Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations 233 (2007) 87–104

Journal of Differential Equations

www.elsevier.com/locate/jde

Invariant measures and regularity properties of perturbed Ornstein–Uhlenbeck semigroups ☆

Abdelhadi Es-Sarhir, Bálint Farkas *

Technische Universität Darmstadt, Fachbereich Mathematik, AG4, Schloßgartenstraße 7, D-64289 Darmstadt, Germany

Received 26 October 2005; revised 21 June 2006

Available online 1 November 2006

Dedicated to Rainer Nagel on the occasion of his 65th birthday

Abstract

This paper deals with perturbations of the Ornstein–Uhlenbeck operator on L^2-spaces with respect to a Gaussian measure μ. We perturb the generator of the Ornstein–Uhlenbeck semigroup by a certain unbounded, non-linear drift, and show various properties of the perturbed semigroup such as compactness and positivity. Strong Feller property, existence and uniqueness of an invariant measure are discussed as well.

MSC: 46A03; 47D06; 47D07; 47D99; 47A55

Keywords: Existence and uniqueness of invariant measures

1. Introduction and preliminaries

The starting point in this work is the following second-order problem in an infinite-dimensional Hilbert space H

$$\begin{cases} u'(t,x) = \frac{1}{2}\operatorname{Tr} D^2u(t,x) + \langle Ax, Du(t,x)\rangle + \langle F(x), (-A^*)^\gamma Du(t,x)\rangle, & t \geqslant 0, \\ u(0,x) = f(x), & x \in H. \end{cases} \tag{1}$$

☆ The support by the European Community in the framework of the Research Training Network "Evolution equations for deterministic and stochastic systems;" contract number HPRN-CT-2002-00, is gratefully acknowledged.

* Corresponding author.
E-mail addresses: essarhir@mathematik.tu-darmstadt.de (A. Es-Sarhir), farkas@mathematik.tu-darmstadt.de (B. Farkas).

0022-0396/$ – see front matter
doi:10.1016/j.jde.2006.09.022

Here A is a linear operator on H, D denotes the Fréchet derivative and F is a function mapping H into itself (the precise assumptions and ingredients will be formulated later). Such problems are successfully handled by functional analytic, actually operator-semigroup theoretic methods (as presented, e.g., in [12]). As a general reference to this approach we mention the monographs by Da Prato [7] and Da Prato, Zabczyk [10,11].

The Ornstein–Uhlenbeck semigroups corresponding to problem (1) for $F \equiv 0$ is studied in, e.g., Da Prato [5,6], Da Prato, Zabczyk [9–11], van Neerven [25] and van Neerven, Zabczyk [26]. For $\gamma = 0$ the general case, using perturbation arguments, is then treated by Da Prato in [5, 6] and also by Chojnowska-Michalik [2], Chojnowska-Michalik, Goldys [3,4], Goldys, Kocan [16], Rhandi [23]. Our investigation is inspired by Da Prato's paper [6] where he considers the critical case $\gamma = 1/2$ and works in the space of bounded uniformly continuous functions on H. His work concentrates rather on the corresponding elliptic problem and uses perturbation and dissipativity methods. Our idea is to work on the space $L^2(H,\mu)$, where μ is the invariant measure for the Ornstein–Uhlenbeck semigroup, and to obtain the solution of the above problem by perturbation techniques, more precisely by using the Miyadera–Voigt perturbation theorem, from semigroup theory.

We also obtain qualitative properties such as compactness, positivity and under certain conditions the irreducibility of $(P_t)_{t\geqslant 0}$ (see Sections 2 and 4). These imply immediately, thanks to a spectral theoretic argument, the existence of an invariant measure ν of the above Cauchy problem. We stress here that the L^2-approach makes it also possible to show the strong Feller property of the semigroup $(P_t)_{t\geqslant 0}$, which means that for all bounded, measurable functions f and $t > 0$ we have $P_t f \in C_b(H)$ (where $C_b(H)$ denotes the Banach space of bounded and continuous function on H). This particularly involves that the space $C_b(H)$ is invariant under the semigroup $(P_t)_{t\geqslant 0}$.

We will also see that the restricted semigroup $(\tilde{P}_t)_{t\geqslant 0}$ is a Markov semigroup on $C_b(H)$. Denote by $X(t)$ the corresponding stochastic process on H. Then for suitable functions F, $X(t)$ is a solution of the following stochastic differential equation (see Zambotti [31])

$$\begin{cases} \mathrm{d}X(t) = AX(t)\,\mathrm{d}t + (-A)^{\gamma} F(X(t)) + \mathrm{d}W(t), \\ X(0) = x \in H, \end{cases}$$

here $W(t)$ is a cylindrical Wiener process on the real, separable Hilbert space H, with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Further, the invariant measure ν is proved to have finite second moment (because it is absolutely continuous with respect to the Gaussian measure μ—having all finite moments—and its density ρ is in $L^2(H,\mu)$), so this means that the above stochastic equation admits a stationary solution with stationary distribution ν.

Of course, $(P_t)_{t\geqslant 0}$ is a contractive semigroup on $C_b(H)$, but using a spectral property of this semigroup, we can even show that under some additional conditions it is bounded on $L^2(H,\mu)$ (see Theorems 4.2 and 4.3). In this way we generalize the results obtained in [3] and [23], where only the case $\gamma = 0$ was treated.

Now we comment on the case $\gamma = 0$. It was proved in [3] that the invariant measure is unique without the assumption of F being Lipschitz. However, for the sake of the simple, more or less functional analytic treatment we included this additional assumption (see Corollary 4.4). In [2, Corollary 6.3], it is shown that the semigroup $(P_t)_{t\geqslant 0}$ is quasi-contractive, our result of (polynomial) boundedness (Theorem 4.2) complements this, showing that the semigroup *cannot* grow exponentially. Polynomial boundedness is improved to boundedness in presence of the irreducibility of $(P_t)_{t\geqslant 0}$ (see Theorem 4.3).

Actually, we will quickly leave behind our starting problem, and concentrate on the semigroup theoretic results, whose interpretation concerning the properties of the equation above is fairly standard.

Let us set up now the framework for our investigations. Consider the stochastic differential equation (corresponding to the case $F \equiv 0$ in the above problem)

$$\begin{cases} \mathrm{d}Z(t) = AZ(t)\,\mathrm{d}t + \mathrm{d}W(t), \\ Z(0) = x \in H, \end{cases} \tag{2}$$

where A is a linear operator in H satisfying the following hypothesis.

Hypothesis 1.1.

(a) The operator $(A, D(A))$ generates an analytic semigroup $(e^{tA})_{t \geqslant 0}$.
(b) For $t > 0$ the operator e^{tA} is Hilbert–Schmidt and

$$\int_0^\infty \|e^{tA}\|_{\mathrm{HS}}^2\,\mathrm{d}t < +\infty,$$

where $\|\cdot\|_{\mathrm{HS}}$ stands for the Hilbert–Schmidt operator norm.

It is well known that under this hypothesis, Eq. (2) has an H-valued mild solution (see, e.g., Da Prato, Zabczyk [9]) which is a Gaussian Markov process given by

$$Z(t,x) = e^{tA}x + \int_0^t e^{(t-s)A}\,\mathrm{d}W(s). \tag{3}$$

The distribution of the process $Z(t,x)$ is easily described. To that purpose define

$$Q_t = \int_0^t e^{sA}e^{sA^*}\,\mathrm{d}s, \quad t > 0.$$

From the assumptions above it follows that for $t > 0$ the operators Q_t, and even $Q_\infty := \lim_{t\to\infty} Q_t$, are of trace class. The Gaussian measures $\mathcal{N}_{e^{tA}x, Q_t}$ on H with mean $e^{tA}x$ and covariance Q_t exist for all $t > 0$ and $x \in H$. This family of measures gives the distribution of the process $Z(t,x)$. The transition semigroup associated to this process is called the *Ornstein–Uhlenbeck semigroup* and is defined by

$$(R_t\varphi)(x) := \mathbf{E}\big(\varphi\big(Z(t,x)\big)\big) \quad \text{for all } \varphi \in C_b(H), \tag{4}$$

where $\mathbf{E}$ denotes the expectation. Using the distribution of the process $Z(t,x)$, we can write (4) as

$$(R_t\varphi)(x) = \int_H \varphi(y)\,\mathrm{d}\mathcal{N}_{e^{tA}x, Q_t}(y), \quad \varphi \in C_b(H),\ x \in H.$$

A change of variables gives the expression

$$(R_t\varphi)(x) = \int_H \varphi\big(e^{tA}x + y\big)\, \mathrm{d}\mathcal{N}_{Q_t}(y), \quad t \geqslant 0,\ \varphi \in C_b(H),\ x \in H, \tag{5}$$

where $\mathcal{N}_{Q_t}$ is the Gaussian measure on H of mean 0 and covariance operator Q_t.

