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Stochastic optimal control and algorithm of the trajectory of horizontal wells

An Li*, Enmin Feng, Xuelian Sun

Department of Applied Mathematics, Dalian University of Technology, Dalian 116024, PR China

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Abstract

This paper presents a nonlinear, multi-phase and stochastic dynamical system according to engineering background. We show that the stochastic dynamical system exists a unique solution for every initial state. A stochastic optimal control model is constructed and the sufficient and necessary conditions for optimality are proved via dynamic programming principle. This model can be converted into a parametric nonlinear stochastic programming by integrating the state equation. It is discussed here that the local optimal solution depends in a continuous way on the parameters. A revised Hooke–Jeeves algorithm based on this property has been developed. Computer simulation is used for this paper, and the numerical results illustrate the validity and efficiency of the algorithm.

MSC: 93E20; 49L20; 90C30

Keywords: Horizontal well; Stochastic differential equation; Optimal control; Nonlinear programming; Hooke–Jeeves algorithm

1. Introduction

The research on designing the trajectory of horizontal wells considerably developed over these last years. Many methods in dealing with specific problems have been put forward, respectively. In general, the well path is a three-dimensional curve that reaches a given target from a given starting location subject to several constraints. There are some well planning programs available commercially to solve the problem of designing an appropriate trajectory of a horizontal well. However, those methods belong to the category of trial-and-error or human–computer interaction essentially, in which the identification of some control parameters depends to a great extent on the designers' experience and intuition. In recent years, very few references have discussed the horizontal well planning in the mathematical literature. Foreign and domestic experts mainly put forward nonlinear programming models [2,4,6], a fuzzy model [9] and an optimization model [8]. In fact, there are more unknown parameters for complete well than there are defining equations. Consequently, the problem of finding a well path is underdetermined in existing results. Due to the effects of some factors such as stratum and tools, the real trajectory of horizontal wells is deviant from the theoretically optimal one in drilling. But such perturbations have been ignored or only been given a little qualitative consideration in the previous designs. If the parameters provided by the optimal design are applied into practice, the trajectory may not

* Corresponding author.
E-mail addresses: leean1980@163.com (A. Li), emfeng@dlut.edu.cn (E. Feng).

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achieve optimal, or even deviates from the target. Therefore, we establish a nonlinear, multi-phase and stochastic control system of the trajectory of horizontal wells based on the dynamic model [8] Stochastic control is the study of dynamical systems subject to random perturbations and which can be controlled in order to optimize some performance criterion. Our chief concern is to derive some tractable characterization of the value function and optimal control. This article is intended to prove the sufficient and necessary conditions of optimal solution and that the optimal solution depends in a continuous way on the parameters (perturbations). In addition, a revised Hooke–Jeeves algorithm is proposed and the corresponding software is programmed to calculate the practical problems. The numerical results demonstrate the correctness and effectiveness of the stochastic control model and algorithm.

The rest of this paper is organized as follows. Section 2 consist of the problem description and the mathematical model. The existence and uniqueness of the solution of stochastic differential equation are discussed in Section 3. Section 4 give the key results on the characterization of optimality. In Section 5, the parametric nonlinear stochastic programming problem is introduced and some important properties are proved. Finally, the optimization algorithm is proposed to solve the nonlinear stochastic programming with a numerical issue and the conclusions of the paper are mentioned.

2. Problem formulation

As is shown in Fig. 1, the trajectory of a horizontal well can be described in a cartesian coordinate system having its origin at the initial point (Kick-off Point), with x-axis representing North, y-axis representing East, and z-axis representing the vertical depth. Any point on the curve is completely described by its inclination φ, azimuth ϕ, and coordinates x, y, z. In order to simplify the problem, we idealize the trajectory of horizontal wells to be a combination of alternately n constant-curvature smooth quasi-helix segments. Tool-face angle w and curvature K are key parameters to drill a horizontal well, which are governed by the general build-up rate of bottle-hole assembly (BHA) in the drilling operation. Under such assumptions, the rate of change of inclination K_φ and the rate of change of azimuth K_ϕ obey the following rules, respectively,

$$K_\varphi = K\cos w, \quad K_\phi = \frac{K\sin w}{\sin\varphi}.$$

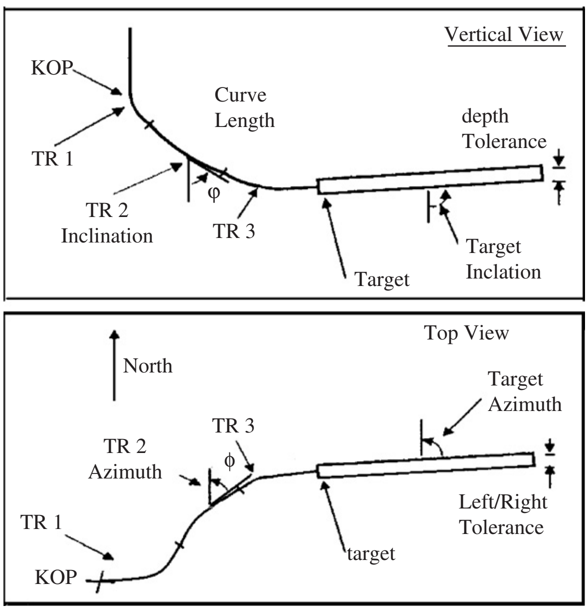

Fig. 1. Horizontal Well's terminology.

Differential geometry shows the rate of change of coordinates with respect to arc length,

$$\frac{dx}{ds} = \sin\varphi\cos\phi, \quad \frac{dy}{ds} = \sin\varphi\sin\phi, \quad \frac{dz}{ds} = \cos\varphi.$$

Anypoint of the trajectory can be completely determined by ordinary equations in terms of several independent parameters such as tool-face angle, curvature.

