Asymptotic stability of ground states in 2D nonlinear Schrödinger equation including subcritical cases

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**Abstract**

We consider a class of nonlinear Schrödinger equations in two space dimensions with an attractive potential. The nonlinearity is local but rather general encompassing for the first time both subcritical and supercritical (in \( L^2 \)) nonlinearities. We study the asymptotic stability of the nonlinear bound states, i.e. periodic in time localized in space solutions. Our result shows that all solutions with small initial data, converge to a nonlinear bound state. Therefore, the nonlinear bound states are asymptotically stable. The proof hinges on dispersive estimates that we obtain for the time-dependent, Hamiltonian, linearized dynamics around a carefully chosen one-parameter family of bound states that “shadow” the nonlinear evolution of the system. Due to the generality of the methods we develop we expect them to extend to the case of perturbations of large bound states and to other nonlinear dispersive wave type equations.

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**1. Introduction**

In this paper we study the long-time behavior of solutions of the nonlinear Schrödinger equation (NLS) with potential in two space dimensions (2D):

\[
\begin{align*}
    i \partial_t u(t, x) &= \left[ -\Delta_x + V(x) \right] u + g(u), \quad t \in \mathbb{R}, \; x \in \mathbb{R}^2, \\
    u(0, x) &= u_0(x)
\end{align*}
\]  

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where the local nonlinearity is constructed from the real-valued, odd, $C^2$ function $g : \mathbb{R} \mapsto \mathbb{R}$ satisfying
\[
g(0) = g'(0) = 0 \quad \text{and} \quad |g''(s)| \leq C(|s|^{\alpha_1} + |s|^{\alpha_2}), \quad s \in \mathbb{R}, \quad \frac{1}{2} < \alpha_1 \leq \alpha_2 < \infty, \quad (1.3)
\]
which is then extended to a complex function via the gauge symmetry:
\[
g(e^{i\theta}s) = e^{i\theta}g(s), \quad \theta \in \mathbb{R}. \quad (1.4)
\]
The equation has important applications in statistical physics, optics and water waves. For $g(s) = s^3$, it describes certain limiting behavior of Bose–Einstein condensates, see [8,14]. This case has already been analyzed in [13] but the result in this paper gives a more detailed description of the dynamics, in particular we show that the solutions eventually converge to a ground state. For $g(s) = c_3 s^3 + c_5 s^5 + \cdots$ the equation describes propagation of time harmonic waves in wave guides made out of centrosymmetric materials, see [15,18]. In this case, $t$ plays the role of the coordinate along the axis of symmetry of the wave guide and we can infer that beyond a transition region all (small) waves converge to a propagating mode. For quartz, liquid crystals, and other materials which are not centrosymmetric the nonlinearity becomes $g(s) = c_2 s^2 + c_5 s^5 + \cdots$ and our result does not apply due to the quadratic term which dominates for $s$ small. However, the techniques we develop in this paper allow us to control lower power nonlinearities compared to the previous methods, hence they are an important step forward. Due to their generality we also expect them to be adaptable and play an important role in studying the long-time dynamics of other dispersive wave equations. A more detailed discussion follows.

It is well known that the above nonlinear equation admits periodic in time, localized in space solutions (bound states or solitary waves). They can be obtained via both variational techniques [1,22,28] and bifurcation methods [13,20,22]. Moreover, the set of periodic solutions can be organized as a $C^2$ manifold (center manifold), see next section or [11,12]. Orbital stability of solitary waves, i.e. stability modulo the group of symmetries $u \mapsto e^{-i\theta}u$, was first proved in [22,30], see also [9,10,24].

The main result of this paper is that solutions of (1.1)–(1.2) with small initial data asymptotically converge to a bound state, see Theorem 3.1. Asymptotic stability studies for bound states in NLS were initiated in the work of A. Soffer and M.I. Weinstein [25,26] and their method was further developed in [2–4,7,20,29]. But these results cannot be extended to our case. Indeed, [25] uses spherical symmetry of solutions to infer their space localization and employ exclusively integrable in time weighted estimates. A similar idea is implemented in [29] where the nonlinearity is localized in space. To compensate for delocalization, [7,20,26] use stronger, integrable in time, $L^1 \mapsto L^\infty$ dispersion estimates for Schrödinger operators in $3$D and higher dimensions, compared to the ones available in $2$D see (1.5). The technique of virial theorem allows bootstrapping of high power nonlinearities in [2–4] in spite of the weak dispersion in $1$D, but its adaptation to the $2$D case would require at least a quintic nonlinearity. [16], see also [11], employs Strichartz and Kato smoothing type estimates to show asymptotic stability for data in the energy space but the method is restricted to critical and supercritical nonlinearities: $\alpha_1 \geq 1$.

In the present paper we use linearization around a time-varying profile to rigorously control the long-time behavior of solutions. At each time the profile is given by the bound state that “best approximates” the solution at that time, see Lemma 2.1. By best approximation we mean that at each time the correction is a superposition of “radiation (dispersive) modes” for the linearized equation. This forces us to continuously change the bound state, hence the linearization, according to the evolution of the actual solution. Previously, linearization around a fixed bound state has been used, see the papers cited above. By continuously adapting the linear dynamics to the nonlinear evolution of the solution we can more precisely capture the effective potential induced by the nonlinearity $g$ into a time-dependent linear operator. Once we have a good understanding of this time-dependent linear dynamics, i.e. we have good dispersive estimates for its semigroup of operators, see Section 4, we obtain rigorous estimates for the nonlinear dynamics via Duhamel formula and contraction principles for integral equations, see Section 3. This two-step approach completely separates the analysis of
linear terms (even when they depend on time) from the nonlinear terms and allows us to use dispersive estimates tailored to each of them. Consequently, we can control a wider range of nonlinearities including for the first time subcritical ones: \( \alpha_1 < 1 \).

The main challenge for our approach is to obtain dispersive estimates for Schrödinger type operators with time-dependent coefficients. This is accomplished in Section 4 via a perturbative method. It relies on the fact that the time-dependent coefficients are small and localized in space to obtain first estimates with time-dependent coefficients. This is accomplished in Section 4 via a perturbative method. It relies on the fact that the time-dependent coefficients are small and localized in space to obtain first estimates. Then we remove the weights in two steps by employing non-weighted estimates for the nearby constant coefficient operator:

\[
\| e^{i(\Delta - V)t} \|_{L^1 \to L^\infty} \sim |t|^{-1}. \tag{1.5}
\]

Unfortunately, for the last two steps it is crucial that \(|t|^{-1}, |t| \geq 1\), is almost integrable in time. We do not know yet how to adapt the method for operators with weaker dispersive properties such as the 1D Schrödinger case:

\[
\| e^{i(\Delta^2 - V(x))t} \|_{L^1 \to L^\infty} \sim |t|^{-1/2}.
\]

We could have obtained sharper estimates in Section 4 by using a generalized Fourier multiplier technique to remove the singularity of (1.5) at \( t = 0 \) see [12, Appendix and Section 4]. We chose not to do it because it requires stronger hypotheses on \( V \) without allowing us to enlarge the spectrum of nonlinearities that we can treat.

Note that we have recently used a similar technique to show that in the critical (cubic) case, (1.1) with \( g(s) = s^3, s \in \mathbb{R} \), the center manifold of bound states is an attractor for small initial data, see [13]. In the current paper the technique is much refined, we use a better projection of the dynamics on the center manifold and sharper estimates for the linear dynamics. The refinements not only allow us to treat a much larger spectrum of nonlinearities including, for the first time, the subcritical ones.

Finally, we remark that our method is quite general, based solely on linearization around a time-varying profile and estimates for integral operators with dispersive kernels. Therefore we expect it to generalize to the case of large nonlinear 2D ground states, see for example [7], to the case of multiple families of bound states, see for example [27], or to the case of time-dependent nonlinearity, see [6]. In all three cases our method will not only allow to treat the less dispersive environment, 2D compared to 3D, but it should greatly reduce the restrictions on the nonlinearity. Together with collaborators we are currently working on adapting the method to other dimensions and other dispersive wave type equations. The work in 3D is complete, see [12].

Notations. \( H = -\Delta + V \);

\( L^p = \{ f : \mathbb{R}^2 \mapsto \mathbb{C} \mid f \text{ measurable and } \int_{\mathbb{R}^2} |f(x)|^p \, dx < \infty \} \), \( 1 \leq p < \infty \), endowed with the standard norm \( \| f \|_{L^p} = (\int_{\mathbb{R}^2} |f(x)|^p \, dx)^{1/p} \), while for \( p = \infty \), \( L^\infty = \{ f : \mathbb{R}^2 \mapsto \mathbb{C} \mid f \text{ measurable and ess sup } |f(x)| < \infty \} \), and it is endowed with the norm \( \| f \|_{L^\infty} = \text{ess sup } |f(x)| \);

\( (x) = (1 + |x|^2)^{1/2} \), and for \( \sigma \in \mathbb{R} \), \( 1 \leq p < \infty \), \( L_{\sigma}^p \) denotes the \( L^p \) space with weight \( (x)^\sigma \), i.e. the space of functions \( f(x) \) such that \( (x)^\sigma f(x) \) are integrable endowed with the norm \( \| f(x) \|_{L_{\sigma}^p} = \| (x)^\sigma f(x) \|_p \) while for \( p = \infty \), \( L_{\sigma}^\infty \) denotes the vector space of measurable functions \( f(x) \) such that \( \text{ess sup } |(x)^\sigma f(x)| < \infty \) endowed with the norm \( \| f(x) \|_{L_{\sigma}^\infty} = \| (x)^\sigma f(x) \|_{L^\infty} \);

\( \langle f, g \rangle = \int_{\mathbb{R}^2} \overline{f(x)} g(x) \, dx \) is the scalar product in \( L^2 \) where \( \overline{z} \) is the complex conjugate of the complex number \( z \);

\( P_c \) is the projection associated to the continuous spectrum of the self-adjoint operator \( H \) on \( L^2 \), range \( P_c = \mathcal{H}_0 \);

\( H^n \) denote the Sobolev spaces of measurable functions having all distributional partial derivatives up to order \( n \) in \( L^2 \), \( \| \cdot \|_{H^n} \) denotes the standard norm in these spaces.
2. Preliminaries. The center manifold

The center manifold is formed by the collection of periodic solutions for (1.1):

$$u_E(t, x) = e^{-iEt} \psi_E(x)$$

(2.1)

where $E \in \mathbb{R}$ and $0 \neq \psi_E \in H^2(\mathbb{R}^2)$ satisfy the time-independent equation

$$[-\Delta + V] \psi_E + g(\psi_E) = E \psi_E.$$  

(2.2)

Clearly the function constantly equal to zero is a solution of (2.2) but (iii) in the following hypotheses on the potential $V$ allows for a bifurcation with a nontrivial, one (complex) parameter family of solutions:

(H1) Assume that

(i) there exist $C > 0$ and $\rho > 3$ such that

$$|V(x)| \leq C \langle x \rangle^{-\rho}, \quad \text{for all } x \in \mathbb{R}^2;$$

(ii) 0 is a regular point\(^1\) of the spectrum of the linear operator $H = -\Delta + V$ acting on $L^2$:

(iii) $H$ acting on $L^2$ has exactly one negative eigenvalue $E_0 < 0$ with corresponding normalized eigenvector $\psi_0$. It is well known that $\psi_0(x)$ can be chosen strictly positive and exponentially decaying as $|x| \to \infty$.

