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Similarity of analytic Toeplitz operators on the Bergman spaces ☆

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Abstract

In this paper we give a function theoretic similarity classification for Toeplitz operators on weighted Bergman spaces with symbol analytic on the closure of the unit disk.

Keywords: Bergman spaces; Toeplitz operator; Similarity; Blaschke product; Strongly irreducible

1. Introduction

Let dA denote Lebesgue area measure on the unit disk $\mathbb{D}$, normalized so that the measure of $\mathbb{D}$ equals 1. For $\alpha > -1$, the *weighted Bergman space* A^2_α is the space of analytic functions on $\mathbb{D}$ which are square-integrable with respect to the measure $dA_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha\, dA(z)$. For $u \in L^\infty(\mathbb{D}, dA)$, the *Toeplitz operator* T_u with symbol u is the operator on A^2_α defined by $T_u f = P(uf)$; here P is the orthogonal projection from $L^2(\mathbb{D}, dA_\alpha)$ onto A^2_α. T_g is called to be the analytic Toeplitz operator if $g \in H^\infty$ (the set of bounded analytic functions on $\mathbb{D}$). In this case, T_g is just the operator of multiplication by g on A^2_α. In this paper we study the similarity of Toeplitz operators with symbol analytic on the closure of the unit disk on A^2_α. On the Hardy

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space, Cowen showed that two Toeplitz operators with symbol analytic on the closure of the unit disk are similar if and only if they are unitarily equivalent [3]. However this is not true on the Bergman space. In [17], Sun showed that for two functions f and g analytic on the closure of the unit disk, the analytic Toeplitz operator T_f on the Bergman space A_0^2 is unitarily equivalent to T_g if and only if there is an inner function χ of order one such that $g = f \circ \chi$. Also in [18], Sun and Yu showed that if the direct sum of two analytic Toeplitz operators is unitarily equivalent to an analytic Toeplitz operator, then they must be constants. So the Bergman space is rigid. But in [12], Jiang and Li showed that if f is a finite Blaschke product, then the analytic Toeplitz operator T_f is similar to the direct sum of finite copies of the Bergman shift T_z on the unweighted Bergman space A_0^2. In this paper, we will completely determine when two Toeplitz operators with symbol analytic on the closure of the unit disk are similar on the weighted Bergman spaces in terms of symbols, which is analogous to the result on the Hardy space [3]. While the Beurling theorem plays an important role on the Hardy space [3] and the Beurling theorem does not hold on the weighted Bergman spaces [8], we apply the general results [9–11] on similarity classification of the Cowen–Douglas classes to analytic Toeplitz operators. It was shown in [9,11] that two strongly irreducible members of Cowen–Douglas operator class $\mathbf{B}_n(\Omega)$ [4] are similar if and only if the respective commutant algebras have isomorphic K_0 groups and strongly irreducible decomposition operators give the similarity classification for Cowen–Douglas operator classes. As the adjoint of Toeplitz operators with symbol analytic on the closure of the unit disk is in the Cowen–Douglas operator classes, our main ideas are to identify the commutant of analytic Toeplitz operators as the commutant of Toeplitz operators with some finite Blaschke products and to use the strongly irreducible decomposition of analytic Toeplitz operators and K_0-groups of the commutants.

The following theorem is our main result. It gives a function theoretic similarity classification of Toeplitz operators with symbol analytic on the closure of the unit disk.

Theorem 1.1. *Suppose that f and g are analytic on the closure of the unit disk $\mathbb{D}$. T_f is similar to T_g on the weighted Bergman spaces A_α^2 if and only if there are two finite Blaschke products B and B_1 with the same order and a function h analytic on the closure of the unit disk such that*

$$f = h \circ B \quad \textit{and} \quad g = h \circ B_1.$$

2. Toeplitz operators with symbol as a finite Blashcke product

A finite Blashcke product B is given by

$$B = \prod_{k=1}^{n} \frac{z - a_k}{1 - \overline{a_k} z}$$

for n numbers $\{a_k\}_{k=1}^n$ in the unit disk and some positive integer n. Here n is said to be the order of the Blaschke product B. In this section we will show that two Toeplitz operators with symbols as finite Blaschke products are similar on the weighted Bergman spaces if and only if their symbols have the same order. This was conjectured in [6] and proved in [12] on the unweighted Bergman space. For another proof, see [7]. Even for a very special Toeplitz operator T_{z^n} on the weighted Bergman space A_α^2, the following result was established in [14] recently.

Theorem 2.1. *Let B be a Blaschke product with order n. Then T_B on the weighted Bergman space A^2_α is similar to the direct sum $\bigoplus_{j=1}^n T_z$ on $\bigoplus_{j=1}^n A^2_\alpha$.*

We define the composition operator C_B on A^2_α by

$$C_B(f)(z) = f\big(B(z)\big)$$

for $f \in A^2_\alpha$. Since B is a finite Blaschke product, the Nevanlinna counting function of B is equivalent to $(1-|z|^2)$ on the unit disk. Thus C_B is bounded below and hence has the closed range. For each k, define

$$M_k = \left\{ \frac{f(B)}{1-\overline{a_k}z} \colon f \in A^2_\alpha \right\}.$$

Since $T_{\frac{1}{1-\overline{a_k}z}}$ is invertible, M_k is a closed subspace of A^2_α.

Assume that the n-th order Blaschke B has n distinct zeros. In [16], Stessin and Zhu showed that the Bergman space A^2_0 is spanned by $\{M_1, \dots, M_n\}$. In [12], Jiang and Li showed that the Bergman space A^2_0 is the Banach direct sum $M_1 \dotplus M_2 \dotplus \cdots \dotplus M_n$. The following theorem extends Jiang and He's result to the weighted Bergman spaces, which immediately gives Theorem 2.1.

Theorem 2.2. *Let B be an n-th order Blaschke product with distinct zeros $\{a_k\}_{k=1}^n$ in the unit disk. Then the weighted Bergman space A^2_α is the Banach direct sum of $M_1, \dots, M_n$, i.e.,*

$$A_\alpha = M_1 \dotplus M_2 \dotplus \cdots \dotplus M_n.$$

Before going to the proof of the above theorems we need the following simple lemma.

Lemma 2.3. *Suppose that $\{a_k\}_{k=1}^n$ are n distinct nonzero numbers in the unit disk. Then the following system*

$$\begin{pmatrix} \frac{1}{1-|a_1|^2} & \frac{1}{1-\overline{a_2}a_1} & \frac{1}{1-\overline{a_3}a_1} & \cdots & \frac{1}{1-\overline{a_n}a_1} \\ \frac{1}{1-\overline{a_1}a_2} & \frac{1}{1-|a_2|^2} & \frac{1}{1-\overline{a_3}a_2} & \cdots & \frac{1}{1-\overline{a_n}a_2} \\ \frac{1}{1-\overline{a_1}a_3} & \frac{1}{1-\overline{a_2}a_3} & \frac{1}{1-|a_3|^2} & \cdots & \frac{1}{1-\overline{a_n}a_3} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{1-\overline{a_1}a_n} & \frac{1}{1-\overline{a_2}a_n} & \frac{1}{1-\overline{a_3}a_n} & \cdots & \frac{1}{1-|a_n|^2} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{2.1}$$

has only the trivial solution.

