Gap Embedding for Well-Quasi-Orderings [1]

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Abstract

Given a quasi-ordering of labels, a labelled ordered tree s is *embedded with gaps* in another tree t if there is an injection from the nodes of s into those of t that maps each edge in s to a unique disjoint path in t with greater-or-equivalent labels, and which preserves the order of children. We show that finite trees are well-quasi-ordered with respect to gap embedding when labels are taken from an arbitrary well-quasi-ordering such that each tree path can be partitioned into a bounded number of subpaths of comparable nodes. This extends Kříž's result [3] and is also optimal in the sense that unbounded incomparability yields a counterexample.

1 Introduction

Kruskal's Tree Theorem [4], stating that finite trees are well-quasi-ordered under homeomorphic embedding, and its extensions, have played an important rôle in both logic and computer science. In proof theory, it was shown to be independent of certain logical systems by exploiting its close relationship with ordinal notation systems [7], while in computer science it provides a common tool for proving the termination of many rewrite-systems via the *recursive path* and related orderings [1].

A term ordering is said to have the *subterm property* if terms are always bigger than all their subterms. Term orderings with the "replacement" property (reducing subterms reduces the whole term) that also have the subterm property are called *simplification orderings* [1]. Simplification orderings perforce include the homeomorphic embedding relation. Nevertheless, it is sometimes necessary to prove termination of rewrite systems that are not "simplifying" in this sense. In term rewriting, the tree-label ordering corresponds to a precedence ordering of the function symbols pertaining to a given signature. For demonstrating termination of rewriting, it is beneficial to use a *partial* (or *quasi-*) ordering on labels, rather than a total one.

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In [8], it was shown that many important order-theoretic properties of the well-partial-ordered precedence relations on function symbols carry over to the induced termination ordering. This is done by defining a general framework for precedence-based termination orderings via (so-called) *relativized ordinal notations*. Based on a few examples, it is further conjectured that every such application of a partial-order to an ordinal notation system carries the order-theoretic properties of the partial-order to the relativized notation system. An example of such a construction, using Takeuti's ordinal diagrams, is introduced in [6] under the name *quasi-ordinal-diagrams*. The definition of these diagrams is the only result we know of that deals with gap embedding of trees and *quasi*-ordered labels.

Kříž's result in [3] is of a purely combinatorial nature. It verifies a conjecture of Harvey Friedman that states that finite trees labelled by ordinals are well-quasi-ordered under gap embedding, which is a homeomorphic embedding equipped with further stipulations regarding the labels of the path pertaining to the embedding tree.

This work extends the result of Kříž's to finite trees with well-quasi-ordered labels. Indeed, finite trees ordered by embeddability (without the gap condition) with well-quasi-ordered labels is the result proven originally by Kruskal [4]. It shows that when each tree path contains only comparable labels, the well-quasi-order property of the set of trees is preserved. By simple induction, our result extends also to the case where every path in the tree can be partitioned into some bounded number of subpaths with comparable labels. Moreover, since the absence of such a bound yields a bad sequence with respect to gap-embedding, this is actually the canonical counterexample: every bad sequence with respect to gap embedding must contain paths of unbounded incomparability.

2 Preliminaries

A *quasi-ordering* is a set Q together with a reflexive and transitive binary relation $\precsim$. Given a quasi-ordering $(Q, \precsim)$ and two elements $a, b \in Q$, we say that a and b are *comparable* if either $a \precsim b$ or $b \precsim a$; otherwise we say that they are *incomparable*. We denote by $\prec$ the strict part of $\precsim$.

A quasi-ordering $(Q, \precsim)$ is a *well-quasi-ordering* (*wqo*) if for every infinite sequence $a_1, a_2, a_3, \ldots$ from Q there exist $i < j \in \mathbf{N}$ such that $a_i \precsim a_j$. An infinite sequence from Q is referred to as *bad* if for all $i < j$, $a_i \not\precsim a_j$ holds; otherwise it is called *good*. If, for all $i, j \in \mathbf{N}$, a_i is incomparable to a_j, the sequence is an *antichain*.