It is well known that the semigroup $(R_t)_{t\geqslant 0}$ is not strongly continuous on the spaces $C_b(H)$ and $UC_b(H)$ (see Cerrai [1] and van Neerven, Zabczyk [26]). In order to get around this, we will consider the L^2-space with respect to an *invariant measure*. From [10, Section 6.2.1] we know that the measure $\mu = \mathcal{N}_{0,Q_\infty}$ is invariant for $(R_t)_{t\geqslant 0}$ (and it is even unique), which means that

$$\int_H R_t\varphi(x)\, \mathrm{d}\mu(x) = \int_H \varphi(x)\, \mathrm{d}\mu(x) \quad \text{for all } \varphi \in C_b(H). \tag{6}$$

Therefore, we can extend $(R_t)_{t\geqslant 0}$ to a C_0-semigroup of contractions on $L^2(H,\mu)$.

We denote by $(L, D(L))$ the generator of the semigroup $(R_t)_{t\geqslant 0}$, and we define the operator $(L_0, D(L_0))$ with $D(L_0) := \mathcal{F}_b^\infty(H)$, and for $x \in H$

$$L_0\varphi(x) := \frac{1}{2}\operatorname{Tr} D^2\varphi(x) + \big\langle x, A^* D\varphi(x)\big\rangle, \quad \varphi \in D(L_0), \tag{7}$$

where $\mathcal{F}_b^\infty(H)$ is the space of cylindrical functions:

$$\mathcal{F}_b^\infty(H) := \big\{ f \in C_b^\infty(H),\ f(x) = f_m\big(\langle x, h_1\rangle, \ldots, \langle x, h_m\rangle\big),\ f_m \in C_b^\infty\big(\mathbb{R}^m\big)\big\},$$

where $(h_i)_{1\leqslant i\leqslant m}$ are elements from $D(A^*)$. For a function $\varphi \in \mathcal{F}_b^\infty(H)$, we denote by $D\varphi$ its Fréchet derivative.

It follows from [3, Lemma 3] that $(D, \mathcal{F}_b^\infty(H))$ is closable in $L^2(H,\mu)$ and if we denote again by D its closure then its domain is the Sobolev space $W^{1,2}(H,\mu)$ which is defined as the completion of $\mathcal{F}_b^\infty(H)$ in the $L^2(H,\mu)$ with respect to the norm defined through

$$\|\varphi\|_{W^{1,2}}^2 = \int_H \big|\varphi(x)\big|^2\, \mathrm{d}\mu(x) + \int_H \big\|D\varphi(x)\big\|^2\, \mathrm{d}\mu(x).$$

It was proved in [8] that

$$W^{1,2}(H,\mu) \hookrightarrow L^2(H,\mu) \quad \text{is a compact embedding.} \tag{8}$$

The following proposition from [3] gives a description of the generator of the Ornstein–Uhlenbeck semigroup.

Proposition 1.2. *Assume Hypothesis* 1.1. *Then $D(L_0)$ is invariant under the action of R_t and the generator L of the Ornstein–Uhlenbeck semigroup $(R_t)_{t\geqslant 0}$ on $L^2(H,\mu)$, is a unique extension of L_0 to a generator of a C_0-semigroup. Moreover, L is the closure of L_0.*

For the sake of completeness we repeat here the already classical theorem Miyadera [19] and Voigt [27, Theorem 1] (see also [12, Corollary III.3.16]).

Theorem 1.3. *Let $(R_t)_{t\geqslant 0}$ be a C_0-semigroup on a Banach space E with generator $(A, D(A))$. Consider an A-bounded linear operator $(B, D(B))$ such that there are constants $\alpha > 0$, $q \in [0,1)$ with*

$$\int_0^\alpha \|BR_t\varphi\|\,\mathrm{d}t \leqslant q\|\varphi\| \quad \text{for } \varphi \in D(A). \tag{9}$$

Then the following assertions hold.

(a) *The operator $A+B$ with domain $D(A)$ generates a C_0-semigroup $(P_t)_{t\geqslant 0}$ on E given by the Dyson–Phillips series*

$$P_t = \sum_{n=0}^{\infty} U_n(t), \quad t \geqslant 0, \tag{10}$$

where the operators $U_0(t) := R_t$ and $U_{n+1}(t)\varphi := \int_0^t U_n(t-s)BR_s\varphi\,\mathrm{d}s$ for $t \geqslant 0$ and $\varphi \in D(A)$ extend continuously to the whole space E. The series in (10) *converges in the operator norm uniformly for $t \geqslant 0$ in compact intervals.*

(b) *For $\varphi \in D(A)$ and $t \geqslant 0$, we have the integral equations*

$$P_t\varphi = R_t\varphi + \int_0^t P_{t-s}BR_s\varphi\,\mathrm{d}s, \tag{11}$$

$$P_t\varphi = R_t\varphi + \int_0^t R_{t-s}BP_s\varphi\,\mathrm{d}s. \tag{12}$$

The estimate $\|P_t\| \leqslant Me^{\omega t}$ holds with constants $M > 0$, $\omega \in \mathbb{R}$ dependent only on the semigroup $(R_t)_{t\geqslant 0}$ and the constants α and q.

2. Perturbation of the Ornstein–Uhlenbeck semigroup

In this section we consider perturbations of the Ornstein–Uhlenbeck semigroup by non-linear drifts as, e.g., in [3] and [23]. We consider the linear operator $(B, D(B))$ defined by

$$D(B) := D(L) \quad \text{and} \quad B\varphi(x) := \bigl\langle F(x), (-A^*)^\gamma D\varphi(x)\bigr\rangle, \quad x \in H,$$

where $F \in C_b(H,H)$ and $\gamma \in [0, 1/2)$. This definition is indeed meaningful: Lemma 2.3 below shows that for $\varphi \in D(L)$ and $x \in H$ we have $D\varphi(x) \in D((-A^*)^\gamma)$. This perturbing operator appears as the third term in (1), thus taking into account the description of the Ornstein–Uhlenbeck operator (7), our ultimate aim is to show the generator property of $L+B$. For this purpose we will check the appropriate assumptions of the Miyadera–Voigt perturbation Theorem 1.3.

