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Note

On van der Waerden's Theorem on Arithmetic Progressions

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In "How the Proof of Baudet's Conjecture was Found" [6] van der Waerden explains how he, Artin and Schreier proved: "If the sequence of integers 1, 2, 3,... is divided into two classes, at least one of the classes contains an arithmetic progression of l terms:

$$a, a+b, \ldots, a+(l-1)\,b$$

no matter how large the given length l is."

He notes that at one point of the proof "Artin expected—and he proved right—that the generalization from 2 to k classes would be an advantage in the induction proof." In their recent book [1], which gives a nice survey on partition theory, Graham *et al.* noted that this idea "is crucial to all known proofs of van de Waerden's theorem." This leads to the question as to whether Artin's idea is crucial for all possible combinatorial proofs, which is also interesting as in the case of Ramsey's theorem [4] Artin's idea is of no particular advantage. In this note we give a combinatorial proof of van der Waerden's theorem without Artin's generalization, i.e., using divisions with 2 classes only.

Notation. $\mathbb{N}$ is the set of nonnegative integers. A coloring of a set X with two colors is a mapping $\Delta: X \to \{0, 1\}$. A subset Y of X to which the restriction $\Delta | Y$ is constant is called monochromatic. Finally for $X, Y \subset \mathbb{N}$ let $X + Y$ be the set of all sums $x + y$ with $x \in X$, $y \in Y$. In particular if $X = \{x\}$ is a singleton then $X + Y = x + Y$.

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DEFINITION. Let $l, m \in \mathbb{N}$. A subset X of $\mathbb{N}$ is an m-fold arithmetic progression of length l iff there exists a "basis" $(a_0, \ldots, a_m) \in \mathbb{N}^{m+1}$ such that

$$X = \left\{ a_0 + \sum_{i=1}^{m} \lambda_i a_i \mid \lambda_i \in \{0, \ldots, l-1\} \right\}.$$

Let X, Y be arithmetic progressions of length l with bases $(a_0, \ldots, a_m)$, $(b_0, \ldots, b_n)$ resp. Y is a canonical subprogression of X iff $Y \subseteq X$ and there exists a partition $0 \in I_0, \ldots, I_n$ of $\{0, \ldots, m\}$ into n pairwise disjoint nonempty sets, such that $b_j = \sum_{i \in I_j} a_i$ for $j = 0, \ldots, n$. Note that the property to be a canonical subprogression is transitive

THEOREM 1. *Let $l, m \in \mathbb{N}$. There exists n with the following property. Let X be any n-fold arithmetic progression of length l and $\Delta: X \to \{0, 1\}$ be a coloring. Then there exists a canonical monochromatic m-fold subprogression Y of X of length l.*

Notation. The minimal n satisfying the theorem for l, m is denoted by $n(l, m)$.

The proof of Theorem 1 will be by double induction, primarily on l, secondarily on m. We need two lemmas.

LEMMA 2. *Assume that the theorem holds for some $l \in \mathbb{N}$ and arbitrary m. Let γ, $m \in \mathbb{N}$. There exists n with the following property: Let*

— *$C \subset \mathbb{N}$ be any set of cardinality at most γ.*

— *X be any n-fold arithmetic progression of length l.*

— *$\Delta: C + X \to \{0, 1\}$ be a coloring.*

Then X contains a canonical m-fold arithmetic subprogression Y of length l, such that for each $c \in C$ the set $c + Y$ is colored monochromatically.

Notation. The minimal n satisfying lemma 2 for l, γ, m is denoted by $n(l, \gamma, m)$.

Proof of Lemma 2. Let $n_\gamma = m$ and $n_{i-1} = n(l, n_i)$. We claim that $n = n_0$ has the desired property. In order to see this let $C = \{c_1, \ldots, c_\gamma\}$, X, $\Delta: C + X \to \{0, 1\}$ be as in the lemma. Consider the restriction of Δ to $c_1 + X$. By definition there exists a canonical n_1-fold subprogression $X_1 \subset X$ of length l such that $c_1 + X_1$ is colored monochromatically. Consider next the restriction of Δ to $c_2 + X_1$ obtaining a canonical n_2-fold subprogression $X_2 \subset X_1$ such that $c_2 + X_2$, and by inclusion, $c_1 + X_2$ are colored monochromatically. By iterating this argument one obtains an n_γ-fold canonical subprogression $Y = X_\gamma$ with the desired property.