For a pair of nodes u, v in a rooted tree, we denote by $u \sqcap v$ the *closest common ancestor* of u and v; we write $u \sqsubset v$ if u is to the left (descendent of elder sibling of ancestor) of v. The following is the definition of the (homeomorphic) tree embedding:

Definition 2.1 [Tree embedding] For two labelled ordered trees s, t we say that s is *embedded* in t (with respect to $\precsim$) if there is an injection $\iota : s \to t$ such that:

- Label increasing: for all nodes x in s, $x \precsim \iota(x)$;
- Ancestry preserving: for all nodes x, y in s, $\iota(y \sqcap x) = \iota(y) \sqcap \iota(x)$;
- Sibling order preserving: for all nodes x, y in s, $x \sqsubset y$ implies $\iota(x) \sqsubset \iota(y)$.

In the next section, we begin by dealing with an abstract embedding relation $\hookrightarrow$ on finite rooted trees $\mathcal{T}$. Later (in Section 3.3), we deal explicitly with the set of trees of interest, namely *ordered* (rooted, planted-plane) finite trees, with nodes well-quasi-ordered by $\precsim$, and such that *every node is comparable with all its ancestor nodes.*

Remark. A more intuitive definition of gap embedding can be given for trees with labels on *edges* instead of nodes. Denote by $s \hookrightarrow' t$ an embedding of an edge-labelled tree s in a likewise labelled tree t, such that each edge of s is mapped to a path in t all labels of which are greater than or equivalent to (with respect to the node ordering $\precsim$) the label of the edge in s. It is not hard to show that, if ordered rooted trees with labels on nodes is wqo under the gap embedding of Definition 2.1, then also the set of edge labelled trees is wqo under this edge-based embedding (cf. [3] Section 1.3).

3 The Main Theorem

We first introduce two abstract relations over finite rooted trees $\mathcal{T}$: A "gap-embedding" relation and a "gap-subtree" relation. These relations are abstract for now, as we only stipulate the existence of a tree embedding relation and a subtree relation equipped with five additional (gap) conditions (see Definition 3.2, 3.3 for the explicit relations). We then show the main construction of the *minimal bad sequence*, required in order to apply the usual Nash-Williams [5] method.

Let $t^\bullet$ denote the root of tree t. There is a *gap subtree* relation $\trianglerighteq$ which is included in the regular subtree relation on trees with the following additional requirements:

(A) $s \trianglerighteq t \trianglerighteq u \wedge t^\bullet \succsim u^\bullet \Rightarrow s \trianglerighteq u$

(B) $s \trianglerighteq t \trianglerighteq u \wedge s^\bullet \precsim t^\bullet \Rightarrow s \trianglerighteq u$

(C) $s \trianglelefteq t \Rightarrow s^\bullet \precsim t^\bullet \vee t^\bullet \precsim s^\bullet$

We denote by $\triangleright$ the proper gap subtree relation. There is also a *gap embedding* quasi-ordering $\hookrightarrow$ on trees with the following additional properties:

(D) $s \hookrightarrow t \trianglelefteq u \wedge t^\bullet \precsim u^\bullet \Rightarrow s \hookrightarrow u$

(E) $s \hookrightarrow t \trianglelefteq u \wedge s^\bullet \precsim u^\bullet \Rightarrow s \hookrightarrow u$

A set of trees is well-quasi-ordered under the gap embedding relation $\hookrightarrow$ if every infinite sequence of trees contains a pair of trees s, t one preceding the other, such that $s \hookrightarrow t$.

A *sequence* s is a *partial* function $s : \mathbf{N} \to \mathcal{T}$. If $s(i)$ is not defined we shall write $s(i) = \bot$. It is very convenient to extend the subtree relation and node ordering to empty positions of a sequence, so that: $t \trianglerighteq \bot$ and $t^\bullet \precsim \bot^\bullet$ for all $t \in \mathcal{T}$.