Let $I_n = \{1, \ldots, n\}$, s stands for the trajectory's arc length from the Kick-off point, $s_0 = 0$ stands for the Kick-off point, and s_i stands for the arc length of the ith terminal point. Let $\Lambda = [\alpha, \frac{\pi}{2}] \times [0, 2\pi] \times R^3$, $\alpha > 0$ is a positive small constant which guarantee that the curve first begins to deviate from the vertical. In general, α is chosen on an empirical basis. $X_i(s) = (X_{i1}(s), X_{i2}(s), X_{i3}(s), X_{i4}(s), X_{i5}(s))^{\mathrm{T}} \in \Lambda$ stands for the state variable at any point $s \in [s_{i-1}, s_i]$ (each component represents inclination, azimuth, north coordinate, east coordinate and vertical depth coordinate, respectively); $X_0 \in \Lambda$ and $X_T \in \Lambda$ mean the state of the Kick-off Point and the target, respectively, which are given. Let the radius of curvature, the reciprocal of curvature, be u_{i1}, the tool-face angle be u_{i2}, which are constants in the same curve segment; let the arc length of the ith curve segment be u_{i3}, then $u_{i3} = s_i - s_{i-1}$. Let $u_i = (u_{i1}, u_{i2}, u_{i3}) \in R^3$ be control variable, according to the engineering constraints its control domain is $U_{\mathrm{ad}} = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3] \subset R^3_+$, $a_i < b_i$, $i = 1, 2, 3$. Let $u = (u_1, u_2, \ldots, u_n) \in R^{3n}$, then its control domain is $\tilde{U}_{\mathrm{ad}} = \{u \in R^{3n} : u_i \in U_{\mathrm{ad}},\ i \in I_n\}$. Let $(\Omega, \mathscr{F}, P)$ be a filtered probability space and $\{\mathscr{F}_s, s \in [0, +\infty)\}$ be its filtration satisfying the usual conditions. The stochastic perturbations that we consider in this paper are assumed the Normal Wiener process on R^{5n}, donated by $w(s) = \{(w_1^{\mathrm{T}}(s), \ldots, w_n^{\mathrm{T}}(s))^{\mathrm{T}} : w_i(s) \in \mathscr{F}_s, i \in I_n, s \in [0, +\infty)\}$. We make the problem stochastic by adding a white noise term, with small coefficients $\varepsilon^{1/2}$, in each component of $X_i(s)$. Consider the perturbed state process $X_i(s)$ valued in R^5 satisfying

$$\begin{aligned} &dX_i(s) = f(X_i, u_i)\,ds + \sigma^{\varepsilon}\,dw_i(s), \quad s \in (s_{i-1}, s_i), \quad i \in I_n \\ &X_1(0) = X_0, \quad X_i(s_{i-1}) = X_{i-1}(s_{i-1}), \quad i = 2, \ldots, n \\ &u_i \in U_{\mathrm{ad}}, \quad i \in I_n, \end{aligned} \tag{1}$$

where

$$f(X_i, u_i) = \left(\frac{\cos u_{i2}}{u_{i1}}, \frac{\sin u_{i2}}{u_{i1}\sin X_{i1}}, \sin X_{i1}\cos X_{i2}, \sin X_{i1}\sin X_{i2}, \cos X_{i1}\right)^{\mathrm{T}}, \tag{2}$$

$\sigma^{\varepsilon} = \varepsilon^{1/2} I$, I is the identity matrix.

3. Existence and uniqueness of the solution of stochastic dynamical system

In this section we will prove the existence and uniqueness of the solution of (1) for any given initial condition. First we prove some properties as follows:

Property 1. *The function f in* (2) *is continuous on $\Lambda \times U_{\mathrm{ad}}$, and satisfies the following properties*:

(a) *f satisfies linear growth property, namely, there exists a constant K such that for $s \in [s_{i-1}, s_i]$, $i \in I_n$*

$$\|f(X_i, u_i)\| \leqslant K(\|X_i\| + 1) \quad \forall X_i \in \Lambda, \ \forall u_i \in U_{\mathrm{ad}}.$$

(b) *f is Lipschitz relatively to Λ, namely, there exists a constant L such that for $s \in [s_{i-1}, s_i]$, $i \in I_n$*

$$\|f(X_i', u_i) - f(X_i'', u_i)\| \leqslant L\|X_i' - X_i''\| \quad \forall X_i', X_i'' \in \Lambda, \ \forall u_i \in U_{\mathrm{ad}}.$$

Proof. (a) Obviously from (2) f does not explicitly depend on s, and f is continuous on $\Lambda \times U_{\mathrm{ad}}$. Furthermore,

$$\begin{aligned}\|f(X_i,u_i)\| &= \left(\sum_{j=1}^{5} f_j^2(X_i,u_i)\right)^{1/2}\\ &= \frac{1}{u_{i1}\sin X_{i1}}(\cos^2 u_{i2}\sin^2 X_{i1}+\sin^2 u_{i2}+u_{i1}^2\sin^2 X_{i1})^{1/2}\\ &\leqslant \frac{1}{u_{i1}\sin X_{i1}}(X_{i1}^2+u_{i2}^2+u_{i1}^2X_{i1}^2)^{1/2}\\ &\leqslant \frac{1}{a_1\sin\alpha}(X_{i1}^2+b_2^2+b_1^2X_{i1}^2)^{1/2}.\end{aligned}$$

let $K=\max\{\sqrt{b_1^2+1}/a_1\sin\alpha,\, b_2/a_1\sin\alpha\}$, it follows that

$$\|f(X_i,u_i)\| \leqslant K(|X_{i1}|+1) \leqslant K(\|X_i\|+1) \quad \forall X_i \in \Lambda \ \forall u_i \in U_{\mathrm{ad}}.$$