Proof. Using row reductions and induction we obtain that the determinant of the coefficient matrix of system (2.1) equals

$$\begin{vmatrix} \frac{1}{1-|a_1|^2} & \frac{1}{1-\overline{a_2}a_1} & \frac{1}{1-\overline{a_3}a_1} & \cdots & \frac{1}{1-\overline{a_n}a_1} \\ \frac{1}{1-\overline{a_1}a_2} & \frac{1}{1-|a_2|^2} & \frac{1}{1-\overline{a_3}a_2} & \cdots & \frac{1}{1-\overline{a_n}a_2} \\ \frac{1}{1-\overline{a_1}a_3} & \frac{1}{1-\overline{a_2}a_3} & \frac{1}{1-|a_3|^2} & \cdots & \frac{1}{1-\overline{a_n}a_3} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{1-\overline{a_1}a_n} & \frac{1}{1-\overline{a_2}a_n} & \frac{1}{1-\overline{a_3}a_n} & \cdots & \frac{1}{1-|a_n|^2} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1}{1-|a_1|^2} & \frac{1}{1-\overline{a_2}a_1} & \frac{1}{1-\overline{a_3}a_1} & \cdots & \frac{1}{1-\overline{a_n}a_1} \\ \frac{\overline{a_1}(a_1-a_2)}{(1-\overline{a_1}a_2)(1-|a_1|^2)} & \frac{\overline{a_2}(a_1-a_2)}{(1-|a_2|^2)(1-\overline{a_2}a_1)} & \frac{\overline{a_3}(a_1-a_2)}{(1-\overline{a_3}a_2)(1-\overline{a_3}a_1)} & \cdots & \frac{\overline{a_n}(a_1-a_2)}{(1-\overline{a_n}a_2)(1-\overline{a_n}a_1)} \\ \frac{\overline{a_1}(a_1-a_3)}{(1-\overline{a_1}a_3)(1-|a_1|^2)} & \frac{\overline{a_2}(a_1-a_3)}{(1-\overline{a_2}a_3)(1-\overline{a_2}a_1)} & \frac{\overline{a_3}(a_1-a_3)}{(1-|a_3|^2)(1-\overline{a_3}a_1)} & \cdots & \frac{\overline{a_n}(a_1-a_3)}{(1-\overline{a_n}a_3)(1-\overline{a_n}a_1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\overline{a_1}(a_1-a_n)}{(1-\overline{a_1}a_n)(1-|a_1|^2)} & \frac{\overline{a_2}(a_1-a_n)}{(1-\overline{a_2}a_n)(1-\overline{a_2}a_1)} & \frac{\overline{a_3}(a_1-a_n)}{(1-\overline{a_3}a_n)(1-\overline{a_3}a_1)} & \cdots & \frac{\overline{a_n}(a_1-a_n)}{(1-|a_n|^2)(1-\overline{a_n}a_1)} \end{vmatrix}$$

$$= \frac{1}{1-|a_1|^2}\prod_{j>1}\left[\frac{\overline{a_1}-\overline{a_j}}{1-\overline{a_1}a_j}\right] \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ \frac{\overline{a_1}}{(1-\overline{a_1}a_2)} & \frac{\overline{a_2}}{(1-|a_2|^2)} & \frac{\overline{a_3}}{(1-\overline{a_3}a_2)} & \cdots & \frac{\overline{a_n}}{(1-\overline{a_n}a_2)} \\ \frac{\overline{a_1}}{(1-\overline{a_1}a_3)} & \frac{\overline{a_2}}{(1-\overline{a_2}a_3)} & \frac{\overline{a_3}}{(1-|a_3|^2)} & \cdots & \frac{\overline{a_n}}{(1-\overline{a_n}a_3)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\overline{a_1}}{(1-\overline{a_1}a_n)} & \frac{\overline{a_2}}{(1-\overline{a_2}a_n)} & \frac{\overline{a_3}}{(1-\overline{a_3}a_n)} & \cdots & \frac{\overline{a_n}}{(1-|a_n|^2)} \end{vmatrix}$$

$$= \frac{1}{1-|a_1|^2}\prod_{j>1}\left[\frac{\overline{a_1}-\overline{a_j}}{1-\overline{a_1}a_j}\right] \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & \frac{-(\overline{a_1}-\overline{a_2})}{(1-|a_2|^2)(1-\overline{a_1}a_2)} & \frac{-(\overline{a_1}-\overline{a_3})}{(1-\overline{a_3}a_2)(1-\overline{a_1}a_2)} & \cdots & \frac{-(\overline{a_1}-\overline{a_n})}{(1-\overline{a_n}a_2)(1-\overline{a_1}a_2)} \\ 0 & \frac{-(\overline{a_1}-\overline{a_2})}{(1-\overline{a_2}a_3)(1-\overline{a_1}a_3)} & \frac{-(\overline{a_1}-\overline{a_3})}{(1-|a_3|^2)(1-\overline{a_1}a_3)} & \cdots & \frac{-(\overline{a_1}-\overline{a_n})}{(1-\overline{a_n}a_3)(1-\overline{a_1}a_3)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \frac{-(\overline{a_1}-\overline{a_2})}{(1-\overline{a_2}a_n)(1-\overline{a_1}a_n)} & \frac{-(\overline{a_1}-\overline{a_3})}{(1-\overline{a_3}a_n)(1-\overline{a_1}a_n)} & \cdots & \frac{-(\overline{a_1}-\overline{a_n})}{(1-|a_n|^2)(1-\overline{a_1}a_n)} \end{vmatrix}$$

$$= (-1)^{n-1}\frac{1}{1-|a_1|^2}\prod_{j>1}\left[\frac{|a_1-a_j|^2}{(1-\overline{a_1}a_j)^2}\right] \begin{vmatrix} \frac{1}{1-|a_2|^2} & \frac{1}{1-\overline{a_3}a_2} & \cdots & \frac{1}{1-\overline{a_n}a_2} \\ \frac{1}{1-\overline{a_2}a_3} & \frac{1}{1-|a_3|^2} & \cdots & \frac{1}{1-\overline{a_n}a_3} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{1}{1-\overline{a_2}a_n} & \frac{1}{1-\overline{a_3}a_n} & \cdots & \frac{1}{1-|a_n|^2} \end{vmatrix}$$

$$= (-1)^{\frac{n(n-1)}{2}}\frac{1}{\prod_{j=1}^n(1-|a_j|^2)}\prod_{j=1}^n\prod_{j<k}\left[\frac{|a_j-a_k|^2}{(1-\overline{a_j}a_k)^2}\right] \neq 0.$$