Let Seq be the set of ω-sequences of trees from $\mathcal{T}$. Define:

$$\begin{aligned} Ds &:= \{i \in \mathbf{N} \mid s(i) \neq \bot\} \\ \text{Bad} &:= \{s \in \text{Seq} \mid \forall i < j \in Ds.\ s(i) \not\hookrightarrow s(j)\} \\ \text{Sub}\ h &:= \{s \in \text{Seq} \mid \forall i \in Ds.\ h(i) \triangleright s(i)\} \\ \text{Inc}\ k &:= \{s \overset{\infty}{\subseteq} k \mid \forall i < j \in Ds.\ s^\bullet(i) \precsim s^\bullet(j)\} \end{aligned}$$

where $s \overset{\infty}{\subseteq} k$ denotes that s is an infinite subset of k. A sequence s is *infinite* when its domain of definition, Ds, is. Thus, Bad is the set of infinite bad sequences; Sub h is the set of all infinite subsequences of gap subtrees of h.

Since $\succsim$ is a well-quasi-ordering, Inc k (the set of infinite increasing subsequences of k) is nonempty, as long as k is infinite, by the infinite version of Ramsey's Theorem.

Our goal then, is to prove the following:

Theorem 3.1 (Main Theorem) Bad $= \emptyset$.

This means that the set of trees $\mathcal{T}$ is wqo under $\hookrightarrow$. In other words, for every $s \in \text{Seq}$ there exist $i < j \in Ds$ such that $s(i) \hookrightarrow s(j)$. This extends the result of Kříž [3] for well-orderings to quasi-ordered labels.

3.1 The Construction

Assuming the above theorem is false, and there are bad sequences of trees, the proof constructs a minimal counterexample, that is, a bad sequence $h \in \text{Bad}$, which is minimal in the sense that no infinite sequence of proper gap subtrees of its elements is also bad:

$$\text{Bad} \cap \text{Sub}\, h = \emptyset$$

This, in turn, leads to a contradiction—as in the original proof by Nash-Williams [5] (see Section 3.3).

The construction of such a minimal bad sequence proceeds by ordinal induction as follows (λ is a limit ordinal):

$H(0)$:	$h :\in \text{Bad}$
	if $\text{Bad} \cap \text{Sub}\ h = \emptyset$ **then return** h
	$h_0 :\in \text{Inc lex}(h)$

$H(\alpha+1):$	**if** $\mathrm{Bad} \cap \mathrm{Sub}\ h_\alpha = \emptyset$ **then return** h_α
	$k := \mathrm{lex}(h_\alpha)$
	$\forall i \in \mathbf{N}.\ f(i) := \begin{cases} k(i) & \textbf{if } h^\bullet_\alpha(i) \precsim k^\bullet(i) \\ \bot & \textbf{otherwise} \end{cases}$
	$g :\in \mathrm{Inc}\ f$
	$\forall i \in \mathbf{N}.\ h_{\alpha+1}(i) := \begin{cases} h_\alpha(i) & \textbf{if } i < \min Dg \\ g(i) & \textbf{otherwise} \end{cases}$
$H(\lambda):$	$\forall i \in \mathbf{N}.\ \ell(i) := \lim_{\gamma \to \lambda} h_\gamma(i)$
	if $\mathrm{Bad} \cap \mathrm{Sub}\ \ell = \emptyset$ **then return** ℓ
	$h_\lambda :\in \mathrm{Inc}\ \mathrm{lex}(\ell)$

where the construct $s :\in S$ chooses an arbitrary s from S (and $s = \bot$ if $S = \emptyset$). The function lex : Bad $\to$ Bad chooses a bad sequence of subtrees (that is, $\mathrm{lex}(h) \in \mathrm{Bad} \cap \mathrm{Sub}(h)$) with (lexicographically) minimal labels:

$\mathrm{lex}(h):$	$K := \mathrm{Bad} \cap \mathrm{Sub}\ h$
	for $i := 1$ **to** ∞ **do**
	$\quad t :\in \mathrm{argmin}\{s^\bullet(i) \mid s \in K\}$
	$\quad K := \{s \in K \mid s(i) = t(i)\}$
	$k :\in K$
	return k

where $\mathrm{argmin}\{s^\bullet(i) \mid s \in K\}$ denotes the set of those $s \in K$ for which $s^\bullet(i)$ is minimal.