(b) Let X_i' and X_i'' be any two distinct points in Λ, applying mean value theorem for derivatives we have

$$f_j(X_i',u_i)-f_j(X_i'',u_i)=(X_i'-X_i'')^{\mathrm{T}} f_j'(\theta X_i'+(1-\theta)X_i'',u_i), \quad 0<\theta<1, \quad j\in I_5.$$

Hence,

$$\begin{aligned}\|f(X_i',u_i)-f(X_i'',u_i)\| &= \left(\sum_{j=1}^{5}(f_j(X_i',u_i)-f_j(X_i'',u_i))^2\right)^{1/2}\\ &= \left(\sum_{j=1}^{5}((X_i'-X_i'')^{\mathrm{T}} f_j'(\theta X_i'+(1-\theta)X_i'',u_i))^2\right)^{1/2},\end{aligned}$$

$$\frac{\partial f(X_i,u_i)}{\partial X_i}=\begin{pmatrix} 0 & 0 & 0 & 0 & 0\\ \frac{-\sin u_{i2}\cos X_{i1}}{u_{i1}\sin^2 X_{i1}} & 0 & 0 & 0 & 0\\ \cos X_{i1}\cos X_{i2} & -\sin X_{i1}\sin X_{i2} & 0 & 0 & 0\\ \cos X_{i1}\sin X_{i2} & \sin X_{i1}\cos X_{i2} & 0 & 0 & 0\\ -\sin X_{i1} & 0 & 0 & 0 & 0\end{pmatrix}.$$

Setting $L'=\max_{(X_i,u_i)\in\Lambda\times U_{\mathrm{ad}}}\max_{j,k\in I_5}|\partial f_j(X_i,u_i)/\partial X_{ik}|$, we get L' is finite since $\Lambda\times U_{\mathrm{ad}}$ is bounded, so we can write

$$\|f(X_i',u_i)-f(X_i'',u_i)\| \leqslant \sqrt{5}L'\|X_i'-X_i''\|.$$

let $L=\sqrt{5}L'$, so f is Lipschitz relative to Λ. □

Theorem 1. *For any given $u_i\in U_{\mathrm{ad}}$, $w_i(s)\in\mathscr{F}_s$ and $X_0\in\Lambda$, there exist a unique piecewise continuous solution $X_i(s,u_i,w_i(s))$ of the stochastic differential equation* (1) *with probability* 1, *and $X_i(s,u_i,w_i(s))$ is a Markov process.*

Proof. Using Property 1 and existence and uniqueness theory for stochastic differential equations in [5], the theorem is proved directly. □

4. Optimal control problem and Bellman's optimality principle

Assume $X_i(s,u_i,w_i(s))$, $s\in[s_{i-1},s_i]$, $i\in I_n$ is the piecewise solution of the stochastic differential *equation* (1), and denote the solution on $[0,s_n]$ by $X(s,u,w(s))=(X_1(s,u_1,w_1(s))^{\mathrm{T}},\ldots,X_n(s,u_n,w_n(s))^{\mathrm{T}})^{\mathrm{T}}\in R^{5n}$. Define the solution set of (1) relative to $\tilde{U}_{\mathrm{ad}}$ by $V_x(\tilde{U}_{\mathrm{ad}}, \mathscr{F}_s) = \{X(s, u, w(s)) \in R^{5n} : X(s, u, w(s))$ is the solution of (1) corresponding to $u \in \tilde{U}_{\mathrm{ad}}, w(s) \in \mathscr{F}_s\}$.

The expected performance criterion are the precision of hitting target and the total length of the trajectory, that is

$$J(u) := E\left[\mu_0 \int_0^{s_n} \mathrm{d}s + \sum_{j=1}^{5} \mu_j (X_j(s_n, u, w(s_n)) - X_{Tj})^2\right], \tag{3}$$

where $\mu_j (j = 0, \ldots, 5)$ are weighting scalars. So we establish the stochastic optimal control model of the trajectory of horizontal wells as follows:

$$\begin{aligned} \mathrm{OCP}: \ & \inf J(u) \\ & \text{s.t.} \quad X(s, u, w(s)) \in V_x(\tilde{U}_{\mathrm{ad}}, \mathscr{F}_s) \\ & \qquad\quad u \in \tilde{U}_{\mathrm{ad}}, \quad w(s) \in \mathscr{F}_s. \end{aligned}$$

From the theory on continuous dependence of solutions on parameters we know that $X(s, u, w(s))$ is continuous relative to u, so $J(u)$ is continuous on $u \in \tilde{U}_{\mathrm{ad}}$. Moreover $\tilde{U}_{\mathrm{ad}}$ is a closed bounded convex subset of R^{3n}. Hence we know the optimal control must exist by Theorem V.6.3 in [5], namely, $\exists u^* \in \tilde{U}_{\mathrm{ad}}$ such that $J(u^*) \leqslant J(u), \ \forall u \in \tilde{U}_{\mathrm{ad}}$.