Conditions (i)–(ii) guarantee the applicability of dispersive estimates of Murata [17] and Schlag [23] to the Schrödinger group $e^{-iHt} p_t$. These estimates are used for obtaining Theorems 4.1 and 4.2, see also [13, Section 4]. In particular (i) implies the local well-posedness in $H^1$ of the initial value problem (1.1)–(1.2), see Section 3.

By the standard bifurcation argument in Banach spaces [19] for (2.2) at $E = E_0$, condition (iii) guarantees existence of nontrivial solutions. Moreover, these solutions can be organized as a $C^2$ manifold (center manifold). Following the proofs in [12, Section 2] or [11] which are not affected by the fact that we are now in two space dimension, we have:

**Proposition 2.1.** There exist $\delta > 0$, the $C^2$ function

$$h : \{ a \in \mathbb{C} : |a| < \delta \} \mapsto L^2_\sigma \cap H^2, \quad \sigma \in \mathbb{R},$$

and the $C^1$ function $E : (-\delta, \delta) \mapsto \mathbb{R}$ such that for $|E - E_0| < \delta$ and $|\langle \psi_0, \psi_E \rangle| < \delta$ the eigenvalue problem (2.2) has a unique non-zero solution up to multiplication with $e^{i\theta}$, $\theta \in (0, 2\pi)$, which can be represented as a center manifold:

$$\psi_E = a \psi_0 + h(a), \quad E = E(|a|), \quad \{ \psi_0, h(a) \} = 0, \quad h(e^{i\theta}a) = e^{i\theta}h(a), \quad |a| < \delta.$$  

(2.3)

Moreover $E(|a|) = O(|a|^{1+\alpha_1})$, $(h(a) = O(|a|^{2+\alpha_1})$ as $|a| \to 0$, and for $a \in \mathbb{R}$, $|a| < \delta$, $h(a)$ is a real-valued function with $\frac{d^2h}{d\sigma^2}(a) = O(|a|^{\alpha_1})$ as $|a| \to 0$, and $\frac{dh}{d\sigma}(0) = 0$.

Since $\psi_0(x)$ is exponentially decaying as $|x| \to \infty$ the proposition implies that $\psi_E \in L^2_\sigma$. A regularity argument, see [25], gives a stronger result:

\(^1\) See [23, Definition 7] or $M_\mu = \{0\}$ in relation (3.1) in [17].
Corollary 2.1. For any $\sigma \in \mathbb{R}$, there exists a finite constant $C_\sigma$ such that
\[
\| (x) \sigma \psi_E \|_{H^2} \leq C_\sigma \| \psi_E \|_{H^2}.
\]

We are going to decompose the solution of (1.1)–(1.2) into a projection onto the center manifold and a correction. To ensure that the correction disperses to infinity on long times we require that the correction is always in the invariant subspace of the linearized dynamics at the projection that complements the tangent space to the center manifold. A short description of the decomposition follows, for more details and the proofs see [12].

Consider the linearization of (1.1) at a function on the center manifold $\psi_E = a \psi_0 + h(a)$, $a = a_1 + ia_2 \in \mathbb{C}$, $|a| < \delta$:
\[
\frac{\partial w}{\partial \tau} = -iL_{\psi_E} [w] - iEw
\]
where
\[
L_{\psi_E} [w] = (-\Delta + V - E)w + Dg_{\psi_E} [w] = (-\Delta + V - E)w + \lim_{\varepsilon \to 0} \frac{g(\psi_E + \varepsilon w) - g(\psi_E)}{\varepsilon}.
\]

Remark 2.1. Note that for $a \in \mathbb{R}$ we have $\psi_E = a \psi_0 + h(a)$ is real-valued and
\[
Dg_{\psi_E} [w] = g'(\psi_E) \Re w + \frac{2}{\psi_E} g(\psi_E) \Im w = \frac{1}{2} \left( g'(\psi_E) + \frac{g(\psi_E)}{\psi_E} \right) w + \frac{1}{2} \left( g'(\psi_E) - \frac{g(\psi_E)}{\psi_E} \right) \overline{w}
\]
hence
\[
|Dg_{\psi_E} [w]| \leq |w| \max \left\{ \left| g'(\psi_E) \right|, \left| \frac{g(\psi_E)}{\psi_E} \right| \right\} \leq C \left( |\psi_E|^{1+\alpha_1} + |\psi_E|^{1+\alpha_2} \right) |w| \tag{2.6}
\]
where we used (1.3). For $a = |a|e^{i\theta} \in \mathbb{C}$ we have, using the equivariant symmetry (1.4), $\psi_E = a \psi_0 + h(a) = e^{i\theta} (|a| \psi_0 + h(|a|)) = e^{i\theta} \psi_E^{\text{real}}$, where $\psi_E^{\text{real}}$ is real-valued, and $Dg_{\psi_E} [w] = e^{i\theta} Dg_{\psi_E^{\text{real}}} [e^{-i\theta} w]$, hence (2.6) is valid for any $\psi_E$ on the manifold of ground states.

Properties of the linearized operator:

1. $L_{\psi_E}$ is real linear and symmetric with respect to the real scalar product $\Re \langle \cdot, \cdot \rangle$, on $L^2(\mathbb{R}^2)$, with domain $H^2(\mathbb{R}^2)$.
2. Zero is an eigenvalue for $-iL_{\psi_E}$ and its generalized eigenspace includes $\{ \frac{\partial \psi_E}{\partial a_1}, \frac{\partial \psi_E}{\partial a_2} \}$.
3. span$_{\mathbb{R}} \{ \frac{\partial \psi_E}{\partial a_1}, \frac{\partial \psi_E}{\partial a_2} \}$ and $\mathcal{H}_0 = \{ -i \frac{\partial \psi_E}{\partial a_1}, i \frac{\partial \psi_E}{\partial a_2} \}^\perp$, where orthogonality is with respect to the real scalar product in $L^2(\mathbb{R}^2)$, are invariant subspaces for $-iL_{\psi_E}$ and, by possibly choosing $\delta > 0$ smaller than the one in Proposition 2.1, we have
\[
L^2(\mathbb{R}^2) = \text{span}_{\mathbb{R}} \left\{ \frac{\partial \psi_E}{\partial a_1}, \frac{\partial \psi_E}{\partial a_2} \right\} \oplus \mathcal{H}_a, \quad \text{for all } a \in \mathbb{C}, \ |a| < \delta.
\]

Note that $\mathcal{H}_0$ coincides with the subspace of $L^2$ associated to the continuous spectrum of the self-adjoint operator $H = -\Delta + V$. 
4. The above decomposition can be extended to $H^{-1}(\mathbb{R}^2)$:

\[
H^{-1}(\mathbb{R}^2) = \text{span}_\mathbb{R} \left\{ \frac{\partial \psi_E}{\partial a_1}, \frac{\partial \psi_E}{\partial a_2} \right\} \oplus \mathcal{H}_a, \quad \text{for all } a \in \mathbb{C}, \ |a| < \delta, \tag{2.7}
\]

where

\[
\mathcal{H}_a = \left\{ \phi \in H^{-1} \left| \Re \left( -i \frac{\partial \psi_E}{\partial a_2}, \phi \right) = 0, \text{ and } \Re \left( i \frac{\partial \psi_E}{\partial a_1}, \phi \right) = 0 \right. \right\}.
\]

Our goal is to decompose the solution of (1.1) at each time into

\[
u = \psi_E + \eta = a\psi_0 + h(a) + \eta, \quad \eta \in \mathcal{H}_a.
\]

which insures that $\eta$ is not in the non-decaying directions of the linearized equation (2.4) at $\psi_E$. The fact that this can be done in a unique manner is a consequence of the following lemma.

**Lemma 2.1.** There exists $\delta/2 > \delta_1 > 0$ such that any $\phi \in H^{-1}(\mathbb{R}^2)$ satisfying $\|\phi\|_{H^{-1}} \leq \delta_1$ can be uniquely decomposed:

\[\phi = \psi_E + \eta = a\psi_0 + h(a) + \eta\]

where $a = a_1 + ia_2 \in \mathbb{C}$, $|a| < \delta$, $\eta \in \mathcal{H}_a$. Moreover the maps $\phi \mapsto a$ and $\phi \mapsto \eta$ are $C^1$ and there exists the constant $C$ independent on $\phi$ such that

\[|a| \leq 2\|\phi\|_{H^{-1}}, \quad \|\eta\|_{H^{-1}} \leq C\|\phi\|_{H^{-1}},\]

while for $\phi \in L^2(\mathbb{R}^2)$ with $\|\phi\|_{L^2} \leq \delta_1$ we have $\eta \in L^2(\mathbb{R}^2)$ and

\[|a| \leq 2\|\phi\|_{L^2}, \quad \|\eta\|_{L^2} \leq C\|\phi\|_{L^2}.
\]

**Remark 2.2.** The above lemma uses the implicit function theorem applied to

\[F: \mathbb{R}^2 \times H^{-1}(\mathbb{R}^2) \mapsto \mathbb{R}^2, \quad F(a_1, a_2, \phi) = \begin{bmatrix} \Re(\Psi_1, \psi_E - \phi) \\ \Re(\Psi_2, \psi_E - \phi) \end{bmatrix},\]

where $\psi_E = (a_1 + ia_2)\psi_0 + h(a_1 + ia_2)$ and

\[\Psi_1(a_1, a_2) = -i \frac{\partial \psi_E}{\partial a_2} \left( \Re \left( -i \frac{\partial \psi_E}{\partial a_2}, \frac{\partial \psi_E}{\partial a_1} \right) \right)^{-1}, \]

\[\Psi_2(a_1, a_2) = i \frac{\partial \psi_E}{\partial a_1} \left( \Re \left( i \frac{\partial \psi_E}{\partial a_1}, \frac{\partial \psi_E}{\partial a_2} \right) \right)^{-1},\]

form the basis of $\{ \frac{\partial \psi_E}{\partial a_1}, \frac{\partial \psi_E}{\partial a_2} \}$ with respect to the decomposition (2.7). Note that

\[\frac{\partial F}{\partial (a_1, a_2)}(a_1, a_2, \phi) = I_{\mathbb{R}^2} - M(a_1, a_2, \phi)\]
where the entries of the two by two matrix $M$ are

$$M_{ij} = \Re \left( \frac{\partial \psi_i}{\partial a_j} \cdot \phi - \psi_E \right)$$

and, consequently, $M(0, 0, 0)$ is the zero matrix. Thus the implicit function theorem applies to $F = 0$, in a neighborhood of $(a_1, a_2, \phi) = (0, 0, 0)$ and the number $\delta_1$ in the above lemma is chosen such that

$$\|M_\phi\| = \|M(a_1(\phi), a_2(\phi), \phi)\| \leq \frac{1}{2}, \quad \text{whenever } \|\phi\|_{H^{-1}} \leq \delta_1,$$

and the norm of the matrix $M$ as a linear, bounded operator on $\mathbb{R}^2$ satisfies

$$\|M\phi\| = \|M(a_1(\phi), a_2(\phi), \phi)\| \leq \frac{1}{2}, \quad \text{whenever } \|\phi\|_{H^{-1}} \leq \delta_1,$$ (2.8)

see [12, Section 2] for details.