3.2 Correctness

We show that $\lim_{\gamma \to \lambda} h_\gamma(i)$ converges to some *fixed* tree. By construction, we have (for all α and i):

(6) $Dh_\alpha \supseteq Dh_{\alpha+1}$
(7) $h_\alpha(i) \trianglerighteq h_{\alpha+1}(i)$
(8) $h^\bullet_\alpha(i) \precsim h^\bullet_{\alpha+1}(i)$

For each sequence h_α (for every countable ordinal α and for all $i < j \in Dh_\alpha$):

(9) $h_\alpha(i) \not\hookrightarrow h_\alpha(j)$
(10) $h^\bullet_\alpha(i) \precsim h^\bullet_\alpha(j)$

For α a successor ordinal, (9,10) are proved by induction: The only interesting case is $i < \min Dg \leq j$, when

$$h^\bullet_{\alpha+1}(i) = h^\bullet_\alpha(i) \precsim h^\bullet_\alpha(j) \precsim k^\bullet(j) = f^\bullet(j) = g^\bullet(j) = h^\bullet_{\alpha+1}(j)$$

from which (9) follows using (E). By considering the limit case, it can be seen that for all $\alpha < \beta$:

(11) $Dh_\alpha \supseteq Dh_\beta$

To complete the proof of the construction, it remains only to establish three additional aspects:

(i) The constructed sequences h_α are all infinite.

(ii) The constructed sequences h_α are each distinct.

(iii) The construction eventually terminates with a minimal bad sequence.

Aspect (i) It must be that $|Df| = \infty$ in the successor case: Suppose f is finite at stage $\alpha+1$. Let k be the bad sequence of subtrees of h_α constructed by lex at stage $\alpha+1$, and k_α, the one constructed at the prior step α from subtrees of some sequence h (in case $\alpha = 0$, this k_α is the output of lex h at the $H(0)$ stage). Let $q = k \setminus f$ be those elements of k that have *smaller* root symbols than h_α (see Fig. 1). By supposition and condition (C), q is infinite and bad. Consider

$$p = k_\alpha[0 : n-1] \cup (q \restriction \mathrm{D}k_\alpha)$$

where $n = \min(\mathrm{D}k_\alpha \cap \mathrm{D}q)$. Note that $\mathrm{D}p \subseteq \mathrm{D}k_\alpha$, $\mathrm{D}q \subseteq \mathrm{D}h_\alpha$ and that for all i if $k_\alpha(i) = \perp$ then also $p(i) = \perp$.

We show now that $p \in \mathrm{Bad} \cap \mathrm{Sub}\ h$. Since $k^\bullet_\alpha(n) \succ q^\bullet(n) = p^\bullet(n)$ also holds, this contradicts the picking of $k_\alpha(n)$, rather than $p(n)$, by lex at the α stage.

Thus, for $i \in \mathrm{D}p$, if $i < n$, we have $p(i) = k_\alpha(i) \lhd h(i)$, by construction of k_α. If $i \geq n$

(12) $p(i) = q(i) = k(i) \lhd h_\alpha(i)$

and $h_\alpha(i) = k_\alpha(i) \lhd h(i)$ or $h_\alpha(i) = h(i)$. In the latter case, $p(i) \lhd h(i)$ follows directly from (12), in the former case, $p(i) \lhd h(i)$ follows from $p^\bullet(i) \prec k^\bullet_\alpha(i)$ and (A). Hence $p \in \mathrm{Sub}\ h$.