It is known that in our case $\operatorname{rg} e^{tA} \subseteq \operatorname{rg} Q_t^{1/2}$ for $t > 0$, see [9, Corollaries 9.22, 9.23]. Thus the Cameron–Martin formula [11, Theorem 1.3.6] provides the Radon–Nikodým derivative of the equivalent measures $\mathcal{N}_{Q_t}$ and $\mathcal{N}_{e^{tA}x,Q_t}$

$$\frac{\mathrm{d}\mathcal{N}_{e^{tA}x,Q_t}}{\mathrm{d}\mathcal{N}_{Q_t}}(y) = e^{-1/2\|\Lambda_t x\|^2+\langle Q_t^{-1/2}y,\Lambda_t x\rangle} \quad \text{for } y \in H,$$

$$\text{with } \Lambda_t := Q_t^{-\frac{1}{2}}e^{tA}, \quad t>0.$$

Further, Λ_t satisfies the following norm estimate, see [9, Corollary 9.22]

$$\|\Lambda_t\| \leqslant ct^{-1/2}, \quad t>0. \tag{13}$$

For our purposes however we will need an estimate for $Q_t^{-\frac{1}{2}}e^{\frac{t}{2}A}$ as well. This can be obtained analogously to the case of Λ_t (see [9, Appendix B]), but for the sake of completeness we sketch the argument. Let us fix $t>0$ and consider the Hilbert space $U = L^2([0,t],H)$ and the bounded operator $J: U \to H$, $Ju := \int_0^t e^{(t-s)A}u(s)\,\mathrm{d}s$. Then the adjoint operator $J^*: H \to U$ is given by $(J^*x)(s) = e^{(t-s)A^*}x$, so we have $Q_t = JJ^*$. Now Corollary B.4 in [9] gives that

$$\big\|Q_t^{-\frac{1}{2}}e^{\frac{t}{2}A}x\big\|^2 = \big\|J^{-1}x\big\|^2 = \min\big\{\|u\|_U^2\colon\ Ju = e^{\frac{t}{2}A}x\big\}. \tag{14}$$

Define the function

$$u(s) := \begin{cases} 0, & s < t/2, \\ \frac{2}{t}e^{(s-t/2)A}x, & s \geqslant t/2. \end{cases}$$

Then we have $u \in U$ and

$$Ju = \int_0^t e^{(t-s)A}u(s)x\,\mathrm{d}s = \frac{2}{t}\int_{t/2}^t e^{(t-s)A}e^{(s-t/2)A}x\,\mathrm{d}s = e^{t/2A}x.$$

So from (14) we obtain

$$\big\|Q_t^{-\frac{1}{2}}e^{\frac{t}{2}A}x\big\|^2 \leqslant \|u\|_U^2 = \frac{4}{t^2}\int_0^{t/2}\big\|e^{sA}x\big\|^2\,\mathrm{d}s \leqslant \frac{c_1}{t}\|x\|^2,$$

hence

$$\big\|Q_t^{-\frac{1}{2}}e^{\frac{t}{2}A}x\big\| \leqslant \frac{c_2}{t^{1/2}}\|x\|.$$

For $x \in D((-A)^\gamma)$ we obtain the following estimate

$$\big\|\Lambda_t(-A)^\gamma x\big\| = \big\|Q_t^{-\frac{1}{2}}e^{\frac{t}{2}A}e^{\frac{t}{2}A}(-A)^\gamma x\big\| \leqslant \frac{c_2}{t^{1/2}}\cdot\frac{c}{t^\gamma}\|x\| = \frac{c_\gamma}{t^{1/2+\gamma}}\|x\|, \tag{15}$$

where we used that $(A, D(A))$ generates an analytic semigroup and hence $\|t^\gamma(-A)^\gamma e^{tA}\|$ is bounded near 0 (see [21, Theorem 6.13]).

The following proposition gives the crucial estimates needed for the application of Theorem 1.3. An immediate consequence is also the compactness of the semigroup $(R_t)_{t\geqslant 0}$ (see [11, Proposition 10.3.1]).

Proposition 2.1. *Assume that Hypothesis* 1.1 *holds and take $\varphi \in L^2(H,\mu)$. Then for any $t>0$ we have $DR_t\varphi \in L^2(H,\mu)$, and*

$$\|DR_t\varphi\|_2 \leqslant c_0 t^{-\frac{1}{2}}\|\varphi\|_2. \tag{16}$$

Moreover, R_t is compact for all $t>0$.

Even more can be said: the following lemma gives further estimates on the derivatives of the orbits of the semigroup R_t being vital for the perturbation theorem.

Lemma 2.2. *Let $\varphi \in L^2(H,\mu)$ and $t>0$. Then $DR_t\varphi(x) \in D((-A^*)^\gamma)$ for all $x \in H$. Moreover $\|(-A^*)^\gamma DR_t\varphi(\cdot)\| \in L^2(H,\mu)$, and we have*

$$\big\|(-A^*)^\gamma DR_t\varphi(x)\big\|^2 \leqslant c_\gamma t^{-2(\gamma+1/2)} R_t\varphi^2(x) \quad \textit{for } x \in H,\ t>0 \textit{ and}, \tag{17}$$

$$\big\|(-A^*)^\gamma DR_t\varphi\big\|_2 \leqslant c_\gamma t^{-(\gamma+1/2)}\|\varphi\|_2, \quad t>0. \tag{18}$$

Proof. Let $\varphi \in L^2(H,\mu)$, $t>0$ and $h \in D((-A)^\gamma)$. Then

$$\big\langle DR_t\varphi(x), (-A)^\gamma h\big\rangle = \int_H \big\langle \Lambda_t(-A)^\gamma h, Q_t^{-1/2}y\big\rangle \varphi\big(e^{tA}x+y\big)\,\mathrm{d}\mathcal{N}_{Q_t}(y), \quad x \in H.$$

By the Hölder inequality and then using (15) it follows that

$$\begin{aligned}\big|\big\langle DR_t\varphi(x), (-A)^\gamma h\big\rangle\big|^2 &\leqslant \big|R_t\varphi^2(x)\big| \int_H \big|\big\langle \Lambda_t(-A)^\gamma h, Q_t^{-1/2}y\big\rangle\big|^2\,\mathrm{d}\mathcal{N}_{Q_t}(y)\\ &\leqslant \big|R_t\varphi^2(x)\big|\cdot c_\gamma^2 t^{-2(\gamma+1/2)}\|h\|^2, \quad x \in H.\end{aligned}$$

Hence from the arbitrariness of h we obtain $DR_t\varphi(x) \in D((-A^*)^\gamma)$ and

$$\big\|(-A^*)^\gamma DR_t\varphi(x)\big\| \leqslant c_\gamma t^{-(\gamma+1/2)}\big|R_t\varphi^2(x)\big|^{1/2}, \quad x \in H.$$

By integrating on H, and using the invariance of the measure μ we obtain inequality (18). □

For the resolvent $R(\lambda, L)$ of the generator $(L, D(L))$ we have the following lemma.

Lemma 2.3. *Suppose that Hypothesis* 1.1 *holds, and let $\varphi \in L^2(H,\mu)$. Then $DR(\lambda,L)\varphi(x) \in D((-A^*)^\gamma)$ for $\lambda>0$ and $x \in H$, and we have*

$$\big\|(-A^*)^\gamma DR(\lambda,L)\varphi\big\|_2 \leqslant c_{2,\gamma}\lambda^{-(1/2+\gamma)}\Gamma(1/2-\gamma)\|\varphi\|_2. \tag{19}$$

In particular, we have $BR(\lambda,L) \in \mathcal{L}(L^2(H,\mu))$.

Proof. From Lemma 2.2 we have for each $\varphi \in L^2(H,\mu)$ and $t>0$, $(-A^*)^\gamma DR_t\varphi \in L^2(H,\mu)$ and

$$\begin{aligned}\left\|(-A^*)^\gamma DR_t - (-A^*)^\gamma DR_s\right\|_2 &= \left\|(-A^*)^\gamma DR_s(R_{t-s}\varphi - \varphi)\right\|_2 \\ &\leqslant c_\gamma s^{-(\gamma+1/2)}\|R_{t-s}\varphi - \varphi\|_2\end{aligned}$$

for $t>s>0$. Hence the function

$$0<t\mapsto (-A^*)^\gamma DR_t \quad \text{is strongly continuous.}$$

On the other hand, from the estimate (18) we obtain

$$\int_0^\infty e^{-\lambda t}\left\|(-A^*)^\gamma DR_t\varphi\right\|_2 \mathrm{d}t < +\infty \quad \text{for all } \varphi\in L^2(H,\mu) \text{ and } \lambda>0.$$

So for each $\varphi \in L^2(H,\mu)$ and $\lambda>0$, we have

$$(-A^*)^\gamma DR(\lambda,L)\varphi = \int_0^\infty e^{-\lambda t}(-A^*)^\gamma DR_t\varphi\,\mathrm{d}t, \quad \text{and}$$
$$(-A^*)^\gamma DR(\lambda,L)\varphi \in L^2(H,\mu).$$

Hence estimate (18) and some calculation give the desired inequality in the lemma. The fact that $BR(\lambda,L)$ is a bounded operator on $L^2(H,\mu)$ follows directly, since F is bounded. □

After these preparations we are able to prove the following perturbation theorem.