For any point $s \in [0, s_n]$, define the value function

$$V(s, X) := \inf_{u \in \tilde{U}_{\mathrm{ad}}} E[\mu_0 \int_s^{s_n} \mathrm{d}s + \sum_{j=1}^{5} \mu_j (X_j(s_n, u, w(s_n)) - X_{Tj})^2],$$

and the operator $L_X^u(s)$ takes the form

$$L_X^u(s) = \frac{\varepsilon}{2} \sum_{i,j \in I_5} \frac{\partial^2}{\partial X_i \partial X_j} + \sum_{i \in I_5} f_i(X, u) \frac{\partial}{\partial X_i}.$$

Theorem 2. *Assume that $V(s, X)$ be a solution of the dynamic programming equation*

$$\frac{\partial V}{\partial s} = -\inf_{u \in \tilde{U}_{\mathrm{ad}}} [L_X^u(s) V + \mu_0], \quad (s, X) \in [0, s_n] \times \Lambda$$

with the boundary data

$$V(s_n, X) = \sum_{j=1}^{5} \mu_j (X_j(s_n, u, w(s_n)) - X_{Tj})^2.$$

If u^* is an admissible feedback control, then u^* is optimal if and only if

$$L_X^{u^*}(s) V + \mu_0 = \inf_{u \in \tilde{U}_{\mathrm{ad}}} [L_X^u(s) V + \mu_0].$$

Proof. *Sufficiency*. For each $v \in \tilde{U}_{\mathrm{ad}}(s, X) \in [0, s_n] \times \Lambda$,

$$\frac{\partial V}{\partial s} + L_X^v(s) V + \mu_0 \geqslant 0.$$

Let us replace s, X, v by $t, X(t), u(t) = u(t, X(t)), s \leqslant t \leqslant \tau$. We get

$$\frac{\partial V}{\partial s} + L_X^u(t) V + \mu_0 \geqslant 0. \tag{4}$$

we apply Theorem V.5.1 in [5] with $M=\mu_0, \psi=V$. It is obvious that

$$E\int_s^\tau |M(t,X(t))|\,\mathrm{d}t = E\int_s^\tau \mu_0\,\mathrm{d}t < \infty.$$

we get

$$V(s,X) \leqslant E\left[\mu_0\int_0^{s_n}\mathrm{d}s + \sum_{j=1}^{5}\mu_j(X_j(s_n,u,w(s_n))-X_{Tj})^2\right] = J(u)$$

We have equality in (4) for $u=u^*$. Therefore, $V(s,X)=J(u^*)$ using Theorem V.5.2 in [5]. Thus u^* is optimal.

Necessity. Applying the principle of optimality in dynamic programming we get

$$\begin{aligned}V(s,X) &= \inf_{u\in\tilde{U}_{\mathrm{ad}}} E\left[\mu_0\int_s^{s_n}\mathrm{d}s + \sum_{j=1}^{5}\mu_j(X_j(s_n,u,w(s_n))-X_{Tj})^2\right]\\ &= \inf_{u\in\tilde{U}_{\mathrm{ad}}} E\left[\mu_0\int_s^{s+h}\mathrm{d}s + \mu_0\int_{s+h}^{s_n}\mathrm{d}s + \sum_{j=1}^{5}\mu_j(X_j(s_n,u,w(s_n))-X_{Tj})^2\right]\\ &= \inf_{u\in\tilde{U}_{\mathrm{ad}}} E\left[\mu_0\int_s^{s+h}\mathrm{d}s\right] + V(s+h,X(s+h))\\ &\leqslant E\left[\mu_0\int_s^{s+h}\mathrm{d}s\right] + V(s+h,X(s+h))\end{aligned}$$

that is

$$V(s+h,X(s+h)) - V(s,X) + E[\mu_0\int_s^{s+h}\mathrm{d}s] \geqslant 0. \tag{5}$$

Multiplying h^{-1} on both sides of above formula and letting $h\to 0^+$, we find

$$\lim_{h\to 0^+}\frac{1}{h}E\left[\mu_0\int_s^{s+h}\mathrm{d}s\right] = \mu_0. \tag{6}$$

Noticing that $X(s)$ is controlled by Itô differential equation (1), we can deduce by Itô differential formula

$$\begin{aligned}&\lim_{h\to 0^+}\frac{1}{h}[V(s+h,X(s+h))-V(s,X)]\\ &\quad = \frac{1}{\mathrm{d}s}\lim_{h\to 0^+}\int_s^{s+h}\left\{\frac{\partial V(\tau,X)}{\partial s} + L_X^u(s)V(\tau,X)\right\}\mathrm{d}\tau = \frac{\partial V}{\partial s} + L_X^u(s)V.\end{aligned} \tag{7}$$

From (5)–(7) we get

$$\frac{\partial V}{\partial s} + L_X^u(s)V + \mu_0 \geqslant 0. \tag{8}$$

On the other hand, assume the optimal control u^* can be achieved on $[s, s+h]$, then

$$\frac{\partial V}{\partial s} + L_X^u(s)V + \mu_0 = 0. \tag{9}$$

From (8) and (9) we get

$$\frac{\partial V}{\partial s} = -\inf_{u\in\tilde{U}_{\mathrm{ad}}} [L_X^u(s)V + \mu_0] = -(L_X^{u^*}(s)V + \mu_0). \tag{10}$$

Thus the proof is completed. □

5. Nonlinear stochastic programming of OCP

By integrating the state equation of (1) for $s \in [s_{i-1}, s_i]$, we get that

$$X_i(s, u_i, w_i(s)) = X_{i-1}(s_{i-1}, u_{i-1}, w(s_{i-1})) + \int_{s_{i-1}}^{s} f(X_i, u_i)\,\mathrm{d}\tau + \int_{s_{i-1}}^{s} \sigma^{\varepsilon}\,\mathrm{d}w(\tau), \quad i \in I_n. \tag{11}$$