Furthermore, were $k_\alpha(i) \hookrightarrow q(j)$ for some $i < n \leq j$, then (by D) $k_\alpha(i) \hookrightarrow k_\alpha(j)$, which is in contradiction to $k_\alpha \in \mathrm{Bad}$. Hence, $p \in \mathrm{Bad} \cap \mathrm{Sub}\ h$ and as claimed $h_{\alpha+1}$ is infinite.

In the limit case also, h_λ is infinite: Let $g_{\alpha+1}$ be the g constructed at step $\alpha+1$ and $n_{\alpha+1} = \min Dg_{\alpha+1}$. Since trees have only finitely many subtrees, and $g_{\alpha+1}$ is built of proper subtrees of the prior bad sequence, we have

(13) $\liminf_{\alpha \to \lambda} n_{\alpha+1} \to \omega$

Otherwise, if $\liminf_{\alpha\to\lambda} n_{\alpha+1} \to c$ for some $c \in \mathbf{N}$, then by the Pigeonhole Principle, for some i in $[0, c]$ there would have been infinitely many subtrees taken from $h(i)$.

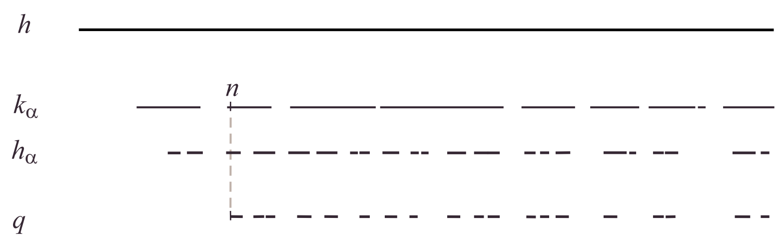

Fig. 1. The bad sequences of the proof of the main theorem. The (dotted) lines represent the domains of the sequences, which are getting sparser as the induction goes on.

Furthermore, once $n_{\alpha+1} < n_\gamma$ for all γ such that $\alpha + 1 < \gamma < \lambda$, we get $h_\gamma[n_{\alpha+1}] = g_{\alpha+1}[n_{\alpha+1}] \neq \perp$ for all such γ, which indeed happens infinitely many times by (13).

Aspect (ii) Distinctness follows from the construction, since, as long as f is infinite, $\min Dg$ is defined and $h_{\alpha+1} \neq h_\alpha$.

Aspect (iii) Termination follows from distinctness by a cardinality argument: There are only countably many sequences h_α, each corresponding to the pair $\langle i, j \rangle$, for the jth time a proper subtree is taken (by lex) in the ith index position.

3.3 Path Comparable Trees

We now make the gap subtree and the gap embedding relations explicit:

Definition 3.2 [Gap subtree] For two trees s, t in $\mathcal{T}$, we say that t is a *gap subtree* of s, and write $s \trianglerighteq t$, if t is a subtree of s and the path $P = [s^\bullet : t^\bullet]$ from $s^\bullet$ to $t^\bullet$ in s meets the following condition:

- $\min_{\precsim} P \in \{s^\bullet, t^\bullet\}$.

Definition 3.3 [Gap embedding] For two trees s, t we say that s is *embedded with gaps* in t and write $s \hookrightarrow t$ if there is an embedding $\iota : s \to t$ satisfying the following additional conditions (see Fig. 2):

- *Edge gap condition:* for all edges $\langle x, y \rangle$ in s (x is the parent of y) and for all nodes z in the path from $\iota(x)$ to $\iota(y)$ in t, $z \succsim y$;
- *Root gap condition:* $x \succsim s^\bullet$ for all nodes x in the path from $t^\bullet$ to $\iota(s^\bullet)$.