Theorem 2.4. *Assume that Hypothesis* 1.1 *holds. Then the operator $N := L+B$ with domain $D(N) := D(L)$ generates a compact, analytic strongly continuous semigroup $(P_t)_{t\geqslant 0}$ on $L^2(H,\mu)$, satisfying the integral equation*

$$P_t\varphi = R_t\varphi + \int_0^t P_{t-s}BR_s\varphi\,\mathrm{d}s \tag{20}$$

for all $t\geqslant 0$ and $\varphi\in L^2(H,\mu)$, and the operator BP_t has a bounded extension to $L^2(H,\mu)$ for each $t>0$. Moreover, we have

$$\int_0^T \|BP_t\varphi\|_2\,\mathrm{d}t < +\infty \quad \textit{for each } T>0. \tag{21}$$

Furthermore, the integral equation

$$P_t\varphi = R_t\varphi + \int_0^t R_{t-s} B P_s \varphi \,\mathrm{d}s \tag{22}$$

holds for all $t \geqslant 0$ *and* $\varphi \in L^2(H,\mu)$. *The generator* $(N, D(N))$ *is the closure of the operator defined for* $\varphi \in \mathcal{F}_b^\infty(H)$ *and* $x \in H$ *by*

$$N_0\varphi(x) := \frac{1}{2}\operatorname{Tr} D^2\varphi(x) + \langle Ax, D\varphi(x)\rangle + \langle F(x), (-A^*)^\gamma D\varphi(x)\rangle.$$

Proof. We use Theorem 1.3, the Miyadera–Voigt perturbation theorem for strongly continuous semigroups. Lemma 2.3 implies that the operator $(B, D(B))$ is L-bounded, hence we only need to check (9) in Theorem 1.3 for B and $(R_t)_{t\geqslant 0}$. From the proof of Lemma 2.3 one can see that the function $0 < t \mapsto BR_t\varphi \in L^2(H,\mu)$ is continuous and by (18) for any $q \in (0,1)$ there exists $\alpha := \alpha(q) > 0$ such that

$$\int_0^\alpha \|BR_t\varphi\|_2 \,\mathrm{d}t \leqslant c_\gamma \|F\|_\infty \|\varphi\|_2 \int_0^\alpha t^{-(\gamma+1/2)} \,\mathrm{d}t \leqslant q\|\varphi\|_2. \tag{23}$$

Therefore $(N, D(L))$ generates a strongly continuous semigroup $(P_t)_{t\geqslant 0}$ on $L^2(H,\mu)$ and the integral equations (20), (22) hold for all $\varphi \in D(L)$. Since (23) is satisfied for all $q \in (0,1)$, by [12, Ex. III.2.18.2] it follows that the L-bound of B is 0. The Ornstein–Uhlenbeck semigroup $(R_t)_{t\geqslant 0}$ is under our assumptions analytic (see [15]), so [12, Theorem III.2.10] shows $(P_t)_{t\geqslant 0}$ to be analytic, too. Since $D(L)$ is dense in $L^2(H,\mu)$, it follows from (18) and Lebesgue's theorem that (20) holds for all $\varphi \in L^2(H,\mu)$.

Let us now prove (21) and (22). It follows from Theorem 1.3 and (18) that the operator P_t is given by

$$P_t\varphi = \sum_{n=0}^\infty U_n(t)\varphi \quad \text{for } \varphi \in L^2(H,\mu),$$

where $U_0(t)\varphi := R_t\varphi$ and $U_{n+1}(t)\varphi := \int_0^t U_n(t-s)BR_s\varphi \,\mathrm{d}s$ for all $t \geqslant 0$ and $\varphi \in D(L)$. By Lemma 2.2 and (13) we have that

$$\left\|(-A^*)^\gamma DR_t\varphi\right\|_2 \leqslant c_{2,\gamma} t^{-(\gamma+1/2)}\|\varphi\|_2$$

for all $t > 0$ and $\varphi \in L^2(H,\mu)$. For U_1 we also have $(-A^*)^\gamma DU_1(t)\varphi \in L^2(H,\mu)$ and

$$\begin{aligned}\left\|(-A^*)^\gamma DU_1(t)\varphi\right\|_2 &\leqslant \int_0^t \left\|(-A^*)^\gamma DR_{t-s}BR_s\varphi\right\|_2 \,\mathrm{d}s \\ &\leqslant c_{2,\gamma}\int_0^t (t-s)^{-(\gamma+1/2)}\|BR_s\varphi\|_2 \,\mathrm{d}s\end{aligned}$$

$$\leqslant c_{2,\gamma}^2 \|F\|_\infty t^{-(\gamma+1/2)} \left[t^{1/2-\gamma} \int_0^1 (1-s)^{-(\gamma+1/2)} s^{-(\gamma+1/2)} \, \mathrm{d}s \right] \|\varphi\|_2$$
$$\leqslant \big(c_{2,\gamma}^2 \|F\|_\infty \tilde{T}^{1/2-\gamma} K\big) t^{-(\gamma+1/2)} \|\varphi\|_2,$$

for $\varphi \in L^2(H,\mu)$ and $t \in (0,\tilde{T}]$ ($1 > \tilde{T} > 0$ is arbitrary), where $K := \int_0^1 (1-s)^{-(\gamma+1/2)} \times s^{-(\gamma+1/2)} \, \mathrm{d}s$.

By induction it follows that for each $n \in \mathbb{N}$, $\varphi \in L^2(H,\mu)$ and $t \in (0,\tilde{T}]$, $\|(-A^*)^\gamma DU_n(t) \times \varphi(x)\| \in L^2(H,\mu)$ and

$$\big\|(-A^*)^\gamma DU_n(t)\varphi\big\|_2 \leqslant c_{2,\gamma} \big(c_{2,\gamma} \|F\|_\infty \tilde{T}^{1/2-\gamma} K\big)^n t^{-(\gamma+1/2)} \|\varphi\|_2.$$

Hence by taking $\tilde{T}$ sufficiently small, we obtain $(-A^*)^\gamma DP_t\varphi \in L^2(H,\mu)$ and

$$\big\|(-A^*)^\gamma DP_t\varphi\big\|_2 \leqslant \sum_{n=0}^{\infty} \big\|(-A^*)^\gamma DU_n(t)\varphi\big\|_2$$
$$\leqslant c_{2,\tilde{T},\gamma} t^{-(\gamma+1/2)} \|\varphi\|_2,$$

for $\varphi \in L^2(H,\mu)$ and $t \in (0,\tilde{T}]$. Since F is bounded we obtain

$$\|BP_t\varphi\|_2 \leqslant c_{3,\tilde{T},\gamma} t^{-(\gamma+1/2)} \|\varphi\|_2, \tag{24}$$

for $\varphi \in L^2(H,\mu)$ and $t \in (0,\tilde{T}]$. This last inequality yields (21) for $T = \tilde{T} > 0$. Using the semigroup property one can prove (21) for arbitrary $T > 0$. From the denseness of $D(L)$ in $L^2(H,\mu)$ and Theorem 1.3, and using again the last inequality, it follows that (22) holds for all $\varphi \in L^2(H,\mu)$.

To show the compactness of $(P_t)_{t\geqslant 0}$, we use the compactness of the semigroup $(R_t)_{t\geqslant 0}$ (see Proposition 2.1) and the integral equation (22). Since for $t > 0$, the operator BP_t is bounded, it follows that the operators $R_{t-s}BP_s$ $(0 < s < t)$ are compact. Hence (24) and a well-known result of Voigt [28] yields the compactness of the operator given by the strong integral

$$\int_0^t R_{t-s}BP_s \, \mathrm{d}s.$$

To prove the last statement it is enough to refer to Proposition 1.2 and notice that by Lemma 2.3 the operator $BR(\lambda, L)$ is bounded, so $\mathcal{F}_b^\infty(H)$ is also a core for $(N, D(L)) = (L+B, D(L))$. □

3. Strong Feller property of the perturbed semigroup

The Ornstein–Uhlenbeck semigroup can also be considered on the space $C_b(H)$. However, there it is not strongly continuous with respect to the supremum norm, see, e.g., van Neerven, Zabzcyk [26]. We tackle this problem by introducing the *mixed topology* τ_m on $C_b(H)$ (or with a different terminology the *strict topology*), which is a sequentially complete, locally convex topology on $C_b(H)$ and on supnorm bounded sets of $C_b(H)$ it coincides with the compact-open topology, see Sentilles [24], Wiweger [29]. (For other approaches relaxing on the notion of supnorm-strong continuity see Cerrai [1] and Priola [22].)