By taking $s = s_i$ in (11), we obtain that

$$X_i(s_i, u_i, w_i(s_i)) = X_{i-1}(s_{i-1}, u_{i-1}, w(s_{i-1})) + \tilde{f}(u_i, \xi_i) + \xi_i, \quad i \in I_n, \tag{12}$$

where

$$\tilde{f}(u_i, \xi_i) := \int_{s_{i-1}}^{s_i} f(X_i, u_i)\,\mathrm{d}\tau = \begin{cases} \hat{f}(u_i, \xi_i) & \cos u_i \neq 0, \\ \bar{f}(u_i, \xi_i) & \cos u_i = 0, \end{cases}$$

$$\xi_i := \int_{s_{i-1}}^{s_i} \sigma^{\varepsilon}\,\mathrm{d}w(\tau) \in V, \quad i \in I_n,$$

$$\hat{f}(u_i, \xi_i) = (\hat{f}_1, \hat{f}_2, \hat{f}_3, \hat{f}_4, \hat{f}_5)^{\mathrm{T}}$$

$$\hat{f}_1 = \frac{u_{i3}\cos u_{i2}}{u_{i1}}, \quad \hat{f}_2 = \tan u_{i2} \ln \frac{\tan(\frac{1}{2}X_{i1}(s_i, u_i, w(s_i)))}{\tan(\frac{1}{2}X_{i-1,1}(s_{i-1}, u_{i-1}, w(s_{i-1})))},$$

$$p(x) = X_{i-1,2}(s_{i-1}, u_{i-1}, w(s_{i-1})) + \tan u_{i2} \ln \frac{\tan(\frac{1}{2}x)}{\tan(\frac{1}{2}X_{i-1,1}(s_{i-1}, u_{i-1}, w(s_{i-1})))}$$

$$\hat{f}_3 = \frac{1}{u_{i1}\cos u_{i2}} \int_{X_{i-1,1}(s_{i-1},u_{i-1},w(s_{i-1}))}^{X_{i1}(s_i,u_i,w(s_i))} \sin x \cos p(x)\,\mathrm{d}x,$$

$$\hat{f}_4 = \frac{1}{u_{i1}\cos u_{i2}} \int_{X_{i-1,1}(s_{i-1},u_{i-1},w(s_{i-1}))}^{X_{i1}(s_i,u_i,w(s_i))} \sin x \sin p(x)\,\mathrm{d}x,$$

$$\hat{f}_5 = \frac{1}{u_{i1}\cos u_{i2}}(\sin X_{i1}(s_i, u_i, w(s_i)) - \sin X_{i-1,1}(s_{i-1}, u_{i-1}, w(s_{i-1}))),$$

$$\bar{f}(u_i, \xi_i) = (\bar{f}_1, \bar{f}_2, \bar{f}_3, \bar{f}_4, \bar{f}_5)^{\mathrm{T}},$$

$$\bar{f}_1 = 0, \quad \bar{f}_2 = \frac{u_{i3}}{u_{i1}\sin X_{i-1,1}(s_{i-1}, u_{i-1}, w(s_{i-1}))},$$

$$\bar{f}_3 = u_{i1}\sin^2 X_{i-1,1}(s_{i-1}, u_{i-1}, w(s_{i-1})) \times (\sin X_{i2}(s_i, u_i, w(s_i)) - \sin X_{i-1,2}(s_{i-1}, u_{i-1}, w(s_{i-1}))),$$

$$\bar{f}_4 = u_{i1}\sin^2 X_{i-1,1}(s_{i-1}, u_{i-1}, w(s_{i-1})) \times (\cos X_{i-1,2}(s_{i-1}, u_{i-1}, w(s_{i-1})) - \cos X_{i2}(s_i, u_i, w(s_i))),$$

$$\bar{f}_5 = u_{i3}\cos X_{i-1,1}(s_{i-1}, u_{i-1}, w(s_{i-1})).$$

By [1, Theorem 10.1, Chapter 3], we infer that ξ_i is an independent normal random variable whose expectancy is zero by making Itô stochastic integral of $\{w_i(s) \in \mathscr{F}_s : s \in [s_{i-1}, s_i], i \in I_n\}$, in which $V = B(0, M) \subset R^5$, M is a positive real number.

From the recurrent formula of (12), we obtain that

$$X(s_n, u, w(s_n)) \equiv X_n(s_n, u_n, w(s_n)) = X_0 + \sum_{i=1}^{n}(\tilde{f}(u_i, \xi_i) + \xi_i). \tag{13}$$

Substitute (13) for $X(s_n, u, w(s_n))$ in (3), and set

$$F(u, \xi) := \mu_0 \sum_{i=1}^{n} u_{i3} + \sum_{j=1}^{5} \mu_j \left(X_{0j} + \sum_{i=1}^{n}(\tilde{f}_j(u_i, \xi_i) + \xi_{ij}) - X_{tj} \right)^2, \tag{14}$$

where $\xi = (\xi_1, \xi_2, \ldots, \xi_n)^{\mathrm{T}} \in \tilde{V}$, $\quad \tilde{V} = \{v = (v_1, v_2, \ldots, v_n) \in R^{5n} : v_i \in V,\ i \in I_n\}$.