LEMMA 3. *Assume that the theorem holds for some $l \in \mathbb{N}$ and arbitrary m. Let $\gamma \in \mathbb{N}$. There exists $n \in \mathbb{N}$ with the following property: Let*

— *$C \subset \mathbb{N}$ be any set of cardinality at most γ.*

— *X be any n-fold arithmetic progression of length $l+1$.*

— *$\Delta: C + X \to \{0, 1\}$ be a coloring.*

Then X contains a canonical 1-fold subprogression Y of length $l+1$ such that for each $c \in C$ the set $c + Y$ is colored monochromatically.

Notation. The minimal n satisfying the lemma for l, γ is denoted by $N(l, \gamma)$.

Proof of Lemma 3. This proof is inspired by Rado [3, p. 433]. Let $p = 2^{\gamma}$. Let $n_p = 1$ and $n_{i-1} - 1 = n(l, \gamma, n_i)$. We claim that $n = n_0$ has the desired property. In order to see this let C, $X = \{a_0^0 + \sum_{i=1}^{n} \lambda_i a_i^0 | \lambda_i \in \{0,..., l\}\}$ and $\Delta: C + X\{0, 1\}$ be as in the lemma. Let $X_0 = \{a_0^0 + \sum_{i=1}^{n_0 - 1} \lambda_i a_i^0 | \lambda_i \in \{0,..., l-1\}\}$. Lemma 2 guarantees the existence of a sequence $(X_j)_{j=0,...,p}$ with the following properties:

(α) $X_{j+1} = \{a_0^{j+1} + \sum_{i=1}^{n_{j+1}} \lambda_i a_i^{j+1} | \lambda_i \in \{0,..., l-1\}\}$ is a canonical subprogression of X_j of length l ($j = 0,..., p-1$).

(β) For every $c \in C$ the set $c + X_{j+1} + l \sum_{i=0}^{j} a_{n_i}^i$ is monochromatic for Δ.

Consider the $p+1$ elements $b_k = \sum_{i=0}^{k} a_{n_i}^i$ ($k = 0,..., p$). For some $k' < k''$ the restrictions $\Delta | C + a_0^p + lb_{k'}$ and $\Delta | C + a_0^p + lb_{k''}$ coincide. By construction the canonical subprogression $Y \subset X$ of length $l+1$

$$Y = \left\{ a_0^p + \lambda \sum_{i=k'+1}^{k''} a_{n_i}^i + lb_{k'} | \lambda \in \{0,..., l\} \right\}$$

has the desired properties.

Proof of Theorem 1. Proceed by induction on l. For $l = 0$ the theorem is obvious. Assume that the theorem holds for some $l \geqslant 0$ and arbitrary m. By induction on m prove it for $l+1$. For $m = 0$ the validity is obvious. Assume that the theorem holds for $l+1$ and some $m \geqslant 0$ with N. Let $\gamma = N^{l+1}$, the obvious maximal cardinality of an N-fold arithmetic progression of length $l+1$. We claim that $n = N + N(l, \gamma)$ has the desired property. In order to see this let $X = \{a_0 + \sum_{i \times 1}^{n} \lambda_i a_i | \lambda_i \in \{0,..., l\}\}$ be an $n-$fold arithmetic progression of length $l+1$. Let

$$C = \left\{ a_0 + \sum_{i=1}^{N} \lambda_i a_i | \lambda_i \in \{0,..., l\} \right\}$$

and

$$X' = \left\{ 0 + \sum_{i=N+1}^{n} \lambda_i a_i \mid \lambda_i \in \{0,..., l\} \right\}. \text{ Thus } X = C + X'.$$

Let $\Delta: X \to \{0, 1\}$ be a coloring. By Lemma 3, X' contains a canonical 1-fold arithmetic subprogression Y' of length $l+1$ such that for each $c \in C$ the set $c + Y'$ is monochromatic for Δ. Define a coloring $\Delta^*: C \to \{0, 1\}$ by $\Delta^*(c) = \Delta(c + Y')$. By hypothesis, C contains a canonical m-fold arithmetic subprogression Y^* of length $l+1$ which is monochromatic for Δ^*. Thus $Y = Y^* + Y'$ is a canonical $m+1$-fold arithmetic subprogression of length $l+1$ which is monochromatic for Δ.

Remark. Canonical subprogressions occured in Hales and Jewett [2]. Theorem 1 may be viewed as the Hales–Jewett theorem rephrased in the language of arithmetic progressions. Thus in fact we have shown that this theorem may be proved using 2-colorings only. By checking the many applications of the Hales–Jewett theorem (Parameter sets, vector spaces, Abelian groups, partition regular systems of equations, graphs, etc.) one sees that the corresponding theorems can be proved using 2-colorings only.

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