Recall that $\mathcal{T}$ is the set of ordered rooted finite trees, with nodes well-quasi-ordered by $\precsim$, and such that *every node is comparable with all its ancestors.* This corresponds to condition (C) in Section 3. We make the following three observations:

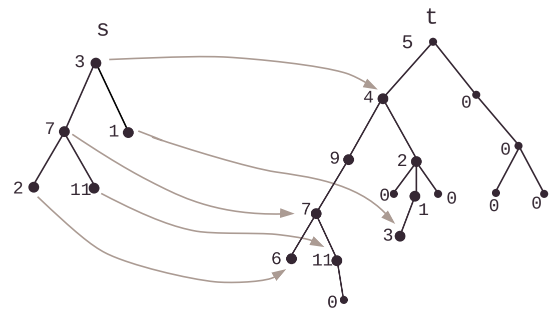

Fig. 2. Gap embedding of s into t.

Observations.

i. The gap subtree conforms to conditions (A,B) given in the previous section.

ii. Gap embedding respects conditions (D,E) of the previous section.

iii. The gap subtree relation includes all immediate subtrees.

Proof of Main Theorem:

Assume by way of contradiction that $\mathrm{Bad} \neq \emptyset$. Hence, by Observations (i) and (ii), we showed in the previous subsection that there is a minimal bad sequence $h \in \mathrm{Bad}$ such that $\mathrm{Bad} \cap \mathrm{Sub}\, h = \emptyset$. Let S be the set of *all immediate subtrees* of trees in h, that is, trees rooted by immediate children of trees in h. Since the labels are taken from a wqo set, there can be at most finitely many trees of only one vertex in h; therefore S is infinite.

For a tree $t \in \mathcal{T}$, we denote by $\langle t_1, \ldots, t_n \rangle$ the finite ordered sequence consisting of its immediate subtrees, in the order they occur as children of $t^\bullet$; by $t^\bullet \langle t_1, \ldots, t_n \rangle$, we denote t itself.

Now, S must be wqo, or else there would be a bad infinite sequence $\mu \subseteq S$. Since, for each tree in h, the number of children of the root is finite, we can assume that μ contains at *most* one subtree for each tree in h. Therefore, $\mu \in \mathrm{Bad} \cap \mathrm{Sub}\, h$, in contradiction to the construction of h.

So, S is a wqo. Let $(s_i)_{i \in \mathrm{D}h}$ be the infinite sequence defined as:

$$\forall i \in \mathrm{D}h.\ s_i := \langle h(i)_1, \ldots, h(i)_{n_i} \rangle$$

where n_i is the number of children of $h^\bullet(i)$. Since S is a wqo, by Higman's Lemma [2], $(s_i)_{i \in \mathrm{D}h}$ is a good sequence with respect to the embedding relation on finite sequences of trees from $\mathcal{T}$ defined by:

$$\begin{aligned}
&\langle s_1, \ldots, s_n \rangle \hookrightarrow \langle t_1, \ldots, t_m \rangle \ \text{ if} \\
&\quad \exists \iota : \{1,\ldots,n\} \to \{1, \ldots, m\}\,.\ \iota \text{ is strictly monotone} \wedge \\
&\qquad\qquad\qquad\qquad\qquad \forall j\, (1 \leq j \leq n).\ s_j \hookrightarrow t_{\iota(j)}
\end{aligned}$$

Therefore, as h is *increasing*, there exists a pair of trees s, t in h, such that s precedes t and $s = s^\bullet \langle s_1, \ldots, s_n \rangle \hookrightarrow t^\bullet \langle t_1, \ldots, t_m \rangle = t$, where the root is mapped to the root and the immediate subtrees of s are embedded in those of t, according to Higman's sequence embedding. Note that this embedding is actually a gap embedding (the fact that ι is strictly monotone is required so that the order of children denoted by $\sqsubset$ is preserved in the embedding); thus, we arrive at a contradiction to the badness of h. □

4 Comparable Subpaths

The condition that each node in a path is comparable to all its ancestors can be relaxed, by allowing each path to be partitioned to only a bounded number of comparable subpaths. By a *comparable subpath* we mean a tree path (that might begin and end in an *internal* node) with all nodes comparable to each other. In what follows we sketch the proof.