The Ornstein–Uhlenbeck semigroup is strongly continuous and locally equicontinuous with respect to τ_m, in other words the Ornstein–Uhlenbeck semigroup is a bi-continuous semigroup on $C_b(H)$ (see [30, Section IX.2–7] for details on semigroups on locally convex spaces, and Farkas [14], Goldys, Kocan [16], Kühnemund [17] for the Ornstein–Uhlenbeck semigroup in this framework).

Let us denote this semigroup by $(\tilde{R}_t)_{t\geqslant 0}$ and its generator as a bi-continuous semigroup by $(\tilde{L}, D(\tilde{L}))$ (see, e.g., [18]). Two facts should be noticed. For $t \geqslant 0$ the operator $\tilde{R}_t$ is the restriction of R_t to $C_b(H)$ (because we have the same explicit formula for the semigroup). Second, if $f \in D(\tilde{L})$, then f belongs also to $D(L)$. This follows from the definition of $D(\tilde{L})$: $\varphi \in D(\tilde{L}) \subseteq C_b(H)$, if and only if $t^{-1}(\tilde{R}_t\varphi - \varphi)$ is supnorm-bounded and converges in the compact-open—or equivalently here, in the mixed—topology; the limit is then $\tilde{L}\varphi$. A similar perturbation result to Theorem 2.4 can be proved for $(\tilde{R}_t)_{t\geqslant 0}$, see [13] for details. In fact, in [13] only the case A is self-adjoint was covered, but the arguments work almost verbatim also in the setting of this paper.

A full analogy to the above presented L^2-results is valid. Namely, if $F : H \to H$ is a bounded and continuous function, then there exists a bi-continuous, i.e., τ_m-strongly continuous, locally τ_m-equicontinuous semigroup $(\tilde{R}_t)_{t\geqslant 0}$ with generator $(\tilde{L} + \tilde{B}, D(\tilde{L}))$, where $\tilde{B}$ is the restriction of B to $D(\tilde{L})$ (part of B in $C_b(H)$). Similarly to (10) the perturbed semigroup can be written as

$$\tilde{P}_t = \sum_{n=0}^{\infty} \tilde{U}_n(t), \quad t \geqslant 0, \tag{25}$$

with $\tilde{U}_0(t) := \tilde{R}_t$ and

$$\tilde{U}_{n+1}(t)\varphi := \int_0^t \tilde{U}_n(t-s)\tilde{B}\tilde{R}_s\varphi \,\mathrm{d}s \quad \text{for } t \geqslant 0 \text{ and } \varphi \in D(\tilde{L}). \tag{26}$$

Here, unlike in the paragraph following (10), the integrals are τ_m-strong integrals. The operators $\tilde{U}_n(t)$ may be extended to bounded operators from $D(\tilde{L})$ to $C_b(H)$, and the series in (25) converges in the operator norm uniformly for $t \geqslant 0$ in compact intervals. It is also proved in [13] that $(\tilde{P}_t)_{t\geqslant 0}$ is a Markov semigroup on $C_b(H)$. We immediately obtain the following

Theorem 3.1. *The semigroup $(\tilde{P}_t)_{t\geqslant 0}$ is the restriction of $(P_t)_{t\geqslant 0}$ to $C_b(H)$.*

Proof. For the sake of clarity we remark that $C_b(H)$ is indeed contained in $L^2(H, \mu)$ since μ is a finite measure. It is clearly enough to show that $U_n(t)$ restricted to $C_b(H)$ coincides with $\tilde{U}_n(t)$ for all $n \in \mathbb{N}$. This follows by an easy induction. Indeed, $\tilde{U}_0(t) = \tilde{R}_t$ is the restriction of $U_0(t) = R_t$. Suppose that for some $n \in \mathbb{N}$ the operator $\tilde{U}_n(t)$ is the restriction of $U_n(t)$. Let $\varphi \in D(\tilde{L})$, then $\varphi \in D(L)$ as remarked above and both $U_{n+1}(t)$ and $\tilde{U}_{n+1}(t)$ are given by strong integrals where the integrands coincide by assumption. The integral (26) defining $\tilde{U}_{n+1}(t)$ converges in the τ_m-topology, hence also in $L^2(H, \mu)$, thus $\tilde{U}_{n+1}(t)\varphi = U_{n+1}(t)\varphi$. Now by density arguments it follows that $\tilde{U}_{n+1}(t)\varphi = U_{n+1}(t)\varphi$ for all $\varphi \in C_b(H)$. □

A part of the following assertion is already implicitly contained in the proof of Theorem 2.4.

Theorem 3.2. *The perturbed semigroup $(P_t)_{t\geqslant 0}$ possesses the* strong Feller property*, that is for all $f: H \to \mathbb{R}$ bounded and measurable function $P_t f \in C_b(H)$ holds for all $t > 0$.*

Proof. Let $\varphi: H \to \mathbb{R}$ bounded and measurable. By (20) we have

$$P_t\varphi = R_t\varphi + \int_0^t P_{t-s}BR_s\varphi\,\mathrm{d}s = R_t\varphi + \int_\varepsilon^t P_{t-s}BR_s\varphi\,\mathrm{d}s + \int_0^\varepsilon P_{t-s}BR_s\varphi\,\mathrm{d}s$$

$$= R_t\varphi + \int_\varepsilon^t \tilde{P}_{t-s}\tilde{B}\tilde{R}_{s-\varepsilon}R_\varepsilon\varphi\,\mathrm{d}s + \int_0^\varepsilon P_{t-s}BR_s\varphi\,\mathrm{d}s.$$

In this last line the first two terms belong to $C_b(H)$ because by our assumption the Ornstein–Uhlenbeck semigroup $(R_t)_{t\geqslant 0}$ *has* the strong Feller property. Hence it suffices to show that $\int_\varepsilon^t P_{t-s}BR_s\varphi\,\mathrm{d}s$ converges to $\int_0^t P_{t-s}BR_s\varphi\,\mathrm{d}s$ in *supremum norm*. This however follows from the estimate in (17). □

4. Invariant measures. Existence and uniqueness

In this closing section we first show the positivity of the semigroup $(P_t)_{t\geqslant 0}$, which leads to the existence of a positive invariant measure. We end by discussing the irreducibility of the semigroup.

We emphasize here that the estimate (21) in Theorem 2.4 enables us to perturb again the semigroup $(P_t)_{t\geqslant 0}$ with operators B as above. This is useful to prove the following theorem.