We need one more technical result relating the spaces $\mathcal{H}_a$ and the space corresponding to the continuous spectrum of $-\Delta + V$:

**Lemma 2.2.** With $\delta_1$ given by the previous lemma we have that for any $a \in \mathbb{C}$, $|a| \leq 2\delta_1$, the linear map $P_c|\mathcal{H}_a : \mathcal{H}_a \mapsto \mathcal{H}_0$ is invertible, and its inverse $R_a : \mathcal{H}_0 \mapsto \mathcal{H}_a$ satisfies

$$\|R_a\zeta\|_1^\sigma \leq C_{-\sigma} \|\zeta\|_1^\sigma, \quad \sigma \in \mathbb{R} \text{ and for all } \zeta \in \mathcal{H}_0 \cap L^2_{-\sigma},$$

$$\|R_a\zeta\|_1^p \leq C_p \|\zeta\|_1^p, \quad 1 \leq p \leq \infty \text{ and for all } \zeta \in \mathcal{H}_0 \cap L^p,$$

$$\overline{R_a\zeta} = R_a\bar{\zeta}$$

where the constants $C_{-\sigma}, C_p > 0$ are independent of $a \in \mathbb{C}$, $|a| \leq 2\delta_1$.

We are now ready to prove our main result.

3. The main result

**Theorem 3.1.** If hypotheses (1.3), (1.4), (H1) hold then for each $q'_0 < \frac{4+2\alpha}{3+2\alpha}$ there exists $\varepsilon_0 > 0$ such that for all initial conditions $u_0(x)$ satisfying

$$\max \left\{ \|u_0\|_{L^{q'_0}}, \|u_0\|_{H^1} \right\} \leq \varepsilon_0$$

the initial value problem (1.1)–(1.2) is globally well-posed in $H^1$, and the solution decomposes into a radiative part and a part that asymptotically converges to a ground state.

More precisely, there exists a $C^1$ function $a : \mathbb{R} \mapsto \mathbb{C}$ such that, for all $t \in \mathbb{R}$, we have

$$u(t, x) = a(t)\psi_0(x) + h(a(t)) + \eta(t, x)$$
where \( \psi_E(t) \) is on the central manifold (i.e. it is a ground state) and \( \eta(t,x) \in \mathcal{H}_a(t) \), see Proposition 2.1 and Lemma 2.1. Moreover, there exist the ground states \( \psi_{E_{\pm \infty}} \) and the \( C^1 \) function \( \tilde{\theta} : \mathbb{R} \rightarrow \mathbb{R} \) such that

\[
\lim_{t \to \pm \infty} \| \psi_E(t) - e^{-it(E_{\pm \tilde{\theta}(t)}(t))} \psi_{E_{\pm \infty}} \|_{H^2 \cap L^2_0} = 0, \quad \sigma \in \mathbb{R},
\]

(3.1)

while \( \eta \) satisfies the following decay estimates. Fix \( p_0 > \max\{ \frac{2}{\alpha_1 - 1/2}, (4 + 2\alpha_2) \frac{q_0 - 2}{q_0 - (4 + 4\alpha_2)} \} \), where \( q_0 = \frac{q_0^*}{q_0 - 2} > 4 + 2\alpha_2 \). Then for \( 2 \leq p \leq \frac{p_0q_0}{p_0 + q_0 - 2} \) we have

\[
\| \eta(t) \|_{L^p} \leq \left\{ \begin{array}{ll}
C_\varepsilon_0 \log \frac{1 - 2/\alpha_0}{1 + |t|^{2/\alpha_0}} & \text{if } \alpha_1 \geq 1 \text{ or } \alpha_1 < 1 \text{ and } p \leq \frac{2}{1 - \alpha_1 + 2/\beta_0}, \\
C_\varepsilon_0 \log \frac{1 - 2/\alpha_0}{1 + |t|^{2/\alpha_0}} & \text{if } \alpha_1 < 1 \text{ and } p > \frac{2}{1 - \alpha_1 + 2/\beta_0}.
\end{array} \right.
\]

(3.2)

for some constant \( C = C(p_0) \).

**Remark 3.1.** The estimates on \( \eta \) show that the component of the solution that does not converge to a ground states disperses like the solution of the free Schrödinger equation except for a logarithmic correction in \( L^p \) spaces for critical and supercritical regimes, \( \alpha_1 \geq 1 \). In subcritical regimes, \( \alpha_1 < 1 \), the decay rate remains comparable to the free Schrödinger one in \( L^p \) spaces for \( 2 \leq p < 2/(1 - \alpha_1) \), while it saturates to \( |t|^{\alpha_1 - 1/2} \) in \( L^p \), \( p > 2/(1 - \alpha_1) \). Note that \( p_0 \) in the previous theorem can be chosen as large as we wish.

**Proof of Theorem 3.1.** It is well known that under hypothesis (H1)(i) the initial value problem (1.1)--(1.2) is locally well-posed in the energy space \( H^1 \) and its \( L^2 \) norm is conserved, see for example [5, Corollary 4.3.3, p. 92]. Global well-posedness follows via energy estimates from \( \| u_0 \|_{H^1} \) small, see [5, Corollary 6.1.5, p. 165].

We choose \( \varepsilon_0 \leq \delta_1 \) given by Lemma 2.1. Then, for all times, \( \| u(t) \|_{H^{-1}} \leq \| u(t) \|_{L^2} \leq \varepsilon_0 \leq \delta_1 \) and, via Lemma 2.1, we can decompose the solution into a solitary wave and a dispersive component:

\[
u(t) = a(t) \psi_E(t) + h(a(t)) + \eta(t) = \psi_E(t) + \eta(t),
\]

where \( |a(t)| = |a_1(t) + ia_2(t)| \leq 2\varepsilon_0 \leq 2\delta_1, \forall t \in \mathbb{R}. \)

(3.3)

Note that since \( a \mapsto h(a) \) is \( C^2 \), see Proposition 2.1, and \( a \) is uniformly bounded in time we deduce that there exists the constant \( C_H > 0 \) such that

\[
\sup_{t \in \mathbb{R}} \left\{ \left\| \psi_E(t) \right\|_{H^2}, \left\| \frac{\partial \psi_E}{\partial a_1}(t) \right\|_{H^2}, \left\| \frac{\partial \psi_E}{\partial a_2}(t) \right\|_{H^2} \right\} \leq C_H \varepsilon_0,
\]

which combined with Corollary 2.1 implies that for any \( \sigma \in \mathbb{R} \) there exists a constant \( C_{H, \sigma} > 0 \) such that

\[
\sup_{t \in \mathbb{R}} \left\{ \left\| \chi^\sigma \psi_E(t) \right\|_{H^2}, \left\| \chi^\sigma \frac{\partial \psi_E}{\partial a_1}(t) \right\|_{H^2}, \left\| \chi^\sigma \frac{\partial \psi_E}{\partial a_2}(t) \right\|_{H^2} \right\} \leq C_{H, \sigma} \varepsilon_0,
\]

(3.4)

Consequently, using the continuous imbedding \( H^2(\mathbb{R}^2) \hookrightarrow L^p(\mathbb{R}^2), 2 \leq p < \infty, \) and \( L^2_0(\mathbb{R}^2) \hookrightarrow L^1(\mathbb{R}^2), \sigma > 1, \) we have that for all \( 1 \leq p \leq \infty \) and all \( \sigma \in \mathbb{R} \), there exist the constants \( C_{p, \sigma} \) such that

\[
\sup_{t \in \mathbb{R}} \left\{ \left\| \psi_E(t) \right\|_{L^p_{\sigma}}, \left\| \frac{\partial \psi_E}{\partial a_1}(t) \right\|_{L^p_{\sigma}}, \left\| \frac{\partial \psi_E}{\partial a_2}(t) \right\|_{L^p_{\sigma}}, \left\| \psi_1(a(t)) \right\|_{L^p_{\sigma}}, \left\| \psi_1(a(t)) \right\|_{L^p_{\sigma}} \right\} \leq C_{p, \sigma} \varepsilon_0.
\]

(3.5)
see Remark 2.2 for the definitions of $\Psi_j(a)$, $j = 1, 2$. In addition, since
\[ u \in C(\mathbb{R}, H^1(\mathbb{R}^2)) \cap C^1(\mathbb{R}, H^{-1}(\mathbb{R}^2)), \]
and $u \mapsto a$ respectively $u \mapsto \eta$ are $C^1$, see Lemma 2.1, we get that $a(t)$ is $C^1$ and $\eta \in C(\mathbb{R}, H^1) \cap C^1(\mathbb{R}, H^{-1})$.

The solution is now described by the $C^1$ function $a : \mathbb{R} \mapsto \mathbb{C}$ and $\eta(t) \in C(\mathbb{R}, H^1) \cap C^1(\mathbb{R}, H^{-1})$. To obtain estimates for them it is useful to first remove their dominant phase. Consider the $C^2$ function:

\[ \theta(t) = \int_0^t E(|a(s)|) \, ds \]  

(3.6)

and

\[ \tilde{u}(t) = e^{i\theta(t)} u(t), \]  

(3.7)

then $\tilde{u}(t)$ satisfies the differential equation

\[ i\partial_t \tilde{u}(t) = -E(|a(t)|) \tilde{u}(t) + (-\Delta + V)\tilde{u}(t) + g(\tilde{u}(t)), \]  

(3.8)

see (1.1) and (1.4). Moreover, like $u(t)$, $\tilde{u}(t)$ can be decomposed:

\[ \tilde{u}(t) = \tilde{a}(t) \psi_0 + h(\tilde{a}(t)) + \tilde{\eta}(t) \]  

(3.9)

where

\[ \tilde{a}(t) = e^{i\theta(t)} a(t), \quad \tilde{\eta}(t) = e^{i\theta(t)} \eta(t) \in \mathcal{H}_{\tilde{a}(t)}. \]  

(3.10)

By plugging in (3.9) into (3.8) we get

\[ i \frac{\partial \tilde{\eta}}{\partial t} + iD \tilde{\psi}_E |_{\tilde{a}} \frac{d \tilde{a}_1}{dt} + iD \tilde{\psi}_E |_{\tilde{a}} \frac{d \tilde{a}_2}{dt} = (-\Delta + V - E(|a|))(\psi_0 + \tilde{\eta}) + g(\psi_0 + \tilde{\eta}) + g(\psi_0 + \tilde{\eta}) - g(\tilde{\psi}_E) \]
\[ = L_{\tilde{\psi}_E} \tilde{\eta} + g_2(\tilde{\psi}_E, \tilde{\eta}), \]

or, equivalently,

\[ \frac{\partial \tilde{\eta}}{\partial t} + \frac{\partial \tilde{\psi}_E}{\partial \tilde{a}_1} \frac{d \tilde{a}_1}{dt} + \frac{\partial \tilde{\psi}_E}{\partial \tilde{a}_2} \frac{d \tilde{a}_2}{dt} = -IL_{\tilde{\psi}_E} \tilde{\eta} - ig_2(\tilde{\psi}_E, \tilde{\eta}) \]  

