4] is different from ours and uses capability trees [8].

(2) Probabilistic language learning.

The probability hierarchy of language learning in the limit [15] has some similarities to FIN and PFIN-hierarchies. It is an interesting open problem whether some analogues of our results can be obtained for language learning in the limit.

(3) What is the computational complexity of decision algorithms for PFIN-hierarchy?

(4) How dense is the probability hierarchy?

Can we prove the result of the following type:

If \( p_1, p_2 \in \left[ \frac{1}{n+1}, \frac{1}{n} \right] \) and \( |p_1 - p_2| < (1/2)^n \), then

\[ \text{PFIN}(p_1) \neq \text{PFIN}(p_2)? \]

Other properties of the whole hierarchy can be studied, too.

Acknowledgments

I would like to thank the referee for the valuable comments that helped to improve this paper. Extended abstract of this paper appeared on the 9th Conference on Computational Learning Theory, Desenzano del Garda, Italy, 1996.

References