This gives that system (2.1) has only the trivial solution and hence

$$c_1 = c_2 = \cdots = c_n = 0. \qquad \square$$

From the above proof we immediately see

$$\begin{vmatrix} \frac{1}{1-\overline{a_1}b_1} & \frac{1}{1-\overline{a_2}b_1} & \frac{1}{1-\overline{a_3}b_1} & \cdots & \frac{1}{1-\overline{a_n}b_1} \\ \frac{1}{1-\overline{a_1}b_2} & \frac{1}{1-\overline{a_2}b_2} & \frac{1}{1-\overline{a_3}b_2} & \cdots & \frac{1}{1-\overline{a_n}b_2} \\ \frac{1}{1-\overline{a_1}b_3} & \frac{1}{1-\overline{a_2}b_3} & \frac{1}{1-\overline{a_3}b_3} & \cdots & \frac{1}{1-\overline{a_n}b_3} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{1-\overline{a_1}b_n} & \frac{1}{1-\overline{a_2}b_n} & \frac{1}{1-\overline{a_3}b_n} & \cdots & \frac{1}{1-\overline{a_n}b_n} \end{vmatrix}$$

$$= (-1)^{\frac{n(n-1)}{2}} \frac{1}{\prod_{j=1}^n (1-\overline{a_j}b_j)} \prod_{j=1}^n \prod_{j<k} \left[\frac{(\overline{a_j}-\overline{a_k})(b_j-b_k)}{(1-\overline{a_j}b_k)^2} \right]. \tag{2.2}$$

The above formula will be used in the proof of Theorem 2.2.

Proof of Theorem 2.2. Without loss of generality we may assume that B has n distinct zeros $\{a_k\}_{k=1}^n$ with $a_k \neq 0$ for each k.

We will show that the sum $\sum_{k=1}^n M_k$ is closed. Suppose that $\{G_m\}$ is a Cauchy sequence in the sum $\sum_{k=1}^n M_k$ and converges to a function G in A_α^2. There are functions f_{km} in A_α^2 such that

$$G_m(z) = \frac{f_{1m}(B(z))}{1-\overline{a_1}z} + \frac{f_{2m}(B(z))}{1-\overline{a_2}z} + \cdots + \frac{f_{nm}(B(z))}{1-\overline{a_n}z}. \tag{2.3}$$

To show that G is in $\sum_{j=1}^n M_j$, we need only to show that for each j, $\{f_{jm}\}$ is Cauchy in A_α^2. Since B is a finite Blaschke product, the Bochner theorem [21] gives that critical points of B in the closed unit disk are contained in a compact subset of the open unit disk. So we may assume that for each point z on the unit circle, there is an open neighborhood $U(z)$ of z such that n (local) inverses $\rho_j : U(z) \to \rho_j(U(z))$ of B^{-1} on $U(z)$ (we have to emphasize that $\{\rho_j\}_{j=1}^n$ depend on $U(z)$) satisfying the following conditions:

(1) $B(\rho_j(w)) = w$ for $w \in U(z)$;
(2) each ρ_j is analytic on $U(z)$;
(3) $\rho_j(w) \neq \rho_k(w)$ for w in the closure of $U(z)$ if $j \neq k$;
(4) each $\rho_j'(w)$ does not vanish on the closure of $U(z)$; and
(5) $\rho_j(U(z) \cap \mathbb{D}) \subset \mathbb{D}$ for $j = 1, \ldots, n$.

Substituting ρ_j into both sides of equality (2.3) by ρ_j and condition (1) give

$$G_m(\rho_j(w)) = \frac{f_{1m}(B(\rho_j(w)))}{1-\overline{a_1}\rho_j(w)} + \frac{f_{2m}(B(\rho_j(w)))}{1-\overline{a_2}\rho_j(w)} + \cdots + \frac{f_{nm}(B(\rho_j(w)))}{1-\overline{a_n}\rho_j(w)}$$

$$= \frac{f_{1m}(w)}{1-\overline{a_1}\rho_j(w)} + \frac{f_{2m}(w)}{1-\overline{a_2}\rho_j(w)} + \cdots + \frac{f_{nm}(w)}{1-\overline{a_n}\rho_j(w)}$$

for $w \in U(z) \cap \mathbb{D}$ and each $j = 1, \dots, n$. We obtain the following system of equations:

$$\begin{pmatrix} \frac{1}{1-\overline{a_1}\rho_1(w)} & \frac{1}{1-\overline{a_2}\rho_1(w)} & \frac{1}{1-\overline{a_3}\rho_1(w)} & \cdots & \frac{1}{1-\overline{a_n}\rho_1(w)} \\ \frac{1}{1-\overline{a_1}\rho_2(w)} & \frac{1}{1-\overline{a_2}\rho_2(w)} & \frac{1}{1-\overline{a_3}\rho_2(w)} & \cdots & \frac{1}{1-\overline{a_n}\rho_2(w)} \\ \frac{1}{1-\overline{a_1}\rho_3(w)} & \frac{1}{1-\overline{a_2}\rho_3(w)} & \frac{1}{1-\overline{a_3}\rho_3(w)} & \cdots & \frac{1}{1-\overline{a_n}\rho_3(w)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{1-\overline{a_1}\rho_n(w)} & \frac{1}{1-\overline{a_2}\rho_n(w)} & \frac{1}{1-\overline{a_3}\rho_n(w)} & \cdots & \frac{1}{1-\overline{a_n}\rho_n(w)} \end{pmatrix} \begin{pmatrix} f_{1m}(w) \\ f_{2m}(w) \\ f_{3m}(w) \\ \vdots \\ f_{nm}(w) \end{pmatrix} = \begin{pmatrix} G_m(\rho_1(w)) \\ G_m(\rho_2(w)) \\ G_m(\rho_3(w)) \\ \vdots \\ G_m(\rho_n(w)) \end{pmatrix}.$$