(3.11)

where $L_{\tilde{\psi}_E}$ is defined by (2.5):

\[ L_{\tilde{\psi}_E} \tilde{\eta} = (-\Delta + V - E(|\tilde{a}|)) \tilde{\eta} + \frac{d}{d\varepsilon} g(\psi_0 + \varepsilon \tilde{\eta}) |_{\varepsilon = 0} \]

and we used $|a| = |\tilde{a}|$, while $g_2$ is defined by

\[ g_2(\tilde{\psi}_E, \tilde{\eta}) = g(\psi_0 + \tilde{\eta}) - g(\tilde{\psi}_E) - \frac{d}{d\varepsilon} g(\psi_0 + \varepsilon \tilde{\eta}) |_{\varepsilon = 0} \]  

(3.12)
and we also used the fact that $\tilde{\psi}_E$ is a solution of the eigenvalue problem (2.2). Note that $g_2$ is at least quadratic in the second variable, more precisely:

**Lemma 3.1.** There exists a constant $C > 0$ such that for all $a, z \in \mathbb{C}$ we have

$$|g_2(a, z)| \leq C(|a|^{\alpha_1} + |a|^{\alpha_2} + |z|^{\alpha_1} + |z|^{\alpha_2})|z|^2.$$  

**Proof.** From the definition (3.12) of $g_2$ we have

$$g_2(a, z) = g(a + z) - g(a) - DG_1[a]z = \int_0^1 (DG_{a+\tau z} - DG_a)[z]d\tau \int_0^1 \int_0^1 D^2 g(x+\tau z)[z]d\tau ds.$$  

Now (1.3) and (1.4) imply that there exists a constant $C_1 > 0$ such that the bilinear form $D^2g$ on $\mathbb{C} \times \mathbb{C}$ satisfies

$$\|D^2g_b\| \leq C_1(|b|^{\alpha_1} + |b|^{\alpha_2}), \quad \forall b \in \mathbb{C}. \quad (3.13)$$

Hence

$$|g_2(a, z)| \leq C_1((2 \max(|a|, |z|))^{\alpha_1} + (2 \max(|a|, |z|))^{\alpha_2}) \frac{1}{2}|z|^2,$$

which proves the lemma. \(\square\)

We now project (3.11) onto the invariant subspaces of $-iL\tilde{\psi}_E$, namely span$_{\mathbb{R}}\{\frac{\partial\tilde{\psi}_E}{\partial t}, \frac{\partial\tilde{\psi}_E}{\partial x}\}$, and $H_2$. More precisely, we evaluate both the left- and right-hand side of (3.11) which are functionals in $H^{-1}(\mathbb{R}^2)$ at $\Psi_j = \Psi_j(\tilde{a}(t)), \ j = 1, 2$, see Remark 2.2, and take the real parts. We obtain

$$\begin{bmatrix} \mathfrak{R}(\Psi_1, \partial_t \tilde{a}) \\ \mathfrak{R}(\Psi_2, \partial_t \tilde{a}) \end{bmatrix} + \frac{d}{dt} \begin{bmatrix} \tilde{a}_1 \\ \tilde{a}_2 \end{bmatrix} = \begin{bmatrix} g_{21}(\tilde{\psi}_E, \tilde{\eta}) \\ g_{22}(\tilde{\psi}_E, \tilde{\eta}) \end{bmatrix}$$

where

$$g_{2j}(\tilde{\psi}_E, \tilde{\eta}) = \mathfrak{R}(\Psi_j, -ig_2(\tilde{\psi}_E, \tilde{\eta})), \quad j = 1, 2. \quad (3.14)$$

Note that from Lemma 3.1 and Hölder inequality we have for all $t \in \mathbb{R}$:

$$|g_{2j}(\tilde{\psi}_E(t), \tilde{\eta}(t))|$$

$$\leq C \int_{\mathbb{R}^2} |\Psi_j(t, x)| (|\tilde{\psi}_E(t, x)|^{\alpha_1} + |\tilde{\psi}_E(t, x)|^{\alpha_2} + |\tilde{\eta}(t, x)|^{\alpha_1} + |\tilde{\eta}(t, x)|^{\alpha_2})|\tilde{\eta}(t, x)|^2 dx$$

$$\leq C \left[ ||\Psi_j(t)||_{L^6} \left( ||\tilde{\psi}_E(t)||_{L^6}^{\alpha_1} + ||\tilde{\psi}_E(t)||_{L^6}^{\alpha_2} \right) ||\tilde{\eta}(t)||_{L^2}^2 \right.$$

$$\left. + ||\Psi_j(t)||_{L^1} ||\tilde{\eta}(t)||_{L^6}^{2+\alpha_1} + ||\Psi_j(t)||_{L^2} ||\tilde{\eta}(t)||_{L^6}^{2+\alpha_2} \right]$$

$$\quad (3.15)$$

where $r_0^{-1} + (p_2/2)^{-1} = 1, r_j^{-1} + (p_2/(2 + \alpha_j))^{-1} = 1, j = 1, 2.$

To calculate $\mathfrak{R}(\Psi_j, \partial_t \tilde{a}(t)), \ j = 1, 2$, we use the fact that $\tilde{\eta}(t) \in H_2$, for all $t \in \mathbb{R}$, i.e.

$$\mathfrak{R}(\Psi_j(\tilde{a}(t)), \tilde{\eta}(t)) \equiv 0.$$
Differentiating the latter with respect to \( t \) we get

\[
\Re\left( \Psi_j, \eta \frac{\partial \eta}{\partial t} \right) = -\Re\left( \frac{\partial \Psi_j}{\partial a_1} \frac{\partial a_1}{\partial t} + \frac{\partial \Psi_j}{\partial a_2} \frac{\partial a_2}{\partial t}, \eta \right), \quad j = 1, 2,
\]

which replaced into above leads to

\[
\frac{d}{dt} \begin{bmatrix} \tilde{a}_1 \\ \tilde{a}_2 \end{bmatrix} = (I - M_{\tilde{a}})^{-1} \begin{bmatrix} g_{21}(\tilde{\Psi}_E, \tilde{\eta}) \\ g_{22}(\tilde{\Psi}_E, \tilde{\eta}) \end{bmatrix}
\]

(3.16)

where the two by two matrix \( M_{\tilde{a}} \) is defined in Remark 2.2. In particular

\[
\begin{bmatrix} \Re(\Psi_1, \eta \frac{\partial \eta}{\partial t}) \\ \Re(\Psi_2, \eta \frac{\partial \eta}{\partial t}) \end{bmatrix} = -M_{\tilde{a}}(I - M_{\tilde{a}})^{-1} \begin{bmatrix} g_{21}(\tilde{\Psi}_E, \tilde{\eta}) \\ g_{22}(\tilde{\Psi}_E, \tilde{\eta}) \end{bmatrix},
\]

which we use to obtain the component in \( \mathcal{H}_{\tilde{a}} = \{ \Psi_1(\tilde{a}), \Psi_2(\tilde{a}) \}^\perp \) of (3.11):

\[
\frac{\partial \tilde{\eta}}{\partial t} + M_{\tilde{a}}(I - M_{\tilde{a}})^{-1} g_3(\tilde{\Psi}_E, \tilde{\eta}) = -iL_{\tilde{\Psi}_E} \tilde{\eta} - ig_2(\tilde{\Psi}_E, \tilde{\eta}) - g_3(\tilde{\Psi}_E, \tilde{\eta}),
\]

or, equivalently:

\[
\frac{\partial \tilde{\eta}}{\partial t} = -iL_{\tilde{\Psi}_E} \tilde{\eta} - ig_2(\tilde{\Psi}_E, \tilde{\eta}) - (I - M_{\tilde{a}})^{-1} g_3(\tilde{\Psi}_E, \tilde{\eta})
\]

where \( g_3 \) is the projection of \(-ig_2\) onto \( \operatorname{span}_{\Re} \{ \frac{\partial \tilde{\Psi}_E}{\partial a_1}, \frac{\partial \tilde{\Psi}_E}{\partial a_2} \} \) relative to the decomposition (2.7):

\[
g_3(\tilde{\Psi}_E, \tilde{\eta}) = g_{21}(\tilde{\Psi}_E, \tilde{\eta}) \frac{\partial \tilde{\Psi}_E}{\partial a_1} + g_{22}(\tilde{\Psi}_E, \tilde{\eta}) \frac{\partial \tilde{\Psi}_E}{\partial a_2},
\]

(3.17)

see (3.14) for the definitions of \( g_{2j}, j = 1, 2 \), and \( I - M_{\tilde{a}} \) is the linear operator on the two-dimensional real vector space \( \operatorname{span}_{\Re} \{ \frac{\partial \tilde{\Psi}_E}{\partial a_1}, \frac{\partial \tilde{\Psi}_E}{\partial a_2} \} \) whose matrix representation relative to the basis \( \{ \frac{\partial \tilde{\Psi}_E}{\partial a_1}, \frac{\partial \tilde{\Psi}_E}{\partial a_2} \} \) is \( I_{\Re^2} - M_{\tilde{a}} \). It is easier to switch back to the variable \( \eta(t) = e^{-i\Theta(t)} \tilde{\eta}(t) \) in \( \mathcal{H}_{\tilde{a}} \):

\[
\frac{\partial \eta}{\partial t} = -i(-\Delta + V)\eta - iD_{\tilde{\Psi}_E} \eta - ig_2(\tilde{\Psi}_E, \eta) - (I - M_{\tilde{a}})^{-1} g_3(\tilde{\Psi}_E, \eta)
\]

(3.18)

where we used the equivariant symmetry (1.4) and its obvious consequences for the symmetries of \( Dg, g_2, g_3 \) and \( M \). Since by Lemma 2.2 it is sufficient to get estimates for \( \zeta(t) = P_c \eta(t) \), we now project (3.18) onto the continuous spectrum of \(-\Delta + V - P_c \tilde{a} \):

\[
\frac{\partial \zeta}{\partial t} = -i(-\Delta + V)\zeta - iP_c D_{\tilde{\Psi}_E} R_{\tilde{a}} \zeta - iP_c g_2(\tilde{\Psi}_E, \tilde{\zeta}) - P_c(I - M_{\tilde{a}})^{-1} g_3(\tilde{\Psi}_E, R_{\tilde{a}} \zeta)
\]

(3.19)

where \( R_{\tilde{a}} : \mathcal{H}_0 \rightarrow \mathcal{H}_a \) is the inverse of \( P_c \) restricted to \( \mathcal{H}_a \), see Lemma 2.2.

Consider the initial value problem for the linear part of (3.19):

\[
\frac{\partial z}{\partial t} = -i(-\Delta + V)z - iP_c D_{\tilde{\Psi}_E(t)} R_{\tilde{a}(t)} z(t),
\]

\[
z(s) = \nu \in \mathcal{H}_0
\]

(3.20)
and write its solution in terms of a family of operators:
\[
\Omega (t, s) : \mathcal{H}_0 \mapsto \mathcal{H}_0, \quad \Omega (t, s) v = z(t), \quad t, s \in \mathbb{R}. \tag{3.21}
\]

In Section 4 we show that such a family of operators exists, is uniformly bounded in \( t, s \) with respect to the \( L^2 \) norm and it has very similar properties with the unitary group of operators \( e^{-i(\Delta + V)(t-s)P_c} \) generated by the Schrödinger operator \(-i(\Delta + V)P_c\). In particular \( \Omega (t, s) \) satisfies certain dispersive decay estimates in weighted \( L^2 \) spaces and \( L^p, p > 2 \), spaces, see Theorems 4.1 and 4.2. For all these results to hold we only need to choose \( \varepsilon_0 \) small enough such that \( \varepsilon_0 C_{\mathcal{H}, 4} \frac{\sigma}{3} \leq \varepsilon_1 \), where \( \sigma > 1 \) and \( \varepsilon_1 > 0 \) are fixed in Section 4 and the constant \( C_{\mathcal{H}, 4, \sigma/3} \) is the one from (3.4).

