A variant of Jensen–Steffensen’s inequality and quasi-arithmetic means

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Abstract

A variant of Jensen–Steffensen’s inequality is proved. Necessary and sufficient conditions for the equality in Jensen–Steffensen’s inequality are established. Several inequalities involving more than two monotonic functions and generalized quasi-arithmetic means with not only positive weights are proved. It is shown that such generalized quasi-arithmetic means have the same comparability properties as those with positive weights.

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1. Introduction

Let $I$ be an interval in $\mathbb{R}$ and $f : I \to \mathbb{R}$ a convex function on $I$. If $\xi = (\xi_1, \ldots, \xi_m)$ is any $m$-tuple in $I^m$ and $p = (p_1, \ldots, p_m)$ any nonnegative $m$-tuple such that $\sum_{i=1}^{m} p_i > 0$, then the well-known Jensen’s inequality (see, for example, [6, p. 43])
\[ f \left( \frac{1}{\sum_{i=1}^{m} p_i} \sum_{i=1}^{m} p_i \xi_i \right) \leq \frac{1}{\sum_{i=1}^{m} p_i} \sum_{i=1}^{m} p_i f(\xi_i) \] (1.1)

holds, where \( P_m = \sum_{i=1}^{m} p_i \).

If \( f \) is strictly convex, then (1.1) is strict unless \( \xi_i = c \) for all \( i \in \{j: p_j > 0\} \).

It is well known that the assumption “\( p \) is a nonnegative \( m \)-tuple” can be relaxed at the expense of more restrictions on the \( m \)-tuple \( \xi \).

If \( p \) is a real \( m \)-tuple such that
\[ 0 \leq P_j \leq P_m, \quad j = 1, \ldots, m; \quad P_m > 0, \] (1.2)
where \( P_j := \sum_{i=1}^{j} p_i \), then for any monotonic \( m \)-tuple \( \xi \) (increasing or decreasing) in \( I^m \) we get
\[ \hat{\xi} = \frac{1}{\sum_{i=1}^{m} p_i} \sum_{i=1}^{m} p_i \xi_i \in I, \] (1.3)
and for any function \( f \) convex on \( I \), (1.1) still holds. Again, for strictly convex function \( f \), (1.1) remains strict under certain additional assumptions on \( \xi \) and \( p \) which we discuss in details in Section 2. Inequality (1.1) considered under conditions (1.2) is known as the Jensen–Steffensen’s inequality (see [6, p. 57]) for convex functions.

Here, as in the rest of the paper, when we say that the \( m \)-tuple \( \xi \) is increasing (decreasing) we mean that \( \xi_1 \leq \xi_2 \leq \cdots \leq \xi_m \) (\( \xi_1 \geq \xi_2 \geq \cdots \geq \xi_m \)). Similarly, when we say that a function \( f : I \rightarrow \mathbb{R} \) is increasing (decreasing) on \( I \) we mean that for all \( u, v \in I \) we have \( u < v \Rightarrow f(u) \leq f(v) \) (\( u < v \Rightarrow f(u) \geq f(v) \)).

In his paper [4], Mercer gave a variant of Jensen’s inequality which is stated in the following theorem.

**Theorem A.** If \( f \) is a convex function on an interval containing an \( n \)-tuple \( x = (x_1, \ldots, x_n) \) such that \( 0 < x_1 \leq x_2 \leq \cdots \leq x_n \) and \( w = (w_1, \ldots, w_n) \) is a positive \( n \)-tuple with \( \sum_{i=1}^{n} w_i = 1 \), then
\[ f \left( \frac{x_1 + x_n - \sum_{i=1}^{n} w_i x_i}{x_1} \right) \leq f(x_1) + f(x_n) - \sum_{i=1}^{n} w_i f(x_i). \] (1.4)

In this paper we first give necessary and sufficient conditions for the equality case in Jensen–Steffensen’s inequality (Section 2). The main results of the paper are stated and proved in Section 3. We prove a variant of Jensen–Steffensen’s inequality (our Theorem 2) which includes Theorem A as a special case. Further generalizations of Theorem A involving two or more functions are given in Theorems 3–5 and Corollary 1.

In Section 4 we consider a generalized quasi-arithmetic means in which the weights need not be nonnegative. The main result concerning two generalized quasi-arithmetic means is proved in Theorem 6. Further generalizations about more than two such means are given in Theorem 7 and Corollary 2. Finally, in Section 5 we give three examples of generalizations of classical power means involving negative weights.
In all the inequalities proved in this paper the equality cases follow from the equality case in Jensen–Steffensen’s inequality.

For concave functions we clearly get the reverse inequalities.

2. Equality case in Jensen–Steffensen’s inequality

As we noted in the introduction, if \( f \) is strictly convex on \( I \), then equality holds in Jensen’s inequality (1.1) if and only if \( \xi_i = c \) for all \( i \in \{ j : p_j > 0 \} \). When considering Jensen–Steffensen’s inequality (1.1) for strictly convex function \( f \), the answer to the question when equality holds in (1.1) is not that simple. Here we offer an answer to that question.

