FINITELY GENERATED FREE TETRAVALENT MODAL ALGEBRAS

Isabel LOUREIRO
C.M.A.F., 2, Av. Prof. Gama Pinto, 1699 Lisboa Codex, Portugal

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The purpose of this work is to prove that the tetravalent modal algebra with a finite set of free generators is finite and to determine its cardinal number.

1. Preliminary definitions and properties

We begin with the following definition introduced by A.A. Monteiro in 1978:

1.1. Definition. A tetravalent modal algebra \((A, \wedge, \vee, \neg, \nabla, 1)\) or simply \(A\), is an equational algebra of type \((2, 2, 1, 1, 0)\) which satisfies the following axioms:

\[
\begin{align*}
(A1) & \quad x \wedge (x \vee y) = x, \\
(A2) & \quad x \wedge (y \vee z) = (z \wedge x) \vee (y \wedge x), \\
(A3) & \quad \neg \neg x = x, \\
(A4) & \quad \neg(x \wedge y) = \neg x \vee \neg y, \\
(A5) & \quad \neg x \vee \neg x = 1, \\
(A6) & \quad x \wedge \neg x = \neg x \vee \neg x.
\end{align*}
\]

It immediately follows that \(A\) is a distributive lattice [8] with least element \(0 = \neg 1\) and a De Morgan algebra [6].

We then assume that the reader is familiar with the basic notions of lattice theory.

We define the unary operator \(\Delta\) on \(A\) by \(\Delta x = \neg \neg x\).

It can be proved that the operators \(\nabla\) and \(\Delta\) satisfy the properties of a closure operator and of an interior operator respectively.

In a corresponding logic, if \(p\) is a proposition, \(\nabla p\) will mean the proposition "\(p\) is possible" and \(\Delta p\) will mean the proposition "\(p\) is necessary".

A tetravalent modal algebra is called trivial if it has only one element.

Three-valued Lukasiewicz algebras are important examples of tetravalent modal algebras [7]. We consider also the example of tetravalent modal algebra,
denoted by $S_3$, whose Hasse diagram and the corresponding operations are:

\[
\begin{array}{c|ccc}
 x & \sim x & \nabla x \\
\hline
0 & 1 & 0 \\
a & a & 1 \\
b & b & 1 \\
1 & 0 & 1 \\
\end{array}
\]

\[a \leq b \iff a \leq \nabla b \leq \nabla a \leq b\]

This algebra has two subalgebras $S_3$ (subalgebras with three elements) formed by the sets $A_1 = \{0, a, 1\}$ and $B_1 = \{0, b, 1\}$ and one subalgebra $S_2$ (subalgebra with two elements) formed by the set $\{0, 1\}$.

We introduce now some definitions and state results that we shall need later. These results will be published in [4].

1.2. Definition ([2]). For each prime filter $P$ of a of a tetravalent modal algebra $A$ (i.e. each prime filter of the subjacent lattice) we define the prime filter $\Phi(P) = C \sim P$, where $C$ denotes the set-theoretical complement and $\sim P = \{\sim x : x \in P\}$.

This mapping $\Phi$ is called the Birula-Rasiowa transformation.

It is easily checked that:

1. $\Phi(\Phi(P)) = P$ for each prime filter $P$ of $A$.
2. If $P$ and $Q$ are both prime filters of $A$ such that $P \subseteq Q$, then $\Phi(Q) \subseteq \Phi(P)$.

Let now $\pi_0$ be a family of prime filters of a tetravalent modal algebra $A$, such that $\Phi(P) \in \pi_0$ for each $P \in \pi_0$. Consider the following definition.

1.3. Definition. For $a, b \in A$ we set $a \equiv b \pmod{\pi_0}$ if the following conditions are satisfied:

1. $\Phi(P) = \Phi(Q) \Rightarrow a \equiv b \pmod{\pi_0}$ for each $P, Q \in \pi_0$.
2. $\Phi(P) = \Phi(Q) \Rightarrow a \equiv b \pmod{\pi_0}$ for each $P, Q \in \pi_0$.