Using Duhamel formula, the solution \( \zeta \in C(\mathbb{R}, H^1 \cap \mathcal{H}_0) \cap C^1(\mathbb{R}, H^{-1}(\mathbb{R}^2) \cap \mathcal{H}_0) \) of (3.19) also satisfies
\[
\zeta (t) = \Omega (t, 0) \zeta (0) - i \int_0^t \Omega (t, s) P_c g_2 (\psi_E (s), R_{a(s)} \zeta (s)) \, ds 
- \int_0^t \Omega (t, s) P_c (\mathbb{I} - M_{a(s)})^{-1} g_3 (\psi_E (s), R_{a(s)} \zeta (s)) \, ds. \tag{3.22}
\]

Note that the right-hand side of (3.22) contains only terms that are quadratic and higher order in \( \zeta \), see Lemma 3.1 and (3.15). As in [12,13] this is essential in controlling low power nonlinearities and it is the main difference between our approach and the existing literature on asymptotic stability of coherent structures for dispersive nonlinear equations, see [13, p. 449] for a more detailed discussion.

To obtain estimates for \( \zeta \) we apply a contraction mapping argument to the fixed point problem (3.22) in the following Banach space. Fix \( p_0 > 2 \) such that
\[
p_0 > \max \left\{ \frac{2}{\alpha_1 - 1/2}, (4 + 2\alpha_2) \frac{q_0 - 2}{q_0 - (4 + 2\alpha_2)} \right\}, \tag{3.23}
\]
and let
\[
p_2 = \frac{p_0 q_0}{p_0 + q_0 - 2}, \tag{3.24}
\]
and
\[
p_1 = \frac{2}{1 - \alpha_1 + 2/p_0} \quad \text{if} \quad \alpha_1 < 1, \tag{3.25}
\]
then:

Case I: if \( \alpha_1 \geq 1 \), or \( 1/2 < \alpha_1 < 1 \) and \( p_1 \geq p_2 \), let
\[
Y = \left\{ v \in C(\mathbb{R}, L^2 \cap L^{p_2}) : \sup_{t \in \mathbb{R}} \| v(t) \|_{L^2} < \infty, \sup_{t \in \mathbb{R}} \frac{(1 + |t|) \| v(t) \|_{L^{p_2}}}{|\log(2 + |t|)|} < \infty \right\};
\]

Case II: if \( 1/2 < \alpha_1 < 1 \) and \( p_1 < p_2 \), let
In a closed ball in the Banach space endowed with the norm

\[ \|v\| = \max \left\{ \sup_{t \in \mathbb{R}} \|v(t)\|_{L^2}, \sup_{t \in \mathbb{R}} \frac{(1 + |t|)^{1 - \frac{2}{p_1}}}{1 - \frac{2}{p_1}} \|v(t)\|_{L^p_1}, \sup_{t \in \mathbb{R}} \frac{(1 + |t|)^{1 - \frac{2}{p_0}}}{1 - \frac{2}{p_0}} \|v(t)\|_{L^p_2} \right\} \]

in Case I, while in Case II

\[ \|v\| = \max \left\{ \sup_{t \in \mathbb{R}} \|v(t)\|_{L^2}, \sup_{t \in \mathbb{R}} \frac{(1 + |t|)^{1 - \frac{2}{p_2}}}{1 - \frac{2}{p_2}} \|v(t)\|_{L^p_1}, \sup_{t \in \mathbb{R}} \frac{(1 + |t|)^{1 - \frac{2}{p_0}}}{1 - \frac{2}{p_0}} \|v(t)\|_{L^p_2} \right\} . \]

Consider now the nonlinear operator in (3.22):

\[ N(v)(t) = -i \int_0^t \Omega(t, s) P_c g_2(\psi_E(s), R_{a(s)} v(s)) \, ds \]

\[ - \int_0^t \Omega(t, s) P_c (\mathbb{I} - M_{u(s)})^{-1} g_3(\psi_E(s), R_{a(s)} v(s)) \, ds. \]

We have:

**Lemma 3.2.** \( N : Y \to Y \) is well defined and locally Lipschitz, i.e. there exists \( \tilde{C} > 0 \), such that

\[ \|Nv_1 - Nv_2\|_Y \leq \tilde{C} \left( \|v_1\|_Y + \|v_2\|_Y + \|v_1\|_{Y^1}^{1+\alpha_1} + \|v_2\|_{Y^1}^{1+\alpha_1} + \|v_1\|_{Y^1}^{1+\alpha_2} + \|v_2\|_{Y^1}^{1+\alpha_2} \right) \|v_1 - v_2\|_Y. \]

Assuming that the lemma has been proven then we can apply the contraction principle for (3.22) in a closed ball in the Banach space \( Y \) in the following way. Let

\[ v = \Omega(t, 0) \zeta(0) \]

then by Theorem 4.2

\[ \|v\|_Y \leq \max\{C_2, C_{p_0, p_1}, C_{p_0, p_2}\} \|\zeta(0)\|_{L^2 \cap L^{q_0'}} \]

where we used the interpolation \( \|\zeta(0)\|_{L^r} \leq \|\zeta(0)\|_{L^2 \cap L^{q_0'}}, q_0' \leq r \leq 2 \) with \( r = q' \) and \( r = p' \) defined in Theorem 4.2 for \( p = p_j, j = 1, 2 \). Recall that

\[ \zeta(0) = P_c \eta(0) = P_c u_0 - h(a(0)) = u_0 - \langle \psi_0, u_0 \rangle \psi_0 - h(a(0)) \]
where \( u_0 = u(0) \) is the initial data, see also (3.3). Hence

\[
\| \xi(0) \|_{L^2 \cap L^\theta'} \leq \| u_0 \|_{L^2 \cap L^\theta'} + \| u_0 \|_{L^2} \| \psi_0 \|_{L^2 \cap L^\theta'} + D_1 \| u_0 \|_{L^2} \leq D \varepsilon_0
\]

where \( D, D > 0 \) are constants independent on \( u_0 \) and the estimate on \( h(a(0)) \) follows from Proposition 2.1 and \( |a(0)| \leq 2 \| u_0 \|_{L^2} \), see Lemma 2.1.

Therefore we can choose \( \varepsilon_0 \) small enough such that \( R = 2 \| v \|_{Y} \) satisfies

\[
\text{Lip} \overset{\text{def}}{=} 2 \tilde{C} (R + R^{1+\alpha_1} + R^{1+\alpha_2}) < 1.
\]

In this case the integral operator given by the right-hand side of (3.22):

\[
K(\xi) = v + N(\xi)
\]

leaves \( B(0, R) = \{ \xi \in Y : \| \xi \|_Y \leq R \} \) invariant and it is a contraction on it with Lipschitz constant \( \text{Lip} \) defined above. Consequently Eq. (3.22) has a unique solution in \( B(0, R) \) and because \( \xi(t) \in C(\mathbb{R}, H^1) \hookrightarrow C(\mathbb{R}, L^2 \cap L^p_1 \cap L^p_2) \) already verified the equation we deduce that \( \xi(t) \) is in \( B(0, R) \), in particular it satisfies the estimates (3.2).

Then \( \eta(t) = R_0(t) \xi(t) \) satisfies (3.2) because of Lemma 2.2. Moreover, the system of ODEs (3.16) has integrable in time right-hand side because the matrix has norm bounded by 2, see (2.8), while \( g_{2j} \) satisfy (3.15) where \( \tilde{\eta}(t) \) differs from \( \eta(t) \) by only a phase and the \( L^p, 1 \leq p \leq \infty \), norms of \( \Psi_i(t), \psi_E(t) \) are uniformly bounded in time, see (3.5). Consequently \( \tilde{a}_1(t) \) and \( \tilde{a}_2(t) \) converge as \( t \to \pm \infty \), and there exist the constants \( C, \varepsilon > 0 \) such that

\[
\lim_{t \to \pm \infty} \tilde{a}(t) = \lim_{t \to \pm \infty} \tilde{a}_1(t) + i \tilde{a}_2(t) \overset{\text{def}}{=} a_{\pm \infty}, \quad \| a(\pm t) - a_{\pm \infty} \| \leq C(1 + t)^{-\varepsilon}, \quad \text{for all } t \geq 0.
\]

We can now define

\[
\psi_{E_{\pm \infty}} = a_{\pm \infty} \psi_0 + h(a_{\pm \infty}), \quad (3.27)
\]

and we have

\[
\lim_{t \to \pm \infty} \| \tilde{\psi}_E(t) - \psi_{E_{\pm \infty}} \|_{H^2 \cap L^2} = 0, \quad \text{for } \sigma \in \mathbb{R}, \quad (3.28)
\]

where we used (3.9) and the continuity of \( h(a) \), see Proposition 2.1. In addition, since \( E : [-2\delta_1, 2\delta_1] \to (-\delta, \delta) \) is a \( C^1 \) function, see Proposition 2.1, the following limits exist together with the constant \( C_1 > 0 \) such that

\[
\lim_{t \to \pm \infty} E(\| a(t) \|) = E_{\pm \infty}, \quad \| E(\| a(\pm t) \|) - E_{\pm \infty} \| \leq C_1 (1 + t)^{-\varepsilon}, \quad \text{for all } t \geq 0.
\]

If we define

\[
\tilde{\theta}(t) = \begin{cases} \frac{1}{\tau} \int_0^\tau E(\| \tilde{a}(s) \|) - E_{\pm \infty} \, ds & \text{if } t > 0, \\ 0 & \text{if } t = 0, \\ \frac{1}{\tau} \int_0^\tau E(\| \tilde{a}(s) \|) - E_{\pm \infty} \, ds & \text{if } t < 0, \end{cases} \quad (3.29)
\]

then
Proof of Lemma 3.2. It suffices to prove the estimate

\[
\| Nv_1 - Nv_2 \|_Y \\
\leq \tilde{C}(\| v_1 \|_Y + \| v_2 \|_Y + \| v_1 \|_Y^{1-\alpha_1} + \| v_2 \|_Y^{1-\alpha_2} + \| v_1 \|_Y^{1+\alpha_1} + \| v_2 \|_Y^{1+\alpha_2}) \| v_1 - v_2 \|_Y, \tag{3.31}
\]

because plugging in \( v_2 \equiv 0 \) and using \( N(0) \equiv 0 \), see (3.26), will then imply \( N(v_1) \in Y \) whenever \( v_1 \in Y \).