Using Cramer's rule to solve the above system of equations, by conditions (1) and (5) and equality (2.2), we have that there are uniformly bounded functions $F_{jk}(w)$ on $U(z)$ such that

$$f_{jm}(w) = \sum_{k=1}^{n} F_{jk}(w) G_m\big(\rho_k(w)\big).$$

Therefore for some positive constants C and C_z and any positive integers m and m',

$$\begin{aligned} &\int_{U(z)\cap\mathbb{D}} \big|f_{jm}(w) - f_{jm'}(w)\big|^2 \, dA_\alpha \\ &\leqslant C\Bigg[\sum_{j=1}^{n} \int_{U(z)\cap\mathbb{D}} \big|G_m\big(\rho_j(w)\big) - G_{m'}\big(\rho_j(w)\big)\big|^2 \, dA_\alpha\Bigg] \\ &\leqslant C\Bigg[\sum_{j=1}^{n} \int_{\rho_j(U(z)\cap\mathbb{D})} \big|G_m(\lambda) - G_{m'}(\lambda)\big|^2 \frac{1}{|\rho_j'(\rho_j^{-1}(\lambda))|^2}\bigg[\frac{1-|\rho_j^{-1}(\lambda)|^2}{1-|\lambda|^2}\bigg]^\alpha \, dA_\alpha\Bigg] \\ &\leqslant CC_z\bigg[\int_{\mathbb{D}} \big|G_m(w) - G_{m'}(w)\big|^2 \, dA_\alpha\bigg]. \end{aligned}$$

The last inequality follows from condition (4) and

$$1 - |w|^2 \approx 1 - \big|\rho_j(w)\big|^2.$$

This comes from condition (1) and the fact that

$$1 - |\lambda|^2 \approx 1 - \big|B(\lambda)\big|^2$$

since

$$\begin{aligned} 1 - \big|B(\lambda)\big|^2 = \big(1 - |\lambda|^2\big)\bigg[&\frac{1-|a_1|^2}{|1-\overline{a_1}\lambda|^2} \\ &+ \big|\phi_{a_1}(\lambda)\big|^2 \frac{1-|a_2|^2}{|1-\overline{a_2}\lambda|^2} + \cdots + \Bigg(\prod_{j=1}^{n-1}\big|\phi_{a_j}(\lambda)\big|^2\Bigg)\frac{1-|a_n|^2}{|1-\overline{a_n}\lambda|^2}\bigg]. \end{aligned}$$

Here $\phi_a(z)$ is the Mobius transform

$$\phi_a(z) = \frac{z-a}{1-\bar{a}z}.$$

Since the unit circle is compact, there are finitely many $\{U(z_k)\}_{k=1}^{l}$ covering the unit circle. Thus there is $0 < r < 1$ such that $\{w \in \mathbb{D}\colon\ r < |w| < 1\}$ is contained in $\bigcup_{k=1}^{l} U(z_j)$ and so for any integers m and m',

$$\begin{aligned}\int\limits_{r<|w|<1} \left|f_{jm}(w) - f_{jm'}(w)\right|^2 dA_\alpha &\leqslant \sum_{k=1}^{l} \int\limits_{U(z_k)\cap\mathbb{D}} \left|f_{jm}(w) - f_{jm'}(w)\right|^2 dA_\alpha \\ &\leqslant lC\Big(\max_{1\leqslant i\leqslant l} C_{z_i}\Big)\Big[\int\limits_{\mathbb{D}} \left|G_m(w) - G_{m'}(w)\right|^2 dA_\alpha\Big].\end{aligned}$$

Since $\chi_{\{r<|w|<1\}}\, dA_\alpha$ is a reversed Carleson measure for A_α^2, by [15], there is a positive constant C_2 such that for all $f \in A_\alpha^2$,

$$\int\limits_{\mathbb{D}} \left|f(w)\right|^2 dA_\alpha \leqslant C_2 \int\limits_{r<|w|<1} \left|f(w)\right|^2 dA_\alpha.$$

Thus for any integers m and m',

$$\begin{aligned}\int\limits_{\mathbb{D}} \left|f_{jm}(w) - f_{jm'}(w)\right|^2 dA_\alpha &\leqslant C_2 \int\limits_{r<|w|<1} \left|f_{jm}(w) - f_{jm'}(w)\right|^2 dA_\alpha \\ &\leqslant lCC_2\Big(\max_{1\leqslant i\leqslant l} C_{z_i}\Big)\Big[\int\limits_{\mathbb{D}} \left|G_m(w) - G_{m'}(w)\right|^2 dA_\alpha\Big].\end{aligned}$$

This implies that for each j, $\{f_{jm}\}$ is Cauchy in A_α^2 since $\{G_m\}$ is Cauchy in A_α^2. We may assume that f_{jm} converges to a function f_j in A_α^2 and hence converges pointwise to f_j. Taking the pointwise limit on both sides of (2.3) gives

$$G(z) = \frac{f_1(B(z))}{1-\overline{a_1}z} + \frac{f_2(B(z))}{1-\overline{a_2}z} + \cdots + \frac{f_n(B(z))}{1-\overline{a_n}z}.$$

Therefore the sum $\sum_{k=1}^{n} M_k$ is a closed subspace of A_α^2.

Next we will show that the sum $\sum_{k=1}^{n} M_k$ is a Banach direct sum

$$M_1 \dotplus M_2 \dotplus \cdots \dotplus M_n$$

on the weighted Bergman space A_α^2.

To do so, let $\frac{f_k(B)}{1-\overline{a_k}z}$ be functions in M_k such that

$$\sum_{k=1}^{n}\frac{f_k(B)}{1-\overline{a_k}z}=0$$

for some functions $f_k=\sum_{m=0}^{\infty}c_{km}z^m$ in the weighted Bergman space A_α^2. On the other hand, easy calculations give

$$\sum_{k=1}^{n}\frac{f_k(B)}{1-\overline{a_k}z}=\sum_{k=1}^{n}\frac{1}{1-\overline{a_k}z}\sum_{m=0}^{\infty}c_{km}B^m$$
$$=\sum_{m=0}^{\infty}\sum_{k=1}^{n}c_{km}\frac{1}{1-\overline{a_k}z}B^m.$$

Taking the inner product of both sides of the above equality with each reproducing kernel k_{a_j} of the weighted Bergman space A_α^2 at a_j and noting that the powers of B vanish at each a_j, we have the following system of equations

$$\begin{pmatrix}\frac{1}{1-|a_1|^2} & \frac{1}{1-\overline{a_2}a_1} & \frac{1}{1-\overline{a_3}a_1} & \cdots & \frac{1}{1-\overline{a_n}a_1}\\ \frac{1}{1-\overline{a_1}a_2} & \frac{1}{1-|a_2|^2} & \frac{1}{1-\overline{a_3}a_2} & \cdots & \frac{1}{1-\overline{a_n}a_2}\\ \frac{1}{1-\overline{a_1}a_3} & \frac{1}{1-\overline{a_2}a_3} & \frac{1}{1-|a_3|^2} & \cdots & \frac{1}{1-\overline{a_n}a_3}\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ \frac{1}{1-\overline{a_1}a_n} & \frac{1}{1-\overline{a_2}a_n} & \frac{1}{1-\overline{a_3}a_n} & \cdots & \frac{1}{1-|a_n|^2}\end{pmatrix}\begin{pmatrix}c_{10}\\ c_{20}\\ c_{30}\\ \vdots\\ c_{n0}\end{pmatrix}=\begin{pmatrix}0\\0\\0\\ \vdots\\0\end{pmatrix}.$$