Throughout this section \( I \) denotes an interval in \( \mathbb{R} \), \( \xi = (\xi_1, \ldots, \xi_m) \) a monotonic \( m \)-tuple in \( I^m \), and \( p = (p_1, \ldots, p_m) \) a real \( m \)-tuple satisfying conditions (1.2). Without loss in generality we may assume that

\[
p_i \neq 0, \quad i = 1, \ldots, m,
\]

since all the terms \( p_i \xi_i \) and \( p_i f(\xi_i) \) with \( p_i = 0 \) can be omitted from (1.1).

Let us define

\[
\bar{P}_1 = P_m, \quad \bar{P}_j = P_m - P_{j-1} = \sum_{i=j}^{m} p_i, \quad j = 2, \ldots, m,
\]

where \( P_j, j = 1, \ldots, m, \) are as in (1.2). Then (1.2) is equivalent to

\[
0 \leq \bar{P}_j \leq P_m, \quad j = 1, \ldots, m; \quad P_m > 0.
\]

Under the above assumptions made on \( p_i \) we have

\[
P_1 = p_1 > 0, \quad \bar{P}_m = p_m > 0,
\]

\[
P_{j-1} + \bar{P}_j = P_m \quad \Rightarrow \quad P_{j-1} > 0 \lor \bar{P}_j > 0, \quad j = 2, \ldots, m.
\]

The case \( m = 2 \) is not interesting since in that case (1.1) is equivalent to the definition of convexity. Therefore, in all what follows we assume that \( m \geq 3 \).

If \( \bar{\xi} \) is defined as

\[
\bar{\xi} = \frac{1}{P_m} \sum_{i=1}^{m} p_i \xi_i,
\]

then the following identities are easily verified to be true:

\[
P_m(\bar{\xi} - \xi_1) = \sum_{j=2}^{m} \bar{P}_j (\xi_j - \xi_{j-1}), \quad P_m(\xi_m - \bar{\xi}) = \sum_{j=1}^{m-1} P_j (\xi_{j+1} - \xi_{j}).
\]

In the sequel we assume \( \xi \) to be increasing, i.e., \( \xi_1 \leq \xi_2 \leq \cdots \leq \xi_m \) (in the case \( \xi \) is decreasing we simply replace \( \xi \) and \( p \) with \( \bar{\xi} = (\xi_m, \xi_{m-1}, \ldots, \xi_1) \) and \( \bar{p} = (p_m, p_{m-1}, \ldots, p_1) \), respectively). In that case, from (1.2), (2.2), and (2.5) it follows

\[
\xi_1 < \bar{\xi} < \xi_m
\]
\[ \bar{\xi} = \xi_1 \iff \forall j = 2, \ldots, m, \quad \bar{P}_j = 0 \vee \xi_j = \xi_{j-1}. \quad (2.6) \]
\[ \bar{\xi} = \xi_m \iff \forall j = 1, \ldots, m-1, \quad P_j = 0 \vee \xi_j = \xi_{j+1}. \quad (2.7) \]

Now, for any function \( f : I \to \mathbb{R} \) it is easy to obtain the identities

\[ P_m f(\xi_1) - \sum_{i=1}^{m} p_i f(\xi_i) = \sum_{j=2}^{m} \bar{P}_j \left[ f(\xi_{j-1}) - f(\xi_j) \right] \quad (2.8) \]

and

\[ P_m f(\xi_m) - \sum_{i=1}^{m} p_i f(\xi_i) = \sum_{j=1}^{m-1} P_j \left[ f(\xi_{j+1}) - f(\xi_j) \right]. \quad (2.9) \]

In case \( \bar{\xi} = \xi_1 \), from (2.6) and (2.8) we get

\[ f(\bar{\xi}) = \frac{1}{P_m} \sum_{i=1}^{m} p_i f(\xi_i) \]

and the same equality follows from (2.7) and (2.9) in case \( \bar{\xi} = \xi_m \). So, we may assume that \( \xi \) and \( p \) are as above and that

\[ \xi_1 < \bar{\xi} < \xi_m, \quad (2.10) \]

in which case \( \bar{\xi} \) is an interior point of \( I \). Therefore, if \( f \) is convex on \( I \), then \( f \) has a line of support \( L(x) = f(\bar{\xi}) + \lambda(x - \bar{\xi}) \) at the point \( \bar{\xi} \) and the following identity is obtained:

\[ P_m f(\bar{\xi}) - \sum_{i=1}^{m} p_i f(\xi_i) = \sum_{j=1}^{k-1} P_j \left[ f(\xi_{j+1}) - f(\xi_j) - \lambda(\xi_{j+1} - \xi_j) \right] \]
\[ + P_k \left[ f(\bar{\xi}) + \lambda(\xi_k - \bar{\xi}) - f(\xi_k) \right] \]
\[ + P_{k+1} \left[ f(\bar{\xi}) + \lambda(\xi_{k+1} - \bar{\xi}) - f(\xi_{k+1}) \right] \]
\[ + \sum_{j=k+2}^{m} P_j \left[ \lambda(\xi_j - \xi_{j-1}) - f(\xi_j) + f(\xi_{j-1}) \right]. \quad (2.11) \]

Here, \( k \in \{1, \ldots, m-1\} \) is chosen so that \( \xi_k \leq \xi < \xi_{k+1} \). In case \( k = 1 \) we assume \( \sum_{j=1}^{k-1} \) to be 0, while in case \( k = m-1 \) we assume \( \sum_{j=k+2}^{m} \) to be 0.