Assume $(R_+^{3n} \times \tilde{V}, \mathscr{L}(R_+^{3n}) \times \mathscr{F}_t, \mu \times P)$ is a measurable space, $\mathscr{L}(R_+^{3n})$ is *Lebesgue σ-field* on R_+^{3n}, μ is *Lebesgue* measure on R_+^{3n}. By (11)–(14) together with differentiability property of compound function, we can easily prove that

Theorem 3. *$F(u, \xi)$ is continuously differentiable on $R_+^{3n} \times \tilde{V}$, and $F(\cdot, \xi)$ is twice continuously differentiable relative to u on R_+^{3n}.*

Let $c=(a_1, a_2, a_3, \ldots, a_1, a_2, a_3) \in R_+^{3n}$, $d=(b_1, b_2, b_3, \ldots, b_1, b_2, b_3) \in R_+^{3n}$, and $k=3(i-1)+j$, $h_k(u)=u_{ij}-d_k$, $g_k(u) = c_k - u_{ij}$, $i \in I_n$, $j = 1, 2, 3$. Then the model OCP equals to

$$\begin{aligned} \min_{u \in R_+^{3n}} \quad & E[F(u, \xi)] \\ \text{s.t.} \quad & h_k(u) \leqslant 0, \quad k \in I_{3n}, \\ & g_k(u) \leqslant 0. \end{aligned} \tag{15}$$

As the system (1) has complete state information, by Lemma 3.1 of Chapter 8 in [1], we know

$$\min_{u \in R_+^{3n}} E[F(u, \xi)] = E\left[\min_{u \in R_+^{3n}} F(u, \xi)\right].$$

So (15) equals to seeking the expectancy of the parametric nonlinear stochastic programming:

$$\begin{aligned} \mathrm{NLP}(\xi) : \min_{u \in R_+^{3n}} \quad & F(u, \xi) \\ \text{s.t.} \quad & h_k(u) \leqslant 0, \quad k \in I_{3n} \\ & g_k(u) \leqslant 0, \end{aligned}$$

where $\xi \sim \mathscr{N}(0, \varsigma^2)$ is a stochastic parameter.

Let the active set of NLP(ξ) at any feasible point u be

$$\begin{aligned} I(u) &= \{k \in I_{3n} : h_k(u) = 0\} = \{i_1, \ldots, i_r\}, \\ J(u) &= \{k \in I_{3n} : g_k(u) = 0\} = \{j_1, \ldots, j_s\}. \end{aligned}$$

From the characterization of $h_k(u)$ and $g_k(u)$, we know that $I(u) \cap J(u) = \Phi$. Let

$$\begin{aligned} \nabla_u G_1(u) &= \{\nabla h_k(u) : k \in I(u)\}, \\ \nabla_u G_2(u) &= \{\nabla g_k(u) : k \in J(u)\}, \\ \nabla_u G(u) &= \nabla_u G_1(u) \cup \nabla_u G_2(u). \end{aligned}$$

By the definition of $h_k(u)$ and $g_k(u)$, the row vectors of $\nabla_u G_1(u)$ and $\nabla_u G_2(u)$ are linearly independent respectively. Since $I(u) \cap J(u) = \Phi$, the row vectors of $\nabla_u G(u)$ are linearly independent.

Let the Lagrangian function of NLP(u, ξ) be

$$L(u, \xi, \lambda, \eta) = F(u, \xi) + \sum_{k=1}^{3n} \lambda_k h_k(u) + \sum_{k=1}^{3n} \eta_k g_k(u),$$

so the KKT conditions of NLP(u, ξ) are

$$\nabla_u F(u,\xi) + \sum_{k=1}^{3n} (\lambda_k - \eta_k) = 0,$$
$$\lambda_k \geqslant 0, \quad \lambda_k h_k(u) = 0; \quad \eta_k \geqslant 0, \quad \eta_k g_k(u) = 0, \quad k \in I_{3n}.$$

Define the following set at any feasible point u^*

$$G = \left\{ d \in R^{3n} \,\middle|\, \begin{array}{cc} \multicolumn{2}{c}{d \neq 0} \\ \nabla h_k(u^*) d = 0, k \in I(u^*) & \lambda_k > 0 \\ \nabla g_k(u^*) d = 0, k \in J(u^*) & \eta_k > 0 \end{array} \right\}.$$

Theorem 4. *Suppose that u^* is a local solution of* NLP(ξ^*), *and let $\lambda^* = (\lambda_1^*, \lambda_2^*, \ldots, \lambda_{3n}^*)$, $\eta^* = (\eta_1^*, \eta_2^*, \ldots, \eta_{3n}^*)$ be Lagrange multipliers such that the* KKT *conditions are satisfied, then*

$$d^{\mathrm{T}} \nabla_{uu}^2 F(u^*, \xi^*) d \geqslant 0 \quad \forall d \in G$$

Proof. Since $\nabla_u G(u)$ is linearly independent, the linear independence constrained qualification is satisfied. Taking ξ^* as a parameter vector, by the same argument as in [7, proof of Theorem 12.5, Chapter 12], we conclude that the proposition is true. $\square$

Suppose that u^* is a local solution of NLP(ξ^*), it is obvious that there exists δ such that NLP(ξ) has a unique solution $\tilde{u^*}(\xi) \in B(u^*, \delta)$ when $\xi \in B(\xi^*, \varepsilon)$, so the function $f : \xi \longrightarrow \tilde{u^*}(\xi)$ can be well defined.

Theorem 5. *Suppose that u^* is a local solution of* NLP(ξ^*), *and let $\lambda^* = (\lambda_1^*, \lambda_2^*, \ldots, \lambda_{3n}^*)$, $\eta^* = (\eta_1^*, \eta_2^*, \ldots, \eta_{3n}^*)$ be Lagrange multipliers such that the* KKT *conditions are satisfied, then there are open neighborhoods U of ξ^* and V' of u^* and a function $\tilde{u^*}(\cdot)$ mapping U to V' such that $\tilde{u^*}(\xi)$ is continuous, and for each $\xi \in U$, $\tilde{u^*}(\xi)$ is the unique local solution of* NLP(ξ) *in V'.*

Proof. By Theorem 3 we get that $F(u, \xi)$ is continuously differentiable in $R_+^{3n} \times \tilde{V}$ and twice continuously differentiable of u in R_+^{3n}. It is clear that $h_k(u)$ and $g_k(u)$ are twice continuously differentiable in R_+^{3n}. So the Assumption (A1) in [10] is satisfied.