We have then the following results:

1.4. Proposition. The relation $\equiv \pmod{\pi_0}$ is a congruence relation on $A$.

1.5. Proposition. The quotient set $A' = A/\equiv = A/\pi_0$ algebraized in the usual way, is a tetravalent modal algebra. Moreover the kernel of the natural homomorphism $h$ from $A$ onto $A'$ (i.e. the set $\{x \in A : h(x) = 1\}$) is the set $N = \bigcap_{P \in \pi_0} P$.

1.6. Proposition. Let $A'$ be a homomorphic image of $A$ by the homomorphism $h$ whose kernel is $N$. Let $\pi'$ be the set of all prime filters of $A'$ and define the set
\[ \pi_0 = \{ h^{-1}(P') : P' \in \pi \}. \] Then we have:

1. \( \pi_0 \) is a family of prime filters of \( A \) such that \( \Phi(P) \in \pi_0 \) for each \( P \in \pi_0 \).
2. \( A' \) is isomorphic to \( A/\pi_0 \).
3. \( N = \cap_{P \in \pi_0} P \).
4. If \( a, b \in A \), \( a \equiv b \pmod{\pi_0} \) iff \( h(a) = h(b) \).
5. \( h = \theta^{-1} \circ h' \), where \( \theta \) is the isomorphism from \( A' \) onto \( A/\pi_0 \) referred into (2) and \( h' \) is the natural homomorphism from \( A \) onto \( A/\pi_0 \).

1.7. Remark. From [5] we have the important result that each prime filter of a tetravalent modal algebra is either an ultrafilter or the image of an ultrafilter by the Birula–Rasiowa transformation \( \Phi \).

In a tetravalent modal algebra \( A \) we can define an implication operator \( \rightarrow \) by the formula: \( a \rightarrow b = \forall \sim a \lor b \). Then we have:

1.8. Definition. A subset \( D \) of \( A \) is a deductive system if \( D \) verifies:

(D1) \( 1 \in D \).
(D2) If \( a, a \rightarrow b \in D \), then \( b \in D \).

We have then:

1.9. Theorem. A subset \( D \) of \( A \) is a maximal deductive system iff there is an ultrafilter \( \mathcal{U} \) of \( A \) such that \( D = \mathcal{U} \cap \Phi(\mathcal{U}) \). Moreover this representation of \( D \) is unique.

Let us suppose that \( D \) is a maximal deductive system of \( A \) and \( \mathcal{U} \) is an ultrafilter of \( A \) such that \( D = \mathcal{U} \cap \Phi(\mathcal{U}) \). Since this representation of \( D \) is unique, by the above result, we denote the tetravalent modal algebra \( A/\equiv(\mathcal{U}, \Phi(\mathcal{U})) \) by \( A/D \).

The following representation theorem should be retained, since it will be needed later:

1.10. Theorem. A nontrivial tetravalent modal algebra \( A \) is isomorphic to a subdirect product of the quotient algebras \( A/M_i \), where \( \{M_i\}_{i \in I} \) is the family of all maximal deductive systems of \( A \).

We have proved also that if \( A \) is finite, then \( A \) is isomorphic to the direct product of the quotient algebras \( A/M_i \).

Let us consider now the following definition:

1.11. Definition. A tetravalent modal algebra \( A \) is simple if \( A \) is nontrivial and each homomorphic image of \( A \) is either trivial or isomorphic to \( A \).
We have obtained the following results:

1.12. **Theorem.** If \( D \) is a maximal deductive system of a tetravalent modal algebra \( A \), the quotient algebra \( A/D \) is isomorphic to \( S_2 \) or \( S_3 \) or \( S_4 \).

1.13. **Theorem.** The only simple tetravalent modal algebras are the algebras \( S_2 \), \( S_3 \) and \( S_4 \).

2. **Finitely generated tetravalent modal algebras**

Let \( G \) be a subset of a tetravalent modal algebra \( A \), we shall denote by \( T(G) \) the tetravalent modal subalgebra generated by \( G \).

We shall prove now that if \( G \) is a finite subset of a tetravalent modal algebra \( A \) with \( n \) elements \( (N(G) = n) \) such that \( T(G) = A \), then \( A \) is finite.