Lemma 2.3 gives that the above system has only the trivial solution and hence

$$c_{10}=c_{20}=\cdots=c_{n0}=0.$$

Let

$$g_k=\sum_{j=1}^{\infty}c_{kj}z^{j-1}.$$

Then $\{g_k\}_{k=1}^n$ are functions in A_α^2 such that $f_k=zg_k$ and

$$0=\sum_{k=1}^{n}\frac{f_k(B)}{1-\overline{a_k}z}$$
$$=\sum_{k=1}^{n}\frac{Bg_k(B)}{1-\overline{a_k}z}$$
$$=B\sum_{k=1}^{n}\frac{g_k(B)}{1-\overline{a_k}z}.$$

So we obtain

$$\sum_{k=1}^{n} \frac{g_k(B)}{1-\overline{a_k}z} = 0.$$

Repeating the above argument gives

$$c_{11} = c_{21} = \cdots = c_{n1} = 0.$$

By the induction we have that $c_{ij} = 0$ for any i, j to get

$$\frac{f_k(B)}{1-\overline{a_k}z} = 0.$$

This implies that 0 in the sum $\sum_{k=1}^{n} M_k$ has the unique decomposition

$$0 = \overbrace{0+0+\cdots+0}^{n}.$$

Therefore the sum $\sum_{j=1}^{n} M_k$ is a Banach direct sum $M_1 \dotplus M_2 \dotplus \cdots \dotplus M_n$.

Next we will show that $M_1 \dotplus M_2 \dotplus \cdots \dotplus M_n$ is dense in the weighted Bergman space A_α^2 and hence equals A_α^2.

Suppose that f is a function in the weighted Bergman space A_α^2 but orthogonal to each M_k. Then

$$\int_{\mathbb{D}} f(z) \frac{1}{1-\bar{z}a_k}\, dA_\alpha = 0$$

for each k. We claim

$$\int_{\mathbb{D}} f(z) \frac{1}{(1-\bar{z}a_k)^m}\, dA_\alpha = 0$$

for each k and all $m \geqslant 0$. Assuming the claim, we have

$$\int_{\mathbb{D}} f(z)\big(1-|z|^2\big)^\alpha \frac{1}{(1-\bar{z}a_k)^m}\, dA = 0 \tag{2.4}$$

for all m. This implies that the Bergman projection $\tilde{P}[f(z)(1-|z|^2)^\alpha]$ of the function $f(z)(1-|z|^2)^\alpha$ into the unweighted Bergman space A_0^2 also equals zero at a_k and all derivatives of $\tilde{P}[f(z)(1-|z|^2)^\alpha]$ equal zero at a_k and so

$$\tilde{P}\big[f(z)\big(1-|z|^2\big)^\alpha\big] = 0.$$

Here $\tilde{P}$ is the Bergman projection from $L^2(\mathbb{D}, dA)$ onto the unweighted Bergman space A_0^2. Taking derivatives of $\tilde{P}[f(z)(1-|z|^2)^\alpha]$ and evaluating at 0 give

$$\int_{\mathbb{D}} f(z)\big(1-|z|^2\big)^\alpha \bar{z}^m \, dA = 0$$

for all $m \geqslant 0$. Thus

$$\int_{\mathbb{D}} f(z)\big(1-|z|^2\big)^\alpha \overline{p(z)} \, dA = 0$$

for any polynomials $p(z)$ and hence

$$\int_{\mathbb{D}} \big|f(z)\big|^2\big(1-|z|^2\big)^\alpha \, dA = 0$$

as polynomials are dense in A_α^2. This gives that f equals 0.

We are going to prove our claim by induction. We warn the reader that although having (2.4) for one point a_k is enough, one needs it for all a_k for the induction step. Clearly, our claim is true for $m = 1$. Assume that the claim is true for $m \leqslant K$, i.e.,

$$\int_{\mathbb{D}} f(z)\frac{1}{(1-\bar{z}a_k)^m} \, dA_\alpha = 0$$

for each k and all $m \leqslant K$. For $m = K + 1$, we note that for each k, the rational function $\frac{B^K}{1-\overline{a_k}z}$ can be written as a sum of partial fractions with the highest power of $\frac{1}{(1-\overline{a_k}z)}$ is $K + 1$ and the highest power of the rest term $\frac{1}{(1-\overline{a_j}z)}$ is K for $j \neq k$. Thus the term $\frac{1}{(1-\overline{a_k}z)^{K+1}}$ is a linear combination of terms $\{\frac{1}{(1-\overline{a_j}z)^m}\}_{j=1,\ldots,n;\, m=1,\ldots,K}$ and $\frac{B^K}{1-\overline{a_k}z}$. So

$$\int_{\mathbb{D}} f(z)\frac{1}{(1-\bar{z}a_k)^{K+1}} \, dA_\alpha = 0.$$

This gives the claim. □

For two bounded operators S and T, we say that T is *similar* to S if there is an invertible operator X such that

$$T = XSX^{-1}.$$

We use $T \sim S$ to denote that T is similar to S. Now we turn to the proof of Theorem 2.1.

Proof of Theorem 2.1. Since for most λ in the unit disk, $\frac{B-\lambda}{1-\bar{\lambda}B}$ is the n-th order Blaschke product with n distinct zeros in the unit disk, without loss of generality we may assume that B has n

distinct zeros $\{a_k\}_{k=1}^n$ with $a_k \neq 0$ for each k. Now we use the same notation as in the above proof. First we will show that on each M_k, T_B is similar to T_z on the weighted Bergman space. To do this, let $Y_k = T_{\frac{1}{1-\overline{a_k}z}} C_B : A^2_\alpha \to M_k$. Clearly, Y_k is a bounded invertible operator from A^2_α onto M_k. Moreover,

$$\begin{aligned} Y_k T_z f &= Y_k(zf) \\ &= \frac{Bf(B)}{1-\overline{a_k}z} \\ &= T_B\left(\frac{f(B)}{1-\overline{a_k}z}\right) \\ &= T_B Y_k f \end{aligned}$$

for each $f \in A^2_\alpha$. By Theorem 2.2, the weighed Bergman spaces

$$A^2_\alpha = M_1 \dotplus M_2 \dotplus \cdots \dotplus M_n.$$

Thus we conclude that

$$T_B \sim \bigoplus_{k=1}^n T_z. \qquad \square$$

3. Some notation and lemmas

In this section we introduce some notation and give a few lemmas which will be used in the proof of our main result in the last section. Let $\{T_f\}'$ denote the *commutant* of T_f, the set of bounded operators commuting with T_f on the Bergman space L^2_a.