Let us slightly change the gap embedding relation $\hookrightarrow$ to allow trees to have leaves labelled by a possibly distinct node ordering: For two trees, the gap embedding of s into t is defined the same as before except for leaves, for which the gap condition is not applicable (eventually we show that it is applicable in order to complete the proof). That is, if $\langle u, v\rangle$ is an edge of s and v is a leaf, then we require that v be mapped to a node with greater or equivalent node, which could only be a leaf of t, since the leaf ordering is disjoint from that of internal nodes (by disjoint orderings we mean that the set of labels are disjoint). No additional condition on the path from $\iota(u)$ to $\iota(v)$ is required. For internal edges of s the conditions remain the same.

We have the following:

Theorem 4.1 *Let $\mathcal{T}_n$ be a set of finite trees with nodes well-quasi-ordered such that each path in a tree can be partitioned into $n \in \mathbf{N}$ or less comparable subpaths then $\mathcal{T}_n$ is a wqo under gap embedding.*

We prove Theorem 4.1 in two steps. First we show that indeed putting an arbitrary well-quasi-ordering on leaves from $\mathcal{T}$ maintains the wqo property of $\mathcal{T}$ with respect to the gap-embedding. Since we can put also trees as labels of leaves, we can choose to label the leaves of $\mathcal{T}$ by some set of trees with nodes well-quasi-ordered by some possibly disjoint ordering than that of $\mathcal{T}$. Hence if we could "unfold" the leaves of $\mathcal{T}$ into subtrees and still keep the set of trees well-quasi-ordered under gap embedding then by induction on n, Theorem 4.1 would follow.

The first step stems easily from the proof of the main theorem: As before, we need a *minimal bad sequence* theorem for the set of trees with two distinct node ordering on internal nodes and leaves. The proof is identical, since the leaf ordering is a wqo then in any induction stage of the construction there can only be finitely many trees with only one node (that is, just leaves), and they are skipped when building f.

The second step consists of showing that using a set of well-quasi-ordered trees to label the leaves, yields again a wqo with respect to the original definition of gap embedding even when we unfold these leaves to form a set of trees such that each path can be partitioned into *two* comparable subpaths. Note that if we have two trees s, t with all internal nodes comparable to their ancestor nodes, and leaves labelled by some set of *comparable paths trees*, such that s is embedded in t according to the relaxed definition above, then unfolding the leaves of s and t would not necessarily yield that the resulting trees have a gap embedding such that *all* the nodes preserve the gap conditions. The reason is that we did not require leaves to have a gap condition in the relaxed gap embedding.

The second step is achieved by forcing the embedding to map each terminal *edge* to a terminal *edge*. (This ensures that leaves trivially preserve the gap conditions.) We do this simply by introducing a new node as a parent of each leaf, labelled with a new maximum element ∞. Since the new maximum element is comparable to all elements of the node ordering, the minimal bad sequence theorem of the previous paragraph applies to the resulting set of trees. Now, any embedding of two trees from this set of trees ought to map a terminal edge to a terminal edge, therefore by the above explanation Theorem 4.1 follows.

5 Conclusions

As noted earlier, a simple counterexample shows that if the paths of trees in $\mathcal{T}$ do not necessarily contain comparable nodes then our Main Theorem might fail, even for strings: Let a, b, c be three incomparable elements of the node ordering. The following is an antichain with respect to gap embedding:

$$c - a - c \qquad c - b - a - c \qquad c - a - b - a - c \qquad c - b - a - b - a - c \ldots$$

Consequently, Theorem 4.1 shows that the above counterexample is *canonical*: Every bad sequence with respect to gap embedding must contain paths of unbounded incomparability.

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