By Theorem 1.10 we know that \( A \) is isomorphic to a subalgebra of the direct product \( \prod_{i \in I} A/M_i \), where \( R = \{ M_i \}_{i \in I} \) is the set of all maximal deductive systems of \( A \). By Theorem 1.12 we have that the quotient algebras \( A/M_i \) \( (M_i \in R) \) are finite and \( N(A/M_i) = 2 \) or \( N(A/M_i) = 3 \) or \( N(A/M_i) = 4 \) according as \( A/M_i = S_2 \) or \( A/M_i = S_3 \) or \( A/M_i = S_4 \). It is sufficient then to prove that \( R \) is finite.

Let us consider \( R_2 = \{ M_i \in R : A/M_i = S_j \} \) for each \( j = 2, 3, 4 \). For each \( j = 2, 3, 4 \), let us denote by \( Epi(A, S_j) \) the set of all epimorphisms from \( A \) onto \( S_j \), by \( F(G, S_j) \) the set of all mappings from \( G \) into \( S_j \) and by \( F^*(G, S_j) \) the set of all mappings \( j \) from \( G \) into \( S_j \) such that \( T(h(G)) = S_j \).

It is clear that we have:

(I) The sets \( R_j \) \( (j = 2, 3, 4) \) form a partition of \( R \).

(II) \( N(F^*(G, S_j)) \leq N(F(G, S_j)) = j^n \) \( (j = 2, 3, 4) \), having the equality for \( j = 2 \).

(III) \( N(Epi(A, S_j)) \leq N(F^*(G, S_j)) \) \( (j = 2, 3, 4) \), since from \( h \in Epi(A, S_j) \) it follows that \( S_j = h(A) = h(T(G)) = T(h(G)) \) and it is well known that every mapping \( f : G \to S_j \) has at most one homomorphic extension.

Let us prove:

(IV) \( N(R_j) \leq N(Epi(A, S_j)) \) \( (j = 2, 3, 4) \).

Let \( h \in Epi(A, S_j) \) \( (j = 2, 3, 4) \) and \( M_i = \text{Ker} \, h \). From Proposition 1.6, Remark 1.7 and Theorem 1.9 it follows that \( M_i \in R_j \).

Let us consider then the mapping \( \psi_j : Epi(A, S_j) \to R_j \) \( (j = 2, 3, 4) \) defined by \( \psi_j(h) = \text{Ker} \, h = M_i \). Let us prove that \( \psi_j \) is surjective \( (j = 2, 3, 4) \) which implies (IV).

Let \( M_i \in R_j \) \( (j = 2, 3, 4) \), then \( A/M_i = S_j \); if \( \delta : A/M_i \to S_j \) is the isomorphism and if \( \alpha : A \to A/M_i \) is the natural homomorphism, then \( h = \delta \circ \alpha \) is an epimorphism from \( A \) onto \( S_j \) whose kernel is \( M_i \), that is \( \psi_j(h) = M_i \). Thus we have (IV).

From (I), (II), (III) and (IV) it follows that \( R \) is finite and then \( A \) is finite, that is:
2.1. Theorem. Every finitely generated tetravalent modal algebra is finite.

In these conditions we have that if $A$ is a finitely generated tetravalent modal algebra, then $A = S_2^{N(R_1)} \times S_3^{N(R_2)} \times S_4^{N(R_3)}$.

Let us prove now:

(V) $N(Epi(A, S_j)) \leq N(R_j) \ (j = 2, 3)$, and

(VI) $N(R_4) = \frac{1}{2} N(Epi(A, S_4))$.

To prove the condition (V) we will show that the mapping $\psi_j \ (j = 2, 3)$ is injective. Consider then $h_1, h_2 \in Epi(A, S_j) \ (j = 2, 3)$, $\psi_j(h_1) = Ker h_1 = M_1$, $\psi_j(h_2) = Ker h_2 = M_2$ and suppose $M_1 = M_2$.

Let $x \in A$. If $x \notin M_1 = M_2$, then we have:

(a) $h_1(x) = 1 = h_2(x)$.