We need the following theorem to study the similarity of analytic Toeplitz operators. The version of the following theorem on the Hardy space was obtained in [20]. For more general results on the Hardy space, see [2] and [19]. The method in [20] also works on the weighted Bergman spaces. For the details of the proof of the following theorem, see the proof of theorem in [20, pp. 524–528] by replacing the reproducing kernel on the Hardy space by one on weighted Bergman spaces A^2_α.

Theorem 3.1. *If f is analytic on the closure of the unit disk, then there are a finite Blaschke product B and a function h analytic on the closure of unit disk such that*

$$f = h \circ B$$

and

$$\{T_f\}' = \{T_B\}'.$$

Let T be a bounded operator on a Hilbert space H. An idempotent Q in the commutant $\{T\}'$ is said to be *minimal* if for every idempotent R in $\{T\}'$, $Q = R$, whenever $\operatorname{Ran} R \subset \operatorname{Ran} Q$. Here $\operatorname{Ran} R$ denotes the range of the operator R.

A bounded operator T on a Hilbert space is said to be *strongly irreducible* if there is no nontrivial idempotent operator in its commutant. In the other words, T is strongly irreducible if and only if XTX^{-1} is irreducible for each invertible operator X.

Lemma 3.2. *If Q is a minimal idempotent in $\{T\}'$, then $T|_{\operatorname{Ran} Q}$ is strongly irreducible.*

Proof. Let $A = T|_{\operatorname{Ran} Q}$. Let R be an idempotent in $\{A\}'$. In other words,

$$AR = RA,$$

and

$$\operatorname{Ran} R \subset \operatorname{Ran} Q.$$

We write H as a Banach direct sum

$$H = \operatorname{Ran} Q \dotplus \operatorname{Ran}(I - Q)$$

and extend R to H by

$$\tilde{R}(x + y) = Rx$$

if $x \in \operatorname{Ran} Q$ and $y \in \operatorname{Ran}(I - Q)$. Clearly, $\tilde{R}$ is an idempotent. For each x in $\operatorname{Ran} Q$,

$$\begin{aligned} TRx &= T|_{\operatorname{Ran} Q} Rx \\ &= ARx \\ &= RAx \\ &= RT|_{\operatorname{Ran} Q} x \\ &= RTx. \end{aligned}$$

For $y \in \operatorname{Ran}(I - Q)$, write $y = (I - Q)z$ and then

$$\begin{aligned} \tilde{R}Ty &= \tilde{R}T(I - Q)z \\ &= \tilde{R}(I - Q)Tz \\ &= 0 \end{aligned}$$

and

$$T\tilde{R}y = 0.$$

This gives

$$T\tilde{R}y = \tilde{R}Ty.$$

Thus we obtain

$$\tilde{R}T = T\tilde{R}.$$

So $\tilde{R}$ is in $\{T\}'$. Since Q is a minimal idempotent in $\{T\}'$, we conclude that $\tilde{R} = Q$ and hence $R = Q$. $\square$

Lemma 3.3. *If f is analytic on the closure of the unit disk and B is a Blaschke product B with order n such that*

$$\{T_f\}' = \{T_B\}',$$

then there is a function h bounded and analytic on the unit disk such that

$$T_f \sim \bigoplus_{j=1}^{n} T_h, \quad \{T_h\}' = \{T_z\}'$$

and T_h is strongly irreducible.

Proof. First we get a decomposition of T_f as a Banach direct sum of strongly irreducible operators. By Theorem 2.1, there is an invertible operator X from A^2_α to $\bigoplus_{j=1}^n A^2_\alpha$ such that

$$XT_BX^{-1} = \bigoplus_{j=1}^{n} T_z.$$

Let P_i be the projection from $\bigoplus_{j=1}^n A^2_\alpha$ onto the i-th component. Clearly,

$$\left[\bigoplus_{j=1}^{n} T_z\right]\Bigg|_{\operatorname{Ran} P_i} = T_z.$$

Since $\{T_z\}' = \{T_g\colon\ g \in H^\infty\}$ is isomorphic to the Banach algebra H^∞ and H^∞ does not contain any nontrivial idempotents, T_z is strongly irreducible. Thus P_i is a minimal idempotent in the commutant $\{\bigoplus_{j=1}^n T_z\}'$. Let $Q_i = X^{-1}P_iX$. So $\{Q_i\}_{i=1}^n$ are n minimal idempotents in $\{T_f\}'$ with

$$\sum_{j=1}^{n} Q_j = I.$$

This follows from the hypothesis:

$$\{T_f\}' = \{T_B\}'$$

and the fact that $\{T_B\}'$ contains exactly n minimal idempotents.

Let $S_i = T_f|_{\operatorname{Ran} Q_i}$ and $T_i = XS_iX^{-1}$. We have

$$XT_fX^{-1} = X\sum_{i=1}^{n} S_iX^{-1} = \bigoplus_{i=1}^{n} T_i.$$

The above direct sum follows from the fact that $\{P_i\}_{i=1}^n$ are orthogonal projections on $\bigoplus_{j=1}^n A_\alpha^2$.

Since

$$\left\{\bigoplus_{j=1}^{n} T_z\right\}' = \{T_F\colon F \in M_n(H^\infty)\},$$

we have

$$T_iT_z = T_zT_i.$$

It is well known that there are functions $f_1, \dots, f_n$ in H^∞ such that

$$T_i = T_{f_i}$$

for each i. This gives the decomposition of T_f as we desired:

$$XT_fX^{-1} = \begin{pmatrix} T_{f_1} & 0 & 0 & \cdots & 0 \\ 0 & T_{f_2} & 0 & \cdots & 0 \\ 0 & 0 & T_{f_3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & T_{f_n} \end{pmatrix}$$

on $\bigoplus_{j=1}^n A_\alpha^2$.

Next we need to show that

$$f_1 = f_2 = \cdots = f_n.$$

The above equalities follow from

$$T_{f_1} = T_{f_2} = \cdots = T_{f_n}.$$

To simplify the proof, we are going to show only that $T_{f_1} = T_{f_2}$ since the same argument gives that $T_{f_i} = T_{f_j}$.