Note that via interpolation in \( L^p \) spaces we have for all \( v \in Y \) and any \( 2 \leq p \leq p_2 \):

\[
\| v(t) \|_{L^p} \leq \begin{cases} \| v \|_Y \log^{1-2/p} \left( \frac{2}{1+|t|} \right) & \text{if } \alpha_1 \geq 1 \text{ or } \alpha_1 < 1 \text{ and } p \leq \frac{2}{1-\alpha_1 + 2/p_0}, \\
\| v \|_Y \log^{\alpha_1-2/p0} \left( \frac{2}{1+|t|} \right) & \text{if } \alpha_1 < 1 \text{ and } p > \frac{2}{1-\alpha_1 + 2/p_0}.
\end{cases} \tag{3.32}
\]

Now, from (3.12), we have for any \( v_1, v_2 \in Y \):

\[
ge_2(\psi_E, R_0 v_1) - g_2(\psi_E, R_0 v_2) \\
= g(\psi_E + R_0 v_1) - g(\psi_E + R_0 v_2) - Dg_{\psi_E} [R_0 (v_1 - v_2)] \\
= \int_0^1 (Dg_{\psi_E + R_0 (\tau v_1 + (1-\tau) v_2)} - Dg_{\psi_E}) [R_0 (v_1 - v_2)] d\tau \\
= \int_0^1 \int_0^1 D^2 g_{\psi_E + s R_0 (\tau v_1 + (1-\tau) v_2)} [R_0 (\tau v_1 + (1-\tau) v_2)] [R_0 (v_1 - v_2)] d\tau ds \\
= A_1(\psi_E, v_1, v_2) + A_2(\psi_E, v_1, v_2) + A_3(\psi_E, v_1, v_2) \tag{3.33}
\]

where we consider \( \chi_j(t, x), \; j = 1, 2, \) to be the characteristic function of the set \( S_1 = \{(t, x) \in \mathbb{R} \times \mathbb{R}^2: |\psi(t, x)| \geq \max(|R_{a(t)} v_1(t, x), |R_{a(t)} v_2(t, x))|), \) respectively \( S_2 = \{(t, x) \in \mathbb{R} \times \mathbb{R}^2: \max(|R_{a(t)} v_1(t, x), |R_{a(t)} v_2(t, x))| \leq 1 \} \) and

\[
A_1(\psi_E, v_1, v_2) = \int_0^1 \int_0^1 \chi_1 D^2 g_{\psi_E + s R_0 (\tau v_1 + (1-\tau) v_2)} [R_0 (\tau v_1 + (1-\tau) v_2)] [R_0 (v_1 - v_2)] d\tau ds,
\]
where we used (3.13). Consequently, for any $x \in \mathbb{R}^2$ we have the pointwise estimates:

\[
\begin{align*}
|A_1(\psi_E(t, x), v_1(t, x), v_2(t, x))| &
\leq C(2^{\alpha_1} |\psi_E(t, x)|^{\alpha_1} + 2^{\alpha_2} |\psi_E(t, x)|^{\alpha_2}) \left( |R_{a(t)}v_1(t, x)| + |R_{a(t)}v_2(t, x)| \right) \\
&\times |R_{a(t)}(v_1(t, x) - v_2(t, x))|,
\end{align*}
\]

\[
\begin{align*}
|A_2(\psi_E(t, x), v_1(t, x), v_2(t, x))| &
\leq 2^{\alpha_1} C \left( |R_{a(t)}v_1(t, x)|^{1+\alpha_1} + |R_{a(t)}v_2(t, x)|^{1+\alpha_1} \right) |R_{a(t)}(v_1(t, x) - v_2(t, x))|,
\end{align*}
\]

\[
\begin{align*}
|A_3(\psi_E(t, x), v_1(t, x), v_2(t, x))| &
\leq 2^{\alpha_2} C \left( |R_{a(t)}v_1(t, x)|^{1+\alpha_2} + |R_{a(t)}v_2(t, x)|^{1+\alpha_2} \right) |R_{a(t)}(v_1(t, x) - v_2(t, x))|,
\end{align*}
\]

where we used (3.13). Consequently, for any $\sigma \in \mathbb{R}$ there exists a constant $C_\sigma > 0$ such that for any $t \in \mathbb{R}$:

\[
\|A_1(\psi_E(t), v_1(t), v_2(t))\|_{L^p_0} \leq C \|2^{\alpha_1} |\psi_E(t)|^{\alpha_1} + 2^{\alpha_2} |\psi_E(t)|^{\alpha_2}\|_{L^p_0} \\
\times \left( \|R_{a(t)}v_1(t)\|_{L^p_2} + \|R_{a(t)}v_2(t)\|_{L^p_2} \right) \|R_{a(t)}(v_1(t) - v_2(t))\|_{L^p_2} \\
\leq \frac{C_\sigma \log^2(2 + |t|)}{(1 + |t|)^{b_1}} (\|v_1\|_Y + \|v_2\|_Y) \|v_1 - v_2\|_Y
\]
see the definition of the Banach space $Y$, and we used Hölder inequality together with (3.5) and Lemma 2.2.

Similarly, for any $1 \leq r' \leq 2$ we have $(2 + \alpha_1) r' \leq (2 + \alpha_2) r' \leq p_2$, hence the above pointwise estimates and (3.32) imply that there exists a constant $C_r' > 0$ such that for any $t \in \mathbb{R}$:

$$
\| A_2(\psi_E(t), v_1(t), v_2(t)) \|_{L^{r'}} \leq 2^{\alpha_1} C \| R_{a(t)} v_1(t) \|_{L^{1+\alpha_1}}^{1+\alpha_1} + |R_{a(t)} v_2(t)|_{L^{1+\alpha_1}}^{1+\alpha_1} \| R_{a(t)} (v_1(t) - v_2(t)) \|_{L^{(2+\alpha_1) r'}}^R \\
\leq \frac{C_r' \log^{a_2(r')}(2 + |t|)}{(1 + |t|)^{b_2(r')}} (\| v_1 \|_{Y}^{1+\alpha_1} + \| v_2 \|_{Y}^{1+\alpha_1}) \| v_1 - v_2 \|_Y 
$$

(3.37)

where

$$
b_2(r') = \alpha_1 + \frac{2}{r'}, \quad a_2(r') = \frac{\alpha_1 + 2/r}{1 - 2/p_0} \quad \text{if } \alpha_1 \geq 1 \text{ or } \alpha_1 < 1 \text{ and } (2 + \alpha_1) r' \leq p_1,
$$

$$
b_2(r') = (2 + \alpha_1) \left( \alpha_1 - \frac{2}{p_0} \right), \quad a_2(r') = (2 + \alpha_1) \frac{\alpha_1 - 2/p_0}{1 - 2/p_0} \quad \text{if } \alpha_1 < 1 \text{ and } (2 + \alpha_1) r' > p_1.
$$

(3.38)

with $1/r + 1/r' = 1$, and

$$
\| A_3(\psi_E(t), v_1(t), v_2(t)) \|_{L^{r'}} \leq 2^{\alpha_2} C \| R_{a(t)} v_1(t) \|_{L^{1+\alpha_2}}^{1+\alpha_2} + |R_{a(t)} v_2(t)|_{L^{1+\alpha_2}}^{1+\alpha_2} \| R_{a(t)} (v_1(t) - v_2(t)) \|_{L^{(2+\alpha_2) r'}}^R \\
\leq \frac{C_r' \log^{a_3(r')}(2 + |t|)}{(1 + |t|)^{b_3(r')}} (\| v_1 \|_{Y}^{1+\alpha_2} + \| v_2 \|_{Y}^{1+\alpha_2}) \| v_1 - v_2 \|_Y 
$$

(3.39)

where

$$
b_3(r') = \alpha_2 + \frac{2}{r'}, \quad a_3(r') = \frac{\alpha_2 + 2/r}{1 - 2/p_0} \quad \text{if } \alpha_1 \geq 1 \text{ or } \alpha_1 < 1 \text{ and } (2 + \alpha_2) r' \leq p_1,
$$

$$
b_3(r') = (2 + \alpha_2) \left( \alpha_1 - \frac{2}{p_0} \right), \quad a_3(r') = (2 + \alpha_2) \frac{\alpha_1 - 2/p_0}{1 - 2/p_0} \quad \text{if } \alpha_1 < 1 \text{ and } (2 + \alpha_2) r' > p_1.
$$

(3.40)

Moreover, using Cauchy–Schwartz inequality and (3.5) we have

$$
|\mathcal{H}(\psi_{a(t)}, -iA_2(\psi_E(t), v_1(t), v_2(t)))| \\
\leq \| \mathcal{H}(\psi_{a(t)}) \|_{L^2} \| A_2(\psi_E(t), v_1(t), v_2(t)) \|_{L^2} \\
\leq C_{2,0} \frac{C_2 \log^{a_2(r')}(2 + |t|)}{(1 + |t|)^{b_2(r')}} (\| v_1 \|_{Y}^{1+\alpha_1} + \| v_2 \|_{Y}^{1+\alpha_1}) \| v_1 - v_2 \|_Y,
$$

(3.41)

and
\[
\|N(\psi_E(t), v_1(t), v_2(t))\| \leq C_{2.0} \frac{C_2 \log^{a_2}(2 + |t|)}{(1 + |t|)^{b_2}} (\|v_1\|_Y^{1+\sigma_2} + \|v_2\|_Y^{1+\alpha_2}) \|v_1 - v_2\|_Y.
\]

(3.42)

Now, from (3.17) and (3.14) we have

\[
g_3(\psi_E, R_n v_1) - g_3(\psi_E, R_n v_2)
= \Re(\langle \psi_1(a), -i(g_2(\psi_E, R_n v_1) - g_2(\psi_E, R_n v_2)) \psi_1 \rangle E_{1s})
+ \Re(\langle \psi_2(a), -i(g_2(\psi_E, R_n v_1) - g_2(\psi_E, R_n v_2)) \psi_2 \rangle E_{2s})

= \Re(\langle \psi_1(a), -i(A_1 + A_2 + A_3)(\psi_E, v_1, v_2) \psi_1 \rangle E_{1s})
+ \Re(\langle \psi_2(a), -i(A_1 + A_2 + A_3)(\psi_E, v_1, v_2) \psi_2 \rangle E_{2s}).
\]

Consequently, for

\[
A_4(\psi_E, v_1, v_2) \stackrel{\text{def}}{=} (I - M_u)^{-1}(g_3(\psi_E, R_n v_1) - g_3(\psi_E, R_n v_2))
\]

(3.43)

we have that for any \( \sigma \in \mathbb{R} \) there exists a constant \( C_\sigma > 0 \) such that

\[
\left\| A_4(\psi_E(t), v_1(t), v_2(t)) \right\|_{L^p_2} \leq \max\left\{ \left\| \frac{\partial \psi_E}{\partial a_1}(t) \right\|_{L^2_2}, \left\| \frac{\partial \psi_E}{\partial a_2}(t) \right\|_{L^2_2} \right\} \\sqrt{2} \left\| (I - M_u(t))^{-1} \right\|_{\mathbb{R}^2 \to \mathbb{R}^2}
\times \left\| \Re(\langle \psi_1(a), -i(A_1 + A_2 + A_3)(t) \psi_1 \rangle E_{1s}) \right\|_2^2 + \left\| \Re(\langle \psi_2(a), -i(A_1 + A_2 + A_3)(t) \psi_2 \rangle E_{2s}) \right\|_2^2
\leq \frac{C_\sigma \log^{a_4}(2 + |t|)}{(1 + |t|)^{b_4}} (\|v_1\|_Y + \|v_2\|_Y + \|v_1\|_Y^{1+\alpha_1})
+ \|v_2\|_Y^{1+\alpha_1} + \|v_1\|_Y^{1+\sigma_2} + \|v_2\|_Y^{1+\sigma_2}) \|v_1 - v_2\|_Y
\]

(3.44)

where

\[
b_4 = \min\{b_1, b_2(2), b_3(2)\}, \quad a_4 = \max\{a_1, a_2(2), a_3(2)\},
\]

(3.45)

and we used (3.5), (2.8), (3.35), (3.41), and (3.42).