If $x \notin M_1 = M_2$, then if $j = 2$ we have $h_1(x) = 0 = h_2(x)$ an therefore $h_1 = h_2$. If $j = 3$ let us suppose that $h_1(x) \neq h_2(x)$. Then we have either:

(b) $h_1(x) = a$ and $h_2(x) = 0$, or

c) $h_1(x) = 0$ and $h_2(x) = a$.

From (b) it follows that $h_1(\neg x) = \neg h_1(x) = \neg a = a$ and $h_2(\neg x) = \neg h_2(x) = \neg 0 = 1$. Then $\neg x \in M_2$ and $\neg x \notin M_1$ and therefore $M_1 \neq M_2$ which contradicts the hypothesis. So we cannot have (b). Similarly we cannot have (c). Hence we have $h_1 = h_2$ and $\psi_j \ (j = 2, 3)$ is an injective mapping, which proves condition (V).

To prove the condition (VI), let $h \in Epi(A, S_4)$, $\psi_4(h) = Ker h = M_i$ and let us prove:

(d) $\psi_4^{-1}(M_i) = \{\alpha \circ h: \alpha \in Aut(S_4)\}$

where $Aut(S_4)$ is the set of all automorphisms of $S_4$. It is easily checked that:

(e) $\{\alpha \circ h: \alpha \in Aut(S_4)\} \subseteq \psi_4^{-1}(M_i)$.

We shall prove then:

(f) $\psi_4^{-1}(M_i) \subseteq \{\alpha \circ h: \alpha \in Aut(S_4)\}$.

Let $h' \in \psi_4^{-1}(M_i)$ that is $h' \in Epi(A, S_4)$ and $\psi_4(h') = Ker h' = M_i = Ker h$.

Since $h \in Epi(A, S_4)$, we have seen that $M_i \in R_4$ and so there is an isomorphism $\theta: A/M_i \rightarrow S_4$.

Let $\phi$ be the natural homomorphism from $A$ onto $A/M_i$.

On the other hand, since $h' \in Epi(A, S_4)$ and $Ker h' = M_i$, there is an isomorphism $\theta': A/M_i \rightarrow S_4$. By Proposition 1.6, we can then consider the following two commutative diagrams:

$$
\begin{array}{ccc}
A & \xrightarrow{h} & S_4 \\
\downarrow{\phi} & \downarrow{\theta} & \\
A/M_i & \xrightarrow{\theta'} & S_4
\end{array}
$$

Since $\theta$ is an isomorphism, $\theta^{-1}$ exists which is also an isomorphism. We have
then:

\[(g) \phi = \theta^{-1} \circ h \quad \text{and} \quad (h) h' = \theta' \circ \phi.\]

From (g) and (h) we obtain:

\[(i) \quad h' = \theta' \circ (\theta^{-1} \circ h) = (\theta' \circ \theta^{-1}) \circ h.\]

Consider \(\alpha = \theta' \circ \theta^{-1}\) which is an isomorphism from \(S_4\) into \(S_4\) and therefore it is an automorphism of \(S_4\). Hence \(h' \in \{\alpha \circ h : \alpha \in \text{Aut}(S_4)\}\) and we have (f). Thus we have (d). From (IV) it follows that \(N(R_4) = N(\text{Epi}(A, S_4))/N(\text{Aut}(S_4))\). But there are only two automorphisms in \(S_4\): the identity automorphism and the automorphism \(\alpha(0) = 0, \alpha(1) = 1, \alpha(a) = b\) and \(\alpha(b) = a\). Therefore we have proved the condition (VI).

3. Finitely generated free tetravalent modal algebras

Given a cardinal number \(\beta > 0\), we shall denote by \(L(\beta)\) the free tetravalent modal algebra with a set \(G\) of free generators, whose cardinal is \(\beta\).

Since tetravalent modal algebras are equational, we can state, by a theorem of universal algebra of G. Birkhoff [3], the existence and uniqueness, up to isomorphisms, of \(L(\beta)\).