Let $I(u^*) = \{i_1^*, \ldots, i_r^*\}$, $J(u^*) = \{j_1^*, \ldots, j_s^*\}$, and represent $\nabla_u G_1(u^*)$ and $\nabla_u G_2(u^*)$ with the matrix, namely

$$\nabla_u G_1(u^*) = \begin{array}{c} \begin{array}{ccccccccc} & & i_1^* & \cdots & i_m^* & \cdots & i_r^* & \cdots & \end{array} \\ \begin{pmatrix} 0 & \cdots & 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 & \cdots & 0 \end{pmatrix} \end{array} \in R^{r \times 3n},$$

$$\nabla_u G_2(u^*) = \begin{array}{c} \begin{array}{ccccccccc} & & j_1^* & \cdots & j_n^* & \cdots & j_s^* & \cdots & \end{array} \\ \begin{pmatrix} 0 & \cdots & -1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & -1 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & -1 & \cdots & 0 \end{pmatrix} \end{array} \in R^{s \times 3n},$$

$$\nabla_u G(u^*) = \nabla_u G_1(u^*) \cup \nabla_u G_2(u^*) = \begin{pmatrix} \nabla_u G_1(u^*) \\ \nabla_u G_2(u^*) \end{pmatrix}.$$

Since $I(u^*) \bigcap J(u^*) = \Phi$, the row vectors of $\nabla_u G(u^*)$ are linearly independent. So the elementary column transformation of $\nabla_u G(u^*)$ is equal to the following matrix

$$R = \left(\begin{array}{cccccc|c} 1 &&&&&& \\ &\ddots&&&&& \\ &&1&&&& \\ &&&-1&&& 0 \\ &&&&\ddots&& \\ &&&&&-1& \end{array}\right) \begin{array}{c} 1 \\ \vdots \\ r \\ 1 \\ \vdots \\ s \end{array}$$

that is, there exists an invertible matrix such that

$$\nabla_u G(u^*) Q = R.$$

By taking $\gamma = (-1, \dots, -1, 1, \dots, 1, 0, \dots, 0) \in R^{3n}$, we yield that $R\gamma < 0$. Let $\gamma' = Q\gamma$, then

$$\nabla_u G(u^*)\gamma' = \nabla_u G(u^*) Q\gamma = R\gamma < 0.$$

So the Assumption (A2) in [10] is satisfied.

Because $\nabla^2_{uu} L(u^*, \xi^*, \lambda^*, \eta^*) = \nabla^2_{uu} F(u^*, \xi^*)$, together with Theorem 4, we get

$$d^{\mathrm{T}} \nabla^2_{uu} L(u^*, \xi^*, \lambda^*, \eta^*) d = d^{\mathrm{T}} \nabla^2_{uu} F(u^*, \xi^*) d \geqslant 0 \quad \forall d \in G$$

So the Assumption (A3) in [10] is satisfied. By Theorem 1 in [10], we obtain that the conclusion is correct. □

6. Optimization algorithm and computer simulation

According to Theorem 5, we know that there exists a $\delta > 0$ such that $\xi \in B(0, \delta)$ implies that the local solution of NLP(ξ) is in the neighborhood of that of NLP(0). Therefore, we can take the local solution of NLP(0) as the initial feasible point to gain that of NLP(ξ).

The uniform designs proposed by Wang and Fang (1981) scatter points uniformly over the experimental domain. They have the advantage of providing a good representation of the experimental domain with fewer runs. Computer experiments using uniform designs have attracted considerable attention in recent years. Traditionally the uniform designs were generated by so-called good lattice point method, cutting method and resolvable balanced incomplete block designs [3] etc.

Hooke–Jeeves algorithm is a pattern search method to unconstrained optimization of nonlinear functions that are not necessarily continuous or differentiable. It does not require the derivatives of the objective function, and the iterative operation is very simple. For each iteration, the algorithm goes through a series of exploratory (directional) searches and one pattern search. However, its convergence rate is slow, and what we want to solve is the optimization in a bounded domain. Moreover, since the objective function is not unimodal, the choice of initial point determines how fast the algorithm converges. So we take some modifications to the Hooke–Jeeves algorithm:

(i) Use uniform design algorithm to generate the initial points in control domain, and decompose the control domain into finite subdomains.

(ii) Assess and modify the iterative point to make it satisfy the constraints and guarantee the descent of objective function. When the iterative point is out of the subdomain in some exploratory search or pattern search, we adopt the bound of subdomain as its value.

(iii) In order to improve the convergence rate, we use an acceleration factor such that the descent degree of objective function is increased. Our revised Hooke–Jeeves algorithm use the following line search scheme in each exploratory search and pattern search. Let $[a, b]$ be the interval on which the line search is to be performed. Let x and d be the initial point and search direction, respectively. Given step-size β, acceleration factor γ, accuracy ε, perform the following steps:

(1) $h^* := h := a$, $\min := f(x + h^* d)$.
(2) $h := h + \beta$. If $h > b$, go to step (4); otherwise go to step (3).