To do so, let

$$V=\begin{pmatrix}0 & I & 0 & \cdots & 0\\ I & 0 & 0 & \cdots & 0\\ 0 & 0 & I & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & I\end{pmatrix}$$

on $\bigoplus_{j=1}^n A_\alpha^2$. Clearly, V is in $\{\bigoplus_{j=1}^n T_z\}'$. Easy computations give

$$\begin{pmatrix}0 & I & 0 & \cdots & 0\\ I & 0 & 0 & \cdots & 0\\ 0 & 0 & I & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & I\end{pmatrix}\begin{pmatrix}T_{f_1} & 0 & 0 & \cdots & 0\\ 0 & T_{f_2} & 0 & \cdots & 0\\ 0 & 0 & T_{f_3} & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & T_{f_n}\end{pmatrix}=\begin{pmatrix}0 & T_{f_2} & 0 & \cdots & 0\\ T_{f_1} & 0 & 0 & \cdots & 0\\ 0 & 0 & T_{f_3} & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & T_{f_n}\end{pmatrix}$$

and

$$\begin{pmatrix}T_{f_1} & 0 & 0 & \cdots & 0\\ 0 & T_{f_2} & 0 & \cdots & 0\\ 0 & 0 & T_{f_3} & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & T_{f_n}\end{pmatrix}\begin{pmatrix}0 & I & 0 & \cdots & 0\\ I & 0 & 0 & \cdots & 0\\ 0 & 0 & I & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & I\end{pmatrix}=\begin{pmatrix}0 & T_{f_1} & 0 & \cdots & 0\\ T_{f_2} & 0 & 0 & \cdots & 0\\ 0 & 0 & T_{f_3} & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & T_{f_n}\end{pmatrix}.$$

Since

$$V\left[\bigoplus_{j=1}^n T_{f_j}\right]=\left[\bigoplus_{j=1}^n T_{f_j}\right]V,$$

we have

$$T_{f_1}=T_{f_2}$$

to obtain

$$XT_fX^{-1}=\bigoplus_j^n T_h$$

where $h=f_1=f_2=\cdots=f_n$. Moreover, since P_j is a minimal idempotent, by Lemma 3.2, T_h is strongly irreducible.

Finally we show that

$$\{T_h\}'=\{T_z\}'.$$

Letting A be in $\{T_h\}'$, we define

$$U = \begin{pmatrix} A & 0 & 0 & \cdots & 0 \\ 0 & A & 0 & \cdots & 0 \\ 0 & 0 & A & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & A \end{pmatrix}.$$

Thus U is in the commutant

$$\left\{ \bigoplus_{j}^{n} T_h \right\}' = \left\{ X T_f X^{-1} \right\}'$$

and so A is an analytic Toeplitz operator. Hence

$$\{T_h\}' = \left\{ T_g \colon g \in H^\infty \right\} = \{T_z\}'. \qquad \square$$

The decomposition of T_f in the above lemma is the so-called "strongly irreducible decomposition" of T_f. In fact the decomposition for the above Toeplitz operator is unique up to the similarity. To state the result in [1] and [13], we need some notation.

Let $\mathcal{B}$ be a Banach algebra and $Proj(\mathcal{B})$ be the set of all idempotents in $\mathcal{B}$. Murray–von Neumann equivalence $\sim_a$ is introduced in $Proj(\mathcal{B})$. Let e and $\tilde{e}$ be in $Proj(\mathcal{B})$. We say that $e \sim_a \tilde{e}$ if there are two elements $x, y \in \mathcal{B}$ such that

$$xy = e, \qquad yx = \tilde{e}.$$

Let $\mathbf{Proj}(\mathcal{B})$ denote the equivalence classes of $Proj(\mathcal{B})$ under Murray–von Neumann equivalence $\sim_a$. Let

$$M_\infty(\mathcal{B}) = \bigcup_{n=1}^{\infty} M_n(\mathcal{B})$$

where $M_n(\mathcal{B})$ is the algebra of $n \times n$ matrices with entries in $\mathcal{B}$. Let

$$\bigvee(\mathcal{B}) = \mathbf{Proj}\big(M_\infty(\mathcal{B})\big).$$

The direct sum of two matrices gives a natural addition in $M_\infty(\mathcal{B})$ and hence induces an addition $+$ in $\mathbf{Proj}(M_\infty(\mathcal{B}))$ by

$$[p] + [q] = [p \oplus q]$$

where $[p]$ denotes the equivalence class of p. $(\bigvee(\mathcal{B}), +)$ forms a semigroup and depends on $\mathcal{B}$ only up to stable isomorphism. $K_0(\mathcal{B})$ is the Grothendieck group of $\bigvee(\mathcal{B})$.

Let A be a bounded operator on a Hilbert space H. We use $H^{(n)}$ to denote the direct sum of n copies of H and $A^{(n)}$ to denote the direct sum of n copies of A acting on $H^{(n)}$.

Let T be a bounded operator on a Hilbert space H and $\mathcal{Q} = \{Q_j\colon\ 1 \leqslant j \leqslant l\}$ be a family of minimal idempotents such that

$$\sum_{j=1}^{l} Q_j = I$$

and

$$Q_j Q_i = 0$$

if $j \neq i$. We say that $\mathcal{Q}$ is a *strongly irreducible decomposition* of T. In fact, let $T_j = T|_{\operatorname{Ran} Q_j}$. Then each T_j is strongly irreducible and T is similar to

$$\sum_{j=1}^{l} \bigoplus T_j.$$

Under similarity, $\{T_1, \ldots, T_n\}$ is classified as equivalence classes $\{[T_{j_1}], \ldots, [T_{j_k}]\}$. Let n_i be the number of elements in $[T_{j_i}]$. Then T is similar to

$$\sum_{i=1}^{k} \bigoplus T_{j_i}^{(n_i)}.$$

If for any two strongly irreducible decompositions $\mathcal{Q}_1 = \{Q_j\colon\ 1 \leqslant j \leqslant l_1\}$ and $\mathcal{Q}_2 = \{\tilde{Q}_j\colon\ 1 \leqslant j \leqslant l_2\}$ of T, we have that $l_1 = l_2$ and there are a permutation π of $\{1, 2, \ldots, l_1\}$ and invertible bounded operators X_j from $\operatorname{Ran} Q_j$ onto $\operatorname{Ran} \tilde{Q}_{\pi(j)}$ such that

$$X_j T|_{\operatorname{Ran} Q_j} = T|_{\operatorname{Ran} \tilde{Q}_{\pi(j)}} X_j,$$

then we say that T has unique strongly irreducible decomposition up to similarity. We need the following theorem [1] and [13].

Theorem 3.4. *Let T be a bounded operator on a Hilbert space H. The following are equivalent*:

(a) *T is similar to $\sum_{i=1}^{k} \bigoplus A_i^{(n_i)}$ under the decomposition of the space*

$$H = \sum_{i=1}^{k} \bigoplus H_i^{(n_i)},$$

where k and n_i are finite, A_i is strongly irreducible and A_i is not similar to A_j if $i \neq j$. Moreover $T^{(n)}$ has a unique strongly irreducible decomposition up to similarity.