From the preceding section it follows that \(L(n)\) is finite for every natural number \(n > 0\). Furthermore:

\[L(n) = S_2^{N(R_2)} \times S_3^{N(R_3)} \times S_4^{N(R_4)}.\]

Let us consider that \(G\) is the set of \(n\) free generators of \(L(n)\). We shall now compute \(N(R_j)\) \((j = 2, 3, 4)\). Let us prove:

\[(\text{VII}) \quad N(F^*(G, S_j)) \leq N(\text{Epi}(L(n), S_j)) \quad (j = 2, 3, 4).\]

Consider then the mapping \(K_j : \text{Epi}(L(n), S_j) \to F^*(G, S_j) \quad (j = 2, 3, 4)\) defined by \(K_j(h) = h \mid G = f \in F^*(G, S_j)\) for each \(h \in \text{Epi}(L(n), S_j)\). Let us prove that \(K_j\) is surjective \((j = 2, 3, 4)\) which implies (VII).

If \(f \in F^*(G, S_j) \quad (j = 2, 3, 4)\) we know that \(f \in F(G, S_j)\) and \(T(f(G)) = S_j\). Since \(L(n)\) is free, \(f\) can be extended to a homomorphism \(h : L(n) \to S_j\), which is an epimorphism because \(h(L(n)) = T(h(G)) = T(f(G)) = S_j\). Therefore \(K_j(h) = h \mid G = f\) and so \(K_j\) is surjective \((j = 2, 3, 4)\) and we have condition (VII).

From (IV), (V), (III) and (VII) it follows:

\[N(R_j) = N(F^*(G, S_j)) \quad (j = 2, 3).\]

From (j) and (II) we obtain: (1) \(N(R_2) = 2^n\).

Let \(F'(G, S_3)\) be the set of all mappings \(f \in F(G, S_3)\) such that \(T(f(G)) = S_2\) (the
unique proper subalgebra of $S_3$, then we have:

(i) \{F'(G, S_3), F^*(G, S_3)\} form a partition of $F(G, S_3)$.

(m) $N(F'(G, S_3)) = N(F(G, S_2))$.

From (j), (II), (l) and (m) it follows: (2) $N(R_3) = 3^n - 2^n$.

Finally let $F'(G, S_a)$ be the set of all mappings $f \in F(G, S_a)$ such that $T(f(G)) = S_2 \cap S_a$. We know that $S_a$ has two subalgebras $S_3$, $A_1 = \{0, a, 1\}$ and $B_1 = \{0, b, 1\}$ with the induced operations. $S_2, A_1$ and $B_1$ are the unique proper subalgebras of $S_a$.

Let $F''(G, S_a)$ be the set of all mappings $f \in F(G, S_a)$ such that $T(f(G)) = A_1$ and let $F'(G, S_a)$ be the set of all mappings $f \in F(G, S_a)$ such that $T(f(G)) = B_1$. Then we have:

(n) \{F'(G, S_4), F''(G, S_4), F'''(G, S_4), F^*(G, S_4)\} form a partition of $F(G, S_4)$,

(o) $N(F'(G, S_4)) = N(F'(G, S_2))$,

(p) $N(F''(G, S_4)) = N(F'''(G, S_4)) = N(F^*(G, S_3))$.

From (II), (j), (2), (n), (o) and (p) it follows:

(q) $N(F''(G, S_4)) = 4^n - (2^n + 2(3^n - 2^n))$.

From (III), (VII), (VI) and (q) we obtain:

(3) $N(R_4) = [4^n - (2^n + 2(3^n - 2^n))] / 2 = 2^{n-1}(2^n + 1) - 3^n$.

From (L), (1), (2) and (3) we get:

$$N(L(n)) = 2^{2n-3n-2}4^{2n-1}(2^n+1)-3^n = 2^{2n+2n-1}-2^{3n-3}-2n.$$  

Remark. The technique used in Sections 2 and 3 is similar to the one employed by M. Abad and L. Monteiro in [1].

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Note added in proof

After the acceptation of this paper, the author learned that Peter Fowler, in his Ph.D. Thesis (Australia), has obtained, independently and with a completely different proof, the last formula contained in Section 3.

References