Theorem 4.1. *The semigroup $(P_t)_{t\geqslant 0}$ generated by $(L + B, D(L))$ is positive on $L^2(H,\mu)$. Moreover, there exists an invariant measure ν for $(P_t)_{t\geqslant 0}$ which is absolutely continuous with respect to the measure μ, i.e.,*

$$\rho := \frac{\mathrm{d}\nu}{\mathrm{d}\mu} \in L^2(H,\mu).$$

Proof. We prove first the positivity of the semigroup $(P_t)_{t\geqslant 0}$. Take $\varepsilon > 0$ such that $\gamma + \varepsilon < 1/2$, and set $A_n := (-A)^\gamma nR(n,A) = -n(-A)^{\gamma-1}AR(n,A)$. So A_n are bounded operators converging pointwise to $(-A)^\gamma$ (see [12, Section 4.10]) and commuting with A. For $n \in \mathbb{N}$ define the operator B_n by $D(B_n) := D(L)$ and

$$B_n\varphi(x) := \langle A_nF(x), D\varphi(x)\rangle = \langle F_n(x), D\varphi(x)\rangle$$
$$= \langle A_n(-A)^{-(\gamma+\varepsilon)}F(x), \big((-A)^{\gamma+\varepsilon}\big)^* D\varphi(x)\rangle$$

for $\varphi \in D(L)$. From this we see that the application of Theorem 2.4 yields the semigroups $(P_t^n)_{t\geqslant 0}$ with generator $(L + B_n, D(L))$. These semigroups are positive on the space $UC_b(H)$

(see Rhandi [23] for the case $F_n \in UC_b(H,H)$ and Chojnowska-Michalik, Goldys [3] or Es-Sarhir, Farkas [13] in case $F_n \in C_b(H,H)$). Furthermore, we have the estimate

$$\|B_n\varphi - B\varphi\|_2 \leqslant \big\|A_n(-A)^{-(\gamma+\varepsilon)} - (-A)^{-\varepsilon}\big\| \cdot \|F\|_\infty \big\|\big((-A)^{\gamma+\varepsilon}\big)^* D\varphi\big\|_2 \tag{27}$$

for all $\varphi \in D(L)$. Note that for $\varphi \in D(L)$ we can write

$$\begin{aligned}(B - B_n)\varphi(x) &= \big\langle F(x) - A_n(-A)^{-\gamma}F(x), \big((-A)^{\gamma}\big)^* D\varphi(x)\big\rangle \\ &= \big\langle G_n(x), \big((-A)^{\gamma}\big)^* D\varphi(x)\big\rangle,\end{aligned}$$

where $G_n \in C_b(H,H)$ is defined by $G_n(x) := F(x) - A_n(-A)^{-\gamma}F(x)$. Hence by Theorem 2.4 the semigroup can be perturbed by the operator $B - B_n$, i.e., the semigroup $(P_t)_{t\geqslant 0}$ is the Miyadera–Voigt perturbation of $(P_t^n)_{t\geqslant 0}$. Thus (20) yields

$$\big(P_t - P_t^n\big)\varphi = \int_0^t P_{t-s}(B - B_n)P_s^n\varphi\,\mathrm{d}s.$$

Combining this with (24) and (27) gives, for sufficiently small $t > 0$,

$$\begin{aligned}\big\|P_t\varphi - P_t^n\varphi\big\|_2 &\leqslant \int_0^t \|P_{t-s}\| \cdot \big\|(B - B_n)P_s^n\varphi\big\|_2\,\mathrm{d}s \\ &\leqslant c_1(t)\big\|A_n(-A)^{-(\gamma+\varepsilon)} - (-A)^{-\varepsilon}\big\| \cdot \|F\|_\infty \int_0^t \big\|\big((-A)^{\gamma+\varepsilon}\big)^* DP_s^n\varphi\big\|_2\,\mathrm{d}s \\ &\leqslant c_2(t)\big\|A_n(-A)^{-(\gamma+\varepsilon)} - (-A)^{-\varepsilon}\big\| \cdot \|F\|_\infty\|\varphi\|_2 \int_0^t \frac{1}{s^{\gamma+\varepsilon+1/2}}\,\mathrm{d}s \\ &= c_3(t)\big\|A_n(-A)^{-(\gamma+\varepsilon)} - (-A)^{-\varepsilon}\big\| \cdot \|\varphi\|_2.\end{aligned} \tag{28}$$

For the operator $(-A)^{-\varepsilon}$ is compact and $A_n(-A)^{-\gamma}$ converges pointwise to the identity I, the operator norm convergence of $A_n(-A)^{-(\gamma+\varepsilon)}$ to $(-A)^{-\varepsilon}$ follows. So the above yield that $P_t^n\varphi \xrightarrow{\|\cdot\|_2} P_t\varphi$ for all $\varphi \in D(L)$ $(n \to \infty)$. Thus $P_t^n\varphi \xrightarrow{\|\cdot\|_2} P_t\varphi$, as $n \to \infty$, for all $\varphi \in L^2(H,\mu)$ by the denseness of $D(L)$ in $L^2(H,\mu)$ and the uniform (exponential) boundedness of $\|P_t^n\|$ with respect to $n \in \mathbb{N}$. The latter property can be easily seen: $\|P_t^n\| \leqslant Me^{\omega t}$ with constants M and ω not depending on n because of the last statement of Theorem 1.3 and the estimate (23). So we see that there exists $t_0 > 0$ such that P_t is positive for $t \in [0, t_0]$. Thus, by the semigroup property, the positivity of P_t follows for all $t \geqslant 0$.

We prove now the last statement of the corollary. First note that $R_t\mathbf{1} = \mathbf{1}$ for all $t \geqslant 0$. Therefore, it follows from (20) that also $P_t\mathbf{1} = \mathbf{1}$ for all $t \geqslant 0$. Since each P_t, $t > 0$, is compact, the same is true for its adjoint P_t^*. Therefore, 1 is also an eigenvalue for P_t^*. Let ϑ be a corresponding eigenvector, i.e., $P_t^*\vartheta = \vartheta$ for all $t \geqslant 0$. Since $(P_t^*)_{t\geqslant 0}$ is positive it follows that $|\vartheta| = |P_t^*\vartheta| \leqslant P_t^*|\vartheta|$ and from

$$\langle P_t^*|\vartheta|, \mathbf{1}\rangle = \langle|\vartheta|, P_t\mathbf{1}\rangle = \langle|\vartheta|, \mathbf{1}\rangle = \langle\left|P_t^*\vartheta\right|, \mathbf{1}\rangle,$$

we obtain

$$\left|P_t^*\vartheta\right| = P_t^*|\vartheta| \quad \text{for all } t \geqslant 0.$$

Hence we deduce

$$|\vartheta| = P_t^*|\vartheta|.$$

If we write $\rho = \frac{|\vartheta|}{\|\vartheta\|_1}$, then $\nu := \rho\mu$ is an invariant probability measure for the semigroup $(P_t)_{t\geqslant 0}$ which has the asserted properties. □

The following proposition describes how fast the semigroup $(P_t)_{t\geqslant 0}$ can grow. Later, after proving the irreducibility, this will give us the boundedness of the semigroup.

Theorem 4.2. *The semigroup $(P_t)_{t\geqslant 0}$ is polynomially bounded on $L^2(H,\mu)$. More precisely $\|P_t\| \leqslant C(1 + t^{n-1})$ for all $t \geqslant 0$, where n is the order of the pole 0 of the resolvent $\lambda \mapsto R(\lambda, L + B)$.*

Proof. First we show that the exponential growth bound $\omega_0(P)$ of $(P_t)_{t\geqslant 0}$ is 0. Since P_t^*, $t > 0$, is compact, its spectrum contains only eigenvalues and 0. Hence to estimate the spectral radius of P_t^* it suffices to estimate the eigenvalues. So let α be an eigenvalue of P_t^* with corresponding eigenvector φ. Then, using also the positivity of P_t, we have

$$|\alpha| \cdot \langle\mathbf{1}, |\varphi|\rangle = \langle\mathbf{1}, \left|P_t^*\varphi\right|\rangle \leqslant \langle\mathbf{1}, P_t^*|\varphi|\rangle = \langle P_t\mathbf{1}, |\varphi|\rangle = \langle\mathbf{1}, |\varphi|\rangle.$$

So $|\alpha| \leqslant 1$, because $\langle\mathbf{1}, |\varphi|\rangle \neq 0$. Therefore $r(P_t^*) \leqslant 1$ holds and so $\omega_0(P) = \omega_0(P^*) \leqslant 0$, but then clearly $\omega_0(P) = 0$.