Table 1
Control parameter

		NLP(0)	$E[\tilde{u^*}(\xi)]$	NLP(0)	$E[\tilde{u^*}(\xi)]$
1	Radius of curvature (m)	59.7383	54.9761	48.0277	48.6275
	Tool-face angle (rad)	0.8717	0.8714	0.5769	0.5672
	Curve length (m)	12.3003	12.3013	21.1415	21.1415
2	Radius of curvature (m)	40.1720	40.7798	40.0000	40.1511
	Tool-face angle (rad)	0.8159	0.8481	0.7309	0.7190
	Curve length (m)	14.3403	14.3391	31.9019	31.9019
3	Radius of curvature (m)	42.7674	42.9243	42.2298	40.1512
	Tool-face angle (rad)	0.2571	0.2579	0.6407	0.7069
	Curve length (m)	53.4009	54.0764	26.9231	26.9231
Precision of hitting target (m)		0.4217	0.2277	0.7933	0.8211
Total curve length (m)		80.0416	80.7170	79.9665	79.9665

(3) If $f(x + hd) < \min$, then $h^* := h$, $\min := f(x + h^*d)$, go to step (2); otherwise $h := h - \beta$, $\beta := 1.1\beta$, go to step (2).
(4) $a := h^* - \gamma$, $b := h^* + \gamma$, $\min := f(x + h^*d)$, $h^* := h := a$, $\beta := 0.9\beta$, $h := h + \beta$. If $h > b$, go to step (5); otherwise go to step (3).
(5) $\gamma := 0.9\gamma$. If $\gamma < \varepsilon$, output h^*, stop; otherwise go to step (4).

The basic steps of the algorithm are given as follows.

Algorithm 1. *Step* 1: Construct m initial points in $\tilde{U}_{\mathrm{ad}}$ by the good lattice point method [3], and decompose $\tilde{U}_{\mathrm{ad}}$ into m subdomains.

Step 2: Use the revised Hooke–Jeeves algorithm to gain the local solution of NLP(0) for each initial point in the corresponding subdomain.

Step 3: Take the local solution of NLP(0) as the initial point of NLP(ξ), and generate a sequence of independent normal random vector $\{\xi_k\}$ whose expectancy is zero. Use the revised Hooke–Jeeves algorithm to gain the local solution of NLP(ξ_k), named as $\tilde{u^*}(\xi_k)$.

Step 4: Calculate $E[u^*(\xi_k)]$ and the expectancy of performance criterion.

Example. According to the model and algorithm mentioned above, we have programmed the software and applied it to the optimal design of several horizontal wells in Liaohe oil field. Here the optimal design of the trajectory of Well $Ci - 16 - Cp146$ is given. It is a short-radius well, and the basic data are listed, respectively, as follows:

Kick-off point:

$X_{01} = 0.18$ rad, $X_{02} = 3.98$ rad, $X_{03} = 102.7$ m, $X_{04} = -156.4$ m, $X_{05} = 1673.2$ m

Target:

$X_{T1} = 1.56$ rad, $X_{T2} = 3.53$ rad, $X_{T3} = 62.5$ m, $X_{T4} = -192.9$ m, $X_{T5} = 1718.0$ m the range of Radius of curvature, tool-face angle and curve length are [40, 60], [0, 1.4], [10, 100], respectively.

We adopt $n = 3$, $m = 50$ and $\alpha = 0.1$ rad in the procedure. We take $E[\tilde{u^*}(\xi)]$ as the local solution of OCP and acquire 16 local solutions. Two groups are arranged in Table 1. to show control parameters' comparisons of NLP(0) and $E[\tilde{u^*}(\xi)]$. By comparison of a large quantity of calculation results, we find that the solution of NLP(ξ) distributes around that of NLP(0) densely, which coincides with the conclusion of Theorem 5. The example shows the revised Hooke–Jeeves algorithm is reliable and efficient.

7. Conclusions

Unlike the approaches in [6,9], the presented model in the paper specifies the effects of random perturbations while drilling. Therefore, the results of this research are much more rational and practical. It is shown from the real examples that the revised Hooke–Jeeves method is efficient and robust. The algorithm is technically superior to the trial-and-error techniques traditionally used for designing the trajectory of horizontal wells.

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References

[1] K.J. Åström, Introduction to Stochastic Control Theory, Academic Press, New York, 1970.
[2] A. Elnagar, A. Hussein, On optimal constrained trajectory planning in 3D environments, Robot. Autonomous Systems 33 (2000) 195–206.
[3] K. Fang, Y. Wang, Number-Theoretic Methods in Statistics, Chapman & Hall, London, 1994.
[4] E. Feng, W. Qian, A stochastic approach to global optimization of nonlinear programming problems with many equality constraints, Appl. Math. Comput. 115 (2004) 111–120.
[5] W.H. Fleming, R.W. Rishel, Deterministic and Stochastic Optimal Control, Springer, Berlin, Heidelberg, New York, 1975.
[6] R.C. McCann, P.V.R. Suryanarayana, Horizontal well path planning and correction using optimization techniques, J. Energy Resour. 123 (2001) 187–193.
[7] J. Nocedal, S.J. Wright, Numerical Optimization, Science Press, Beijing, 2006.
[8] W. Qian, E. Feng, Optimization method of modifying 3-D horizontal wells trajectory while drilling and its application, J. Dalian Univ. Technol. 44 (2002) 610–615.
[9] W. Qian, E. Feng, Multi-objectives fuzzy models for designing 3D trajectory in horizontal wells, J. Appl. Math. Comput. 15 (2004) 265–275.
[10] D. Ralph, S. Dempe, Directional derivatives of the solution of parametric nonlinear program, Math. Programming 70 (1995) 159–172.