(b) *The semigroup $\bigvee(\{T\}')$ is isomorphic to the semigroup $\mathbb{N}^{(k)}$ where $\mathbb{N} = \{0, 1, 2, \ldots\}$ and the isomorphism ϕ sends*

$$[I] \to n_1 e_1 + n_2 e_2 + \cdots + n_k e_k$$

where $\{e_i\}_{i=1}^k$ *are the generators of* $\mathbb{N}^{(k)}$ *and* $n_i \neq 0$.

Remark. The above theorem immediately gives that if $T \sim T_1^{(n_1)} \oplus T_1^{(n_2)}$, both T_1 and T_2 are strongly irreducible and $\bigvee(\{T\}')$ is isomorphic to the semigroup $\mathbb{N}$, then T_1 is similar to T_2.

4. Proof of main result

In this section we give the proof of Theorem 1.1. For a Fredholm operator T on a Hilbert space, we use index T to denote the Fredholm index:

$$\operatorname{index} T = \dim \ker T - \dim \ker T^*.$$

Clearly, the product formula of the Fredholm index [5] gives

$$\operatorname{index} T = k \operatorname{index} S$$

if $T = S^{(k)}$ and both T and S are Fredholm.

Proof of Theorem 1.1. Suppose that there are two finite Blaschke products B and B_1 with the same order n and a function h analytic on the closure of the unit disk such that

$$f = h \circ B,$$
$$g = h \circ B_1.$$

By Theorem 2.1, T_f is similar to $T_{h(z^n)}$ and T_g is similar to $T_{h(z^n)}$. Thus T_f is similar to T_g.

Conversely, suppose that T_f is similar to T_g. Since f and g are analytic functions on the closure of the unit disk, by Theorem 3.1, there are two finite Blaschke products B and $\tilde{B}$ and two functions h and h_1 analytic on the closure of the unit disk such that

$$f = h \circ B, \qquad g = h_1 \circ \tilde{B},$$

and

$$\{T_f\}' = \{T_B\}', \qquad \{T_g\}' = \{T_{\tilde{B}}\}'.$$

Let n be the order of B and n_1 the order of $\tilde{B}$. By Lemma 3.3, T_f has the following strongly irreducible decomposition

$$T_f \sim \bigoplus_{j=1}^{n} T_h = T_h^{(n)}$$

and T_g has the following strongly irreducible decomposition

$$T_g \sim \bigoplus_{j=1}^{n_1} T_{h_1} = T_{h_1}^{(n_1)}.$$

First we show that

$$T_h \sim T_{h_1}.$$

To do so, let $T = T_f \oplus T_g$. Then T is similar to

$$T_h^{(n)} \oplus T_{h_1}^{(n_1)}$$

and hence it is similar to $T_h^{(2n)}$. This means that there is an invertible operator Y such that

$$YTY^{-1} = T_h^{(2n)}.$$

This implies

$$Y\{T\}'Y^{-1} = \left\{T_h^{(2n)}\right\}' = \left\{T_F\colon\ F \in M_{2n}\left(H^\infty\right)\right\}.$$

Thus the Banach algebra $\{T\}'$ is isomorphic to $M_{2n}(H^\infty)$. By Theorem 6.11 in [13, p. 203] or Lemma 2.9 in [1, p. 248], we have

$$\bigvee\left(M_{2n}\left(H^\infty\right)\right) = \bigvee\left(H^\infty\right) = \mathbb{N}.$$

So

$$\bigvee\left(\{T\}'\right) = \bigvee\left(M_{2n}\left(H^\infty\right)\right) = \mathbb{N}.$$

The first equality follows from the fact that the semigroup $\bigvee(\mathcal{B})$ is invariant for Banach algebra isomorphisms. By the remark after Theorem 3.4, we get that T_h is similar to T_{h_1}.

Next we show that $n = n_1$. Since we just proved that T_h is similar to T_{h_1} and T_f is similar to T_g, we have

$$T_f \sim T_h^{(n_1)}.$$

Noting that $T_{f-\lambda}$ and $T_{h-\lambda}$ are Fredholm with nonzero index for some λ, by the Fredholm index product formula, we have

$$\operatorname{index} T_{f-\lambda} = \operatorname{index}\left[T_{h-\lambda}^{(n)}\right] = n \operatorname{index} T_{h-\lambda}$$

and

$$\operatorname{index} T_{f-\lambda} = \operatorname{index}\bigl[T_{h-\lambda}^{(n_1)}\bigr] = n_1 \operatorname{index} T_{h-\lambda}$$

to get that $n = n_1$. Thus B and $\tilde{B}$ have the same order.

Finally we need to show that there is a Mobius transform χ such that

$$h_1 = h \circ \chi.$$

Noting that $\{T_h\}' = \{T_z\}'$ and $\{T_{h_1}\}' = \{T_z\}'$, as T_h is similar to T_{h_1}, we have that there is an invertible operator Z such that

$$Z T_h Z^{-1} = T_{h_1}$$

and

$$Z\{T_h\}' Z^{-1} = \{T_{h_1}\}'.$$

Thus

$$Z\{T_z\}' Z^{-1} = \{T_z\}'.$$

By the fact that

$$\{T_z\}' = \bigl\{T_G\colon\ G \in H^\infty\bigr\},$$

we have

$$Z\bigl\{T_G\colon\ G \in H^\infty\bigr\} Z^{-1} = \bigl\{T_G\colon\ G \in H^\infty\bigr\}.$$

In particular, there is a function χ in H^∞ such that

$$Z T_z Z^{-1} = T_\chi.$$

The spectral picture of T_z forces χ to be a Mobius transform. Since h is analytic on the closure of the unit disk, we can write h as

$$h = \sum_{n=0}^{\infty} a_n z^n$$

with

$$\sum_{j=0}^{\infty} |a_n| r^n < \infty$$

for some $r > 1$. Thus

$$\sum_{n=0}^{\infty} a_n T_z^n = T_h$$

in the norm topology. So

$$\begin{aligned} ZT_h Z^{-1} &= \sum_{n=0}^{\infty} a_n \left(ZT_z Z^{-1}\right)^n \\ &= \sum_{n=0}^{\infty} a_n (T_\chi)^n \\ &= \sum_{n=0}^{\infty} a_n T_{\chi^n} \\ &= T_{\sum_{n=0}^{\infty} a_n \chi^n}. \end{aligned}$$

Since $ZT_h Z^{-1} = T_{h_1}$, we have

$$h_1 = \sum_{n=0}^{\infty} a_n \chi^n = h \circ \chi,$$

to obtain

$$\begin{aligned} f &= h \circ B, \\ g &= h_1 \circ \tilde{B} = h \circ (\chi \circ \tilde{B}) = h \circ B_1 \end{aligned}$$

where $B_1 = \chi \circ \tilde{B}$ is a Blaschke product with order n. □

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