From the positivity and the compactness of the semigroup we have by [12, Corollary VI.1.13] that the boundary spectrum consist of one single point $s(L + B) = 0$, which is also a dominant eigenvalue of $L + B$. Let π be the spectral projection corresponding to the decomposition $\sigma(L+B) = \{0\} \cup (\sigma(L+B) \setminus \{0\})$ (see [12, Section IV.1.16–1.18]). We set $d = \dim(\operatorname{rg}\pi)$, the algebraic multiplicity of the eigenvalue 0 of $L + B$, and $n =$ the order of the pole 0 of the resolvent $R(\lambda, L + B)$. We have $L^2(H,\mu) = \pi L^2(H,\mu) + (I - \pi)L^2(H,\mu)$. Let $(P_t^1)_{t\geqslant 0}$ and $(P_t^2)_{t\geqslant 0}$ be the restricted semigroups to $\pi L^2(H,\mu)$ and $(I - \pi)L^2(H,\mu)$, respectively. Then $(P_t^1)_{t\geqslant 0}$ is polynomially bounded, indeed $\|P_t^1\| \leqslant K(t^{n-1} + 1)$ (because $(L + B)|_{\operatorname{rg}\pi}$ is an $d \times d$-matrix with only 0 as eigenvalue, so the polynomial growth can be read off its Jordan normal form, where the largest Jordan-block is $n \times n$). Further, $(P_t^2)_{t\geqslant 0}$ is uniformly exponentially stable, i.e., $\|P_t^2\| \leqslant Me^{-\omega t}$ with some $\omega > 0$. Let $\varphi \in L^2(H,\mu)$. Then

$$\begin{aligned}\|P_t\varphi\| &\leqslant \big\|P_t^1\pi\varphi\big\| + \big\|P_t^2(I-\pi)\varphi\big\| \\ &\leqslant K\big(t^{n-1}+1\big)\|\pi\varphi\| + Me^{-\omega t}\big\|(I-\pi)\varphi\big\| \leqslant C\big(1+t^{n-1}\big)\|\varphi\|,\end{aligned}$$

for a constant $C > 0$ independent of φ and t. □

Theorem 4.3. *For $F : H \to H$ bounded and Lipschitz continuous the semigroup $(P_t)_{t\geqslant 0}$ is irreducible and bounded.*

Proof. We follow the arguments of [11, Chapter 7] and we also keep the notations therein. Let $(\Omega, \mathcal{M}, (\mathcal{M}_t)_{t\geqslant 0}, \mathbf{P})$ be a filtered probability space with $W(t)$ an H-valued cylindrical Wiener process. Take $0 < T < 1$ and denote by $H^2([0,T])$ and $H^{2,2}([0,T])$ the space of H-valued progressively measurable processes Y with

$$\|Y\|_{H^2} := \sup_{s\in[0,T]} \big(\mathbf{E}\big\|Y(s)\big\|^2\big)^{1/2} < +\infty, \quad \text{and}$$

$$\|Y\|_{H^{2,2}} := \left(\mathbf{E}\int_0^T \big\|Y(s)\big\|^2\,\mathrm{d}s\right)^{1/2} < +\infty, \quad \text{respectively,}$$

both being Banach spaces if endowed with the respective norms.

For each $x \in H$ define the mappings

$$\mathcal{K}_0(Y)(t) := \int_0^t (-A)^\gamma e^{(t-s)A} F\big(Y(s)\big)\,\mathrm{d}s, \quad \text{and}$$

$$\mathcal{K}(x, Y)(t) := e^{tA}x + \int_0^t e^{(t-s)A}\,\mathrm{d}W(s) + \mathcal{K}_0(Y)(t) \quad \text{for } Y \in H^2\big([0,T]\big).$$

We claim that $\mathcal{K}_0$, hence $\mathcal{K}$, is Lipschitz continuous. Indeed, we have $\mathcal{K}_0 = \mathcal{T} \circ \mathcal{F}$, where

$$\mathcal{F}(Y)(t) := F\big(Y(t)\big) \quad \text{is the evaluation map, and}$$

$$\mathcal{T}(Y)(t) := \int_0^t (-A)^\gamma e^{(t-s)A} Y(s)\,\mathrm{d}s \quad \text{is a convolution type map.}$$

By [11, Proposition 7.3.2] we know that $\mathcal{F} : H^2([0,T]) \to H^{2,2}([0,T])$ is Lipschitz continuous, as so is $F : H \to H$ by assumption, moreover the Lipschitz constant of $\mathcal{F}$ depends only on that of F. Now we show that the linear operator $\mathcal{T} : H^{2,2}([0,T]) \to H^2([0,T])$ is bounded, and then the Lipschitz continuity of $\mathcal{K}_0$ follows:

$$\begin{aligned}\big\|\mathcal{K}_0(Y) - \mathcal{K}_0(Y')\big\|_{H^2} &\leqslant \|\mathcal{T}\|_{H^{2,2}\to H^2} \cdot \big\|\mathcal{F}(Y) - \mathcal{F}(Y')\big\|_{H^{2,2}} \\ &\leqslant \ell\|\mathcal{T}\|_{H^2\to H^{2,2}} \cdot \|Y - Y'\|_{H^p},\end{aligned} \tag{29}$$

here ℓ being independent of T. To see the boundedness of $\mathcal{T}$, we can write for $Y \in H^{2,2}([0,T])$ that

$$\begin{aligned}
\mathbf{E}\left\| \int_0^t (-A)^\gamma e^{(t-s)A} Y(s)\,\mathrm{d}s \right\|^2 &\leqslant \mathbf{E}\left(\int_0^t \left\|(-A)^\gamma e^{(t-s)A} Y(s)\right\| \mathrm{d}s \right)^2 \\
&\leqslant \mathbf{E}\left(\int_0^t \left\|(-A)^\gamma e^{(t-s)A}\right\| \cdot \left\|Y(s)\right\| \mathrm{d}s \right)^2 \\
&\leqslant \int_0^t \left\|(-A)^\gamma e^{(t-s)A}\right\|^2 \mathrm{d}s \cdot \mathbf{E}\left(\int_0^t \left\|Y(s)\right\|^2 \mathrm{d}s \right) \\
&\leqslant C t^{2(\frac{1}{2}-\gamma)} \|Y\|^2_{H^{2,2}}.
\end{aligned} \tag{30}$$

This shows $\|\mathcal{T}(Y)\|_{H^2} \leqslant CT^{(\frac{1}{2}-\gamma)}\|Y\|_{H^{2,2}}$, hence the desired estimate. Even more can be seen from this and (29). If we take $T > 0$ sufficiently small, then the Lipschitz constant of $\mathcal{K}_0$ can be chosen arbitrarily small $\alpha \in (0,1)$.

Additionally to this we define for $Y \in H^2([0,T])$

$$\mathcal{K}_n(Y)(t) := e^{tA}x + \int_0^t e^{(t-s)A}\,\mathrm{d}W(s) + \int_0^t e^{(t-s)A} A_n F\big(Y(s)\big)\,\mathrm{d}s,$$

where A_n are bounded operators and approximate $(-A)^\gamma$, as in the proof of Theorem 4.1. It is straightforward that $\mathcal{K}_n \to \mathcal{K}$ strongly in $H^2([0,T])$. Similar computation to the above shows that the Lipschitz constants of $\mathcal{K}$ and $\mathcal{K}_n$ can be made small $\alpha \in (0,1)$ uniformly with respect to $n \in \mathbb{N}$, if $T > 0$ is small enough. Indeed, one has only to notice that $\|A_n e^{(t-s)A}\| = \|(-A)^\gamma nR(n,A)e^{(t-s)A}\| \leqslant M\|(-A)^\gamma e^{(t-s)A}\|$ for $s \in [0,t)$, and the repeat the arguments as in (30). By Theorem 7.1.1 in [11] we obtain that $\mathcal{K}$ and $\mathcal{K}_n$ have unique fixed points X and X_n, $n \geqslant 1$ respectively (if we want to emphasize the dependence on t and x we write $X(t,x)$ and $X_n(t,x)$). Further, Theorem 7.1.5 in [11] shows that $X_n \to X$ in $H^2([0,T])$.