We are now ready to prove the Lipschitz estimate for the nonlinear operator \( N \), (3.31). From its definition (3.26) and (3.33), (3.43) we have for any \( v_1, v_2 \in \mathcal{Y} \), any \( 2 \leq p \leq p_2 \), and a fixed \( \sigma > 1 \):

\[
\| N(v_1)(t) - N(v_2)(t) \|_{L^p} \leq \int_0^t \| \Omega(t, s) P_c (-iA_1 - iA_2 - iA_3 - A_4)(\psi_E(s), v_1(s), v_2(s)) \|_{L^p} ds
\]

\[
\leq \int_0^t \| \Omega(t, s) \|_{L^2 \to L^p}(\| A_1(\psi_E(s), v_1(s), v_2(s)) \|_{L^2})
\]

from (3.36), (3.38), (3.40) and (3.45) for where

\begin{align}
\int_{0}^{t} \| \Omega(t, s) \|_{L^{p'} \cap L^{r}'} \left( \| A_1(\psi_E(s), v_1(s), v_2(s)) \|_{L^p} + \| A_4(\psi_E(s), v_1(s), v_2(s)) \|_{L^p} \right) ds \\
\leq \left( \| v_1 \|_{L^1} + \| v_2 \|_{L^1} \right) \| v_1 - v_2 \|_Y \\
\times \int_{0}^{t} \frac{C_p \log^2(2 + |s|)}{|1 - |s| |^{2/p} + C_p \log^2(2 + |s|)} ds,
\end{align}

while from Theorem 4.2 and estimates (3.37), (3.39) we get

\begin{align}
\int_{0}^{t} \| \Omega(t, s) \|_{L^{p'} \cap L^{r}'} \| A_2(\psi_E(s), v_1(s), v_2(s)) \|_{L^{p'} \cap L^r} ds \\
\leq \left( \| v_1 \|_{L^1} + \| v_2 \|_{L^1} \right) \| v_1 - v_2 \|_Y \\
\times \int_{0}^{t} \frac{C_p \log^2(2 + |s|)}{|1 - |s| |^{2/p} + C_p \log^2(2 + |s|)} ds,
\end{align}

and

\begin{align}
\int_{0}^{t} \| \Omega(t, s) \|_{L^{p'} \cap L^{r}'} \| A_3(\psi_E(s), v_1(s), v_2(s)) \|_{L^{p'} \cap L^r} ds \\
\leq \left( \| v_1 \|_{L^1} + \| v_2 \|_{L^1} \right) \| v_1 - v_2 \|_Y \\
\times \int_{0}^{t} \frac{C_p \log^2(2 + |s|)}{|1 - |s| |^{2/p} + C_p \log^2(2 + |s|)} ds.
\end{align}

In Case I, i.e. \( \alpha_1 \geq 1 \), or \( 1/2 < \alpha_1 < 1 \) and \( p_1 \geq p_2 \), since \( \alpha_2 \geq \alpha_1 \) and \( p_2 \geq 4 + 2\alpha_2 > 4 \), we have from (3.36), (3.38), (3.40) and (3.45) for \( r' \in [q', p', 2] \) and \( 1/r + 1/r' = 1 \):
\[ b_1 = 2 - \frac{4}{p_2} > 1, \quad b_2(r') = \alpha_1 + \frac{2}{r} > 1, \]
\[ b_3(r') = \alpha_2 + \frac{2}{r} > 1, \quad b_4 = \min \{b_1, b_2(2), b_3(2)\} > 1. \]

We now use the following known convolution estimate:
\[
\int_0^{|t|} \frac{\log^d(2 + |t - s|) \log^c(2 + |s|)}{|t - s|^b} \, ds \leq C(a, b, c, d) \frac{\log^d(2 + |t|)}{(1 + |t|)^b}, \quad \text{for } d > 1, \quad b < 1, \quad (3.47)
\]
to bound the integral terms above and obtain for all \(2 \leq p \leq p_2\):
\[
\| N(v_1)(t) - N(v_2)(t) \|_{L^p} \leq C_p \frac{\log^{\frac{1-2/p_0}{p_0}}(2 + |t|)}{(1 + |t|)^{1-2/p}} \times (\| v_1 \|_Y + \| v_2 \|_Y + \| v_1 \|^1_{Y^{1+\alpha_1}} + \| v_2 \|^1_{Y^{1+\alpha_1}} + \| v_1 \|^1_{Y^{1+\alpha_2}} + \| v_2 \|^1_{Y^{1+\alpha_2}}) \| v_1 - v_2 \|_Y \quad (3.48)
\]
which, upon moving the time-dependent terms to the left-hand side and taking supremum over \(t \in \mathbb{R}\) when \(p \in (2, p_2)\), leads to (3.31) for \(\tilde{C} = \max \{C_2, C_{p_2}\}\).

In Case II, i.e. \(1/2 < \alpha_1 < 1\) and \(p_1 < p_2\), we have from (3.36) \(b_1 = 2(\alpha_1 - \frac{2}{p_0}) > 1\) because \(p_0 > 2/(\alpha_1 - 1/2)\), see (3.23). From (3.38), under the restriction \(2 \leq p \leq p_1\), with \(p', q', q\) defined by (3.46), we have either
\[ b_2(p') > b_2(q') = \alpha_1 + 2/q > 1, \]
or
\[ b_2(p') = b_2(q') = (2 + \alpha_1)(\alpha_1 - 2/p_0) > (2 + \alpha_1)/2 > 1. \]

Since \(\alpha_2 \geq \alpha_1\) implies \(b_3(\cdot) \geq b_2(\cdot)\) we deduce that, under the restriction \(2 \leq p \leq p_1\), we also have
\[ b_3(p') \geq b_3(q') \geq b_2(q') > 1. \]

and
\[ b_4 = \min \{b_1, b_2(2), b_3(2)\} > 1. \]

We can again apply (3.47) to the above integral terms and get for \(2 \leq p \leq p_1\) the estimate (3.48). For \(p > p_1\) one can show that \((2 + \alpha_1)q' < p_1\) hence \(b_2(q') = \alpha_1 + 2/q\), and, in the particular case of \(p = p_2\), we get
\[ b_2(q_2') = \alpha_1 + 2/q_2 < 1 \]
where \(q_2', q_2\) are given by (3.46). We now have from convolution estimates:
Consider the linear Schrödinger equation with a potential in two space dimensions:

\[
\begin{aligned}
&\left\{ \begin{array}{l}
    i \frac{\partial u}{\partial t} = (-\Delta + V(x))u, \\
    u(0) = u_0.
\end{array} \right.
\end{aligned}
\]

It is known that if \( V \) satisfies hypothesis (H1)(i) and (ii) then the radiative part of the solution, i.e. its projection onto the continuous spectrum of \( H = -\Delta + V \), satisfies the estimates

\[
\left\| e^{-iHt} P_c u_0 \right\|_{L^2}^2 \leq C_M \frac{1}{(1 + |t|) \log^2 (2 + |t|)} \| u_0 \|_{L^2}, \quad t \in \mathbb{R},
\]

(4.1)

for any \( \sigma > 1 \) and some constant \( C_M > 0 \) depending only on \( \sigma \), see [17, Theorem 7.6 and Example 7.8], and

\[
\left\| e^{-iHt} P_c u_0 \right\|_{L^p} \leq \frac{C_p}{|t|^{1-2/p}} \| u_0 \|_{L^p}
\]

(4.2)

for some constant \( C_p > 0 \) depending only on \( p \geq 2 \) and \( p' \) given by \( p^{-1} + p^{-1'} = 1 \). The case \( p = \infty \) in (4.2) is proven in [23]. The conservation of the \( L^2 \) norm, see [5, Corollary 4.3.3], gives the \( p = 2 \) case:

\[
\left\| e^{-iHt} P_c u_0 \right\|_{L^2} = \| u_0 \|_{L^2}.
\]

The general result (4.2) follows from Riesz–Thorin interpolation.
We would like to extend these estimates to the linearized dynamics around the center manifold. In other words we consider the linear equation (3.20), with initial data at time \( s \):

\[
\frac{\partial z}{\partial t} = -i(-\Delta + V)z - iP_c Dg_{\varphi(t)}Ra(t)z(t),
\]

\[
z(s) = \psi \in \mathcal{H}_0.
\]

Note that this is a nonautonomous problem as the bound state \( \psi_E \) around which we linearize may change with time.

By Duhamel’s principle we have

\[
z(t) = e^{-iH(t-s)}p_c\psi - i\int_s^t e^{-iH(t-\tau)}p_c Dg_{\varphi(t)}Ra(\tau)z(\tau)\,d\tau.
\]

As in (3.21) we denote

\[
\Omega(t, s)v \overset{\text{def}}{=} z(t).
\]

Relying on the fact that \( \psi_E(t) \) is small and localized uniformly in \( t \in \mathbb{R} \), we have shown in [13, Section 4] for the particular case of cubic nonlinearity, \( g(s) = s^3, s \in \mathbb{R} \), that estimates of type (4.1)–(4.2) can be extended to the operator \( \Omega(t, s) \). Due to (2.6) which implies for \( \sigma \geq 0 \) and \( 1 \leq p' \leq 2 \):

\[
\|Dg_{\varphi}Ra\|_{L^2_{\sigma}} \leq C\left(\|\psi_E\|_{L^{1+q}_{\sigma}/(1+\alpha_1)}^{1+\alpha_1} + \|\psi_E\|_{L^{1+q}_{\sigma}/(1+\alpha_1)}^{1+\alpha_2}\right)C_{-\sigma}\|z\|^2_{L^2_{-\sigma}},
\]

\[
\|Dg_{\varphi}Ra\|_{L^p_{\sigma'}} \leq C\left(\|\psi_E\|_{L^{1+q}_{\sigma}/(1+\alpha_1)}^{1+\alpha_1} + \|\psi_E\|_{L^{1+q}_{\sigma'/1+\alpha_1}}^{1+\alpha_2}\right)C_{-\sigma}\|z\|^2_{L^2_{-\sigma}},
\]

\[
\|Dg_{\varphi}Ra\|_{L^p_{\sigma}} \leq C\left(\|\psi_E\|_{L^{1+q}_{\sigma}/(1+\alpha_1)}^{1+\alpha_1} + \|\psi_E\|_{L^{1+q}_{\sigma'/1+\alpha_1}}^{1+\alpha_2}\right)C_{r}\|z\|^2_{H^r},
\]

see also Lemma 2.2, we can use, with obvious modifications, the arguments in [13, Section 4] to show that:

**Theorem 4.1.** Fix \( \sigma > 1 \). There exists \( \varepsilon_1 > 0 \) such that if \( \|\langle x\rangle^{d\sigma/3}\psi_E(t)\|_{H^2} < \varepsilon_1 \) for all \( t \in \mathbb{R} \), then there exist constants \( C, C_p > 0 \) with the property that for any \( t, s \in \mathbb{R} \) the following hold:

\[
\|\Omega(t, s)\|_{L^2_{\sigma}} \leq \frac{C}{(1 + |t - s|)^\log^2(2 + |t - s|)},
\]

\[
\|\Omega(t, s)\|_{L^{p'}_{\sigma}} \leq \frac{C_p}{|t - s|^{\frac{1}{p' - 1}}}, \text{ for any } 2 \leq p < \infty \text{ where } p^{-1} + p^{-1} = 1,
\]

\[
\|\Omega(t, s)\|_{L^p_{\sigma}} \leq \frac{C_p}{|t - s|^{\frac{1}{p} - 1}}, \text{ for any } p \geq 2.
\]

And, for

\[
T(t, s) = \Omega(t, s) - e^{-iH(t-s)}p_c,
\]

(4.9)
**Lemma 4.1.** Assume that \( \| (x)^{A_1/3} \psi_E(t) \|_{H^2} < \varepsilon_1, t \in \mathbb{R} \), where \( \varepsilon_1 \) is the one used in Theorem 4.1. Then for each \( 1 < q' \leq 2 \) and \( 2 < p < \infty \) there exist the constants \( C_{q'}, C_{p,q'} > 0 \) such that for all \( t, s \in \mathbb{R} \) we have

\[
\| T(t, s) \|_{L^1 \cap L^q' \to L^2_{-\sigma}} \leq \frac{C_{q'}}{1 + |t - s|},
\]

\[
\| T(t, s) \|_{L^1 \cap L^q' \to L^p} \leq \frac{C_{p,q'} \log(2 + |t - s|)}{(1 + |t - s|)^{1 - \frac{2}{p}}}.\]

Note that according to the proofs in [13, Section 4] \( C_{q'} \to \infty \) as \( q' \to 1 \) and \( C_{p,q'} \to \infty \) as \( q' \to 1 \) or \( p \to \infty \). These could be prevented and an estimate of the type

\[
\| T(t, s) \|_{L^1 \to L^\infty} \leq \frac{C \log(2 + |t - s|)}{1 + |t - s|} \tag{4.10}
\]

can be obtained by avoiding the singularity of \( \| e^{-iHt} P_c \|_{L^1 \to L^\infty} \sim t^{-1} \) at \( t = 0 \) via a generalized Fourier multiplier technique developed in [12, Appendix and Section 4]. We choose not to use it here because it requires stronger restrictions on the potential \( V(x) \) such as its Fourier transform should be in \( L^1 \) while its gradient should be in \( L^p \), for some \( p \geq 2 \), and convergent to zero as \( |x| \to \infty \).

We now present an improved \( L^2 \) estimate for the family of operators \( T(t, s) \).

**Lemma 4.2.** Assume that \( \| (x)^{A_1/3} \psi_E(t) \|_{H^2} < \varepsilon_1, t \in \mathbb{R} \), where \( \varepsilon_1 \) is the one used in Theorem 4.1. Then there exists the constant \( C_2 > 0 \) such that for all \( t, s \in \mathbb{R} \) we have

\[
\| T(t, s) \|_{L^2 \to L^2} \leq C_2.
\]

**Proof.** We are going to use a Kato type smoothing estimate:

\[
\| (x)^{-\sigma} e^{-iHt} P_c f(x) \|_{L^2_t(\mathbb{R}, L^2_x)} \leq C_K \| f \|_{L^2}, \tag{4.11}
\]

see for example [21]. We claim that the previous estimate still holds if we replace \( e^{-iH(t-s)} P_c \) by \( \Omega(t, s) \), namely, there exists a constant \( \tilde{C}_K > 0 \) such that for any \( s \in \mathbb{R} \):

\[
\| (x)^{-\sigma} \Omega(., s) f \|_{L^2_t(\mathbb{R}, L^2_x)} \leq \tilde{C}_K \| f \|_{L^2}. \tag{4.12}
\]

Indeed, from (4.4) and (4.3), we have

\[
(x)^{-\sigma} \Omega(t, s)v = (x)^{-\sigma} e^{-iH(t-s)} P_c v - i \int_s^t (x)^{-\sigma} e^{-iH(t-\tau)} P_c Dg_{\psi_E(\tau)} [R_{a(\tau)} \Omega(\tau, s)v] d\tau
\]

and using (4.5):

\[
\| \Omega(t, s) v \|_{L^2_{-\sigma}} \leq \| e^{-iH(t-s)} P_c v \|_{L^2_{-\sigma}} + \int_s^t \left( \| e^{-iH(t-\tau)} P_c \|_{L^2_{-\sigma}} \| Dg_{\psi_E(\tau)} R_{a(\tau)} \Omega(\tau, s)v(s) \|_{L^2} d\tau
\]

\[
\leq \| e^{-iH(t-s)} v \|_{L^2_{-\sigma}} + C \sup_{\tau \in \mathbb{R}} \left( \| \psi_E(\tau) \|_{L^2_{-\sigma}(1+\alpha_1)}^{1+\alpha_1} + \| \psi_E(\tau) \|_{L^2_{-\sigma}(1+\alpha_2)}^{1+\alpha_2} \right)
\]

\[
\times \int_{\mathbb{R}} \frac{\| \Omega(\tau, s)v \|_{L^2_{-\sigma}}}{(1 + |t - \tau|) \log^2(2 + |t - \tau|)} d\tau.
\]
By Young inequality \( \| f \ast g \|_{L^2(\mathbb{R})} \leq \| f \|_{L^1(\mathbb{R})} \| g \|_{L^2(\mathbb{R})} \) and (4.11) we get
\[
\| \Omega(\cdot, s)v \|_{L^2(\mathbb{R}, L^2_{\infty, p})} \leq C \| v \|_{L^2_{\infty, p}} + C \varepsilon_1 \| \Omega(\cdot, s)v \|_{L^2(\mathbb{R}, L^2_{\infty, p})}
\]
which implies (4.12).

Finally we turn to the estimate in \( L^2_{\infty} \) for \( T(t, s) \):
\[
\| T(t, s)v \|_{L^2_{\infty}}^2 = \left( \int_s^t \int_s^t e^{-iH(t-\tau)} P_e Dg_{\psi_E}[R_a \Omega(\tau, s)v] d\tau d\tau' \int_s^t e^{-iH(t-\tau')} P_e Dg_{\psi_E}[R_a \Omega(\tau', s)v] d\tau' \right)
\]
\[
= \int_s^t \int_s^t d\tau d\tau' \left( Dg_{\psi_E}[R_a \Omega(\tau, s)v], e^{-iH(t-\tau)} P_e Dg_{\psi_E}[R_a \Omega(t', s)v] \right)
\]
\[
\leq C \sup_{\tau \in \mathbb{R}} \left( \| \psi_E(\tau) \|_{L^2(\mathbb{R})}^{1+\alpha_1} + \| \psi_E(\tau) \|_{L^2(\mathbb{R})}^{1+\alpha_2} \right)^2 \times \left( \int_s^t \int_s^t d\tau d\tau' \| \Omega(\tau, s)v \|_{L^2_{\infty,p}} \| e^{-iH(t-\tau')} P_e \|_{L^1(\mathbb{R})} \| \Omega(\tau', s)v \|_{L^2_{p-\sigma}} \right).
\]

Using (4.1) combined with Young then Hölder inequalities the integral above is bounded by
\[
C_M \| \Omega(\cdot, s)v \|_{L^2(\mathbb{R}, L^2_{\infty, p})}^2 \leq C_M \tilde{C}_K^2 \| v \|_{L^2_{\infty}}^2
\]
where, for the last inequality we employed (4.12). Consequently, there exists a constant \( C_2 \) such that for any \( t, s \in \mathbb{R} \):
\[
\| T(t, s)v \|_{L^2_{\infty}} \leq C_2 \| v \|_{L^2_{\infty}}.
\]
This finishes the proof of the lemma. \( \Box \)

Fix now \( 2 < p_0 < \infty \) and let \( p'_0 = p_0/(p_0 - 1) \). By applying Riesz–Thorin interpolations to the operators \( T(t, s) \) satisfying for all \( t, s \in \mathbb{R} \):
\[
\| T(t, s) \|_{L^1 \rightarrow L^2} \leq C_2,
\]
\[
\| T(t, s) \|_{L^1(\mathbb{R}) \rightarrow L^p_{0 \rightarrow L^p_0}} \leq \frac{C_{p_0} \log(2 + |t-s|)}{(1 + |t-s|)^{1-\frac{2}{p_0}}}
\]
we obtain that for any \( 2 \leq p \leq p_0 \) there exists a constant \( C_{p_0, p} \) between \( C_2 \) and \( C_{p_0} \) such that
\[
\| T(t, s) \|_{L^p_{\infty} \rightarrow L^p_{p' \rightarrow L^p}} \leq \frac{C_{p_0, p} \log^{1-2/p_0}(2 + |t-s|)}{(1 + |t-s|)^{1-\frac{2}{p}}}, \quad \text{where } p' = \frac{p}{p-1}, \ q' = p, \ p_0 - 2 \quad \frac{p_0 - p'}{p_0 - p}.
\]
Finally, using (4.9) and the estimates for the Schrödinger group (4.2) we get:
Theorem 4.2. Fix $2 < p_0 < \infty$ and assume that $\| \langle x \rangle^{\alpha_p/2} \psi_E(t) \|_{\dot{H}^2} < \varepsilon_1$, $t \in \mathbb{R}$, where $\varepsilon_1$ is the constant obtained in Theorem 4.1. Then there exist the constants $C_2, C_{p_0, p} > 0$ such that for all $2 \leq p \leq p_0$ and $t, s \in \mathbb{R}$ the following estimates hold:

$$\| \Omega(t, s) \|_{L^2 \to L^2} \leq C_2,$$

$$\| \Omega(t, s) \|_{L^{p'}(\cap L^p \to L^p)} \leq \frac{C_{p_0, p} \log(2 + |t - s|)}{|t - s|^{1 - \frac{2}{p}}},$$

where $p' = \frac{p}{p - 1}$, $q' = \frac{p_0 - 2}{p_0 - p}$.

Note that the estimates for the family of operators $\Omega(t, s)$ given by the above theorem are similar to the standard $L^p \to L^p$ estimates for Schrödinger operators (4.2) except for the logarithmic correction and a smaller domain of definition $L^q \cap L^p \subset L^p$ where $q' < p'$ when $p' < 2$. If we would have proven (4.10) then we could use $p_0 = \infty$, hence $q' = p'$ in the above theorem and obtain

$$\| \Omega(t, s) \|_{L^{p'} \to L^p} \leq \frac{C_p \log(2 + |t - s|)}{|t - s|^{1 - \frac{2}{p}}},$$

where $p' = \frac{p}{p - 1}$.

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References

2. V.S. Buslaev, G.S. Perel’man, Scattering for the nonlinear Schrödinger equation: States that are close to a soliton, Algebra i Analiz 4 (6) (1992) 63–102.