Consider the semigroups $(P_t^n)_{t\geqslant 0}$ from the proof of Theorem 4.1. They are generated by $(L + B_n, D(L))$, where $B_n\varphi(x) = \langle F_n(x), D\varphi(x)\rangle$, and $F_n(x) = A_n F(x)$ is bounded. It is shown in [3] that for $\varphi \in C_b(H)$ one has $P_t^n\varphi(x) = \mathbf{E}\varphi(X_n(t,x))$. Using a similar argument as in (28), for $\varphi \in C_b(H)$ one can prove that $P_t^n\varphi(x) \to P_t\varphi(x)$ $(n \to \infty)$ for all $x \in H$ uniformly on compact intervals $[0,t_0]$ (see the proof of Theorem 4.4 in [13] for a detailed argument). This means that $X_n \to X$ in distribution. So it follows that $P_t\varphi(x) = \mathbf{E}\varphi(X(t,x))$, where we have

$$X(t,x) = Z(t,x) + \int_0^t (-A)^\gamma e^{(t-s)A} F\big(X(s,x)\big)\,\mathrm{d}s, \tag{31}$$

with $Z(t,x) := e^{tA}x + \int_0^t e^{(t-s)A}\,\mathrm{d}W(s)$ the Ornstein–Uhlenbeck process.

Now we are in the position to show the irreducibility of the semigroup $(P_t)_{t\geqslant 0}$, the idea we follow here appears in [3]. Let $B(x_0, r) \subseteq H$ be an open ball. We show $P_t \mathbf{1}_{B(x_0,r)} = \mathbf{P}(\{\|X(t) - x_0\| < r\}) > 0$ for all $t > 0$. In (31) take $t > 0$ (independent of x) so small that

$$\left\| \int_0^t (-A)^\gamma e^{(t-s)A} F\big(X(s,x)\big)\,\mathrm{d}s \right\| = \int_0^t \big\|(-A)^\gamma e^{(t-s)A}\big\| \cdot \|F\|_\infty \,\mathrm{d}s \leqslant r/2.$$

Then we have

$$\mathbf{P}\big(\big\{\big\|X(t,x) - x_0\big\| < r\big\}\big) \geqslant \mathbf{P}\big(\big\{\big\|Z(t,x) - x_0\big\| < r/2\big\}\big) > 0,$$

which is positive because of the irreducibility of the Ornstein–Uhlenbeck semigroup [10, Theorem 7.3.1]. The proof for irreducibility of $(P_t)_{t\geqslant 0}$ is complete.

The boundedness of the semigroup follows now at once. Because of the irreducibility we known by Perron–Frobenius theory that the spectral projection π is one-dimensional (see [20, Theorem C-III.3.8]), so Theorem 4.2 finishes the proof. □

Corollary 4.4. *For $F : H \to H$ bounded and Lipschitz continuous the invariant measure ν for $(P_t)_{t\geqslant 0}$ is unique.*

Proof. The semigroup $(P_t)_{t\geqslant 0}$ is strong Feller (Theorem 3.2) and irreducible (Theorem 4.3), so the uniqueness follows. □

Acknowledgment

We are indebted to the anonymous referee for the helpful comments and suggestions, which indeed contributed to the improvement of this paper.

References

[1] S. Cerrai, A Hille–Yosida theorem for weakly continuous semigroups, Semigroup Forum 49 (1994) 349–367.
[2] A. Chojnowska-Michalik, Transition semigroups for stochastic semilinear equations on Hilbert spaces, Dissertationes Math. (Rozprawy Mat.) 396 (2001).
[3] A. Chojnowska-Michalik, B. Goldys, Existence, uniqueness and invariant measures for stochastic semilinear equations on Hilbert spaces, Probab. Theory Related Fields 102 (1995) 331–356.
[4] A. Chojnowska-Michalik, B. Goldys, On regularity properties of nonsymmetric Ornstein–Uhlenbeck semigroup in L^p spaces, Stoch. Stoch. Rep. 59 (3–4) (1996) 183–209.
[5] G. Da Prato, Bounded perturbation of Ornstein–Uhlenbeck semigroups, in: Progr. Nonlinear Differential Equations Appl., vol. 50, Birkhäuser, Basel, 2002, pp. 97–114.
[6] G. Da Prato, A new regularity result for Ornstein–Uhlenbeck generators and applications, J. Evol. Equ. 3 (2003) 485–498.
[7] G. Da Prato, Kolmogorov Equations for Stochastic PDEs, Advanced Courses in Mathematics, CRM Barcelona, Birkhäuser, Basel, 2004.
[8] G. Da Prato, P. Malliavin, D. Nualart, Compact families of Wiener functionals, C. R. Acad. Sci. Paris 315 (1992) 1287–1291.
[9] G. Da Prato, J. Zabczyk, Stochastic Equations in Infinite Dimensions, Encyclopedia of Mathematics and its Applications, Cambridge Univ. Press, Cambridge, 1992.
[10] G. Da Prato, J. Zabczyk, Ergodicity for Infinite Dimensional Systems, London Math. Soc. Lecture Notes, vol. 229, Cambridge Univ. Press, Cambridge, 1996.
[11] G. Da Prato, J. Zabczyk, Second Order Partial Differential Equations in Hilbert Spaces, London Math. Soc. Lecture Notes, vol. 283, Cambridge Univ. Press, Cambridge, 2002.

[12] K.-J. Engel, R. Nagel, One-Parameter Semigroups for Linear Evolution Equations, Grad. Texts in Math., vol. 194, Springer-Verlag, New York, 2000.
[13] A. Es-Sarhir, B. Farkas, Perturbation for a class of transition semigroup on the Hölder space $C^{\theta}_{b,\mathrm{loc}}(H)$, J. Math. Anal. Appl. 315 (2006) 666–685.
[14] B. Farkas, Perturbations of bi-continuous semigroups with applications to transition semigroups on $C_b(H)$, Semigroup Forum 68 (2004) 87–107.
[15] B. Goldys, On analyticity of Ornstein–Uhlenbeck semigroups, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. 10 (3) (1999) 131–140.
[16] B. Goldys, M. Kocan, Diffusion semigroups in spaces of continuous functions with mixed topology, J. Differential Equations 173 (2001) 17–39.
[17] F. Kühnemund, Bi-continuous semigroups on spaces with two topologies: Theory and applications, PhD thesis, Eberhard Karls Universität, Tübingen, 2001.
[18] F. Kühnemund, A Hille–Yosida theorem for bi-continuous semigroups, Semigroup Forum 67 (2003) 205–225.
[19] I. Miyadera, On perturbation theory for semigroups of operators, Tôhoku Math. J. 18 (1966) 299–310.
[20] R. Nagel (Ed.), One-Parameter Semigroups of Positive Operators, Lecture Notes in Math., vol. 1184, Springer, New York, 1986.
[21] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Appl. Math. Sci., vol. 44, Springer, New York, 1983.
[22] E. Priola, On a class of Markov type semigroups in spaces of uniformly continuous and bounded functions, Studia Math. 136 (1999) 271–295.
[23] A. Rhandi, On a functional analytic approach for transition semigroups in L^2, Note Mat. 21 (2002) 93–106.
[24] F.D. Sentilles, Bounded continuous functions on a completely regular space, Trans. Amer. Math. Soc. 168 (1972) 311–336.
[25] J.M.A.M. van Neerven, Nonsymmetric Ornstein–Uhlenbeck semigroups in Banach spaces, J. Funct. Anal. 155 (1998) 495–535.
[26] J.M.A.M. van Neerven, J. Zabczyk, Norm discontinuity of Ornstein–Uhlenbeck semigroups, Semigroup Forum 59 (1999) 389–403.
[27] J. Voigt, On the perturbation theory for strongly continuous semigroups, Math. Ann. 229 (1977) 163–171.
[28] J. Voigt, On the convex compactness property for the strong operator topology, Note Mat. 12 (1992) 259–269.
[29] A. Wiweger, Linear spaces with mixed topology, Studia Math. 20 (1961) 47–68.
[30] K. Yosida, Functional Analysis, Classics Math., Springer, Berlin, 1995.
[31] L. Zambotti, An analytic approach to existence and uniqueness for martingale problems in infinite dimensions, Probab. Theory Related Fields 118 (2) (2000) 147–168.