A simple proof of Jensen–Steffensen’s inequality (1.1) based on the identity (2.11) can be found in [6, p. 58] and is a consequence of the fact that for all \( x, y, z \in I \) we have

\[ f(\bar{\xi}) + \lambda(x - \bar{\xi}) - f(x) \leq 0 \]

and

\[ f(z) - f(y) - \lambda(z - y) \leq 0, \quad \text{for } y \leq z \leq \bar{\xi}, \]
\[ \lambda(z - y) - f(z) + f(y) \leq 0, \quad \text{for } \bar{\xi} \leq y \leq z. \]
In case $f$ is strictly convex the first of the above inequalities is strict unless $x = \bar{\xi}$, while the other two inequalities are strict unless $y = z$. That fact enables us to prove the following result.

**Proposition 1.** Let $f : I \to \mathbb{R}$ be a strictly convex function. Let $\xi$ be an increasing $m$-tuple in $I^m$ and $p$ a real $m$-tuple satisfying (1.2) and (2.1). Assume that $m \geq 3$ and that $\bar{\xi}$ satisfies the condition (2.10).

If $\xi_k < \bar{\xi} < \xi_{k+1}$ for some $k \in \{1, \ldots, m-1\}$ or $\xi_1 < \bar{\xi} = \xi_2$ or $\xi_{m-1} = \bar{\xi} < \xi_m$, then

$$P_m f(\bar{\xi}) - \sum_{i=1}^{m} p_i f(\xi_i) < 0.$$  

(2.12)

**Proof.** We take a $k \in \{1, \ldots, m-1\}$ such that $\xi_k \leq \bar{\xi} \leq \xi_{k+1}$ and consider (2.11). All the terms at the right-hand side of (2.11) are $\leq 0$. If $\xi_k < \bar{\xi} < \xi_{k+1}$, then at least one of terms $P_k f(\bar{\xi}) + \lambda(\xi_k - \bar{\xi}) - f(\xi_k)$ and $P_{k+1}(f(\bar{\xi}) + \lambda(\xi_{k+1} - \bar{\xi}) - f(\xi_{k+1}))$ is strictly negative, since from (2.3) we get $P_k > 0 \vee P_{k+1} > 0$, and (2.12) is true. If $\xi_1 < \bar{\xi} = \xi_2$, then $k = 1$ and $P_1 [f(\bar{\xi}) + \lambda(\xi_1 - \bar{\xi}) - f(\xi_1)] < 0$, while in the case $\xi_{m-1} = \bar{\xi} < \xi_m$ we have $k = m - 1$ and $P_m [f(\bar{\xi}) + \lambda(\xi_m - \bar{\xi}) - f(\xi_m)] < 0$. In both last cases we conclude that (2.12) is true. \(\square\)

We summarize the above considerations in the main result in this section which gives a necessary and sufficient condition for the equality case in Jensen–Steffensen’s inequality.

**Theorem 1.** Let $f : I \to \mathbb{R}$ be a strictly convex function and $m \geq 2$. Let $\xi$ be a monotonic $m$-tuple in $I^m$ and $p$ a real $m$-tuple satisfying (1.2) and (2.1).

(a) In the case $m = 2$, Jensen–Steffensen’s inequality (1.1) becomes equality if and only if $\xi_1 = \xi_2$.

(b) In the case $m \geq 3$, Jensen–Steffensen’s inequality (1.1) becomes equality if and only if one of the following two cases occurs:

1. Either $\bar{\xi} = \xi_1$ or $\bar{\xi} = \xi_m$.
2. There exists $k \in \{3, \ldots, m-2\}$ such that $\bar{\xi} = \xi_k$ and

$$\forall j = 1, \ldots, k-1, \quad P_j = 0 \vee \xi_j = \xi_{j+1},$$

$$\forall j = k+1, \ldots, m, \quad \bar{P}_j = 0 \vee \xi_j = \xi_{j-1}. $$

(2.13)

**Proof.** First assume $\xi$ to be increasing. If $m = 2$, then $P_1 > 0$, $P_2 > 0$ and (1.1) is equivalent to the definition of convexity. In case $m \geq 3$, our assertion follows from the above considerations and the fact that in case $\bar{\xi} = \xi_k$, $k \in \{3, \ldots, m-2\}$, (2.11) can be rewritten as

$$P_m f(\bar{\xi}) - \sum_{i=1}^{m} p_i f(\xi_i) = \sum_{j=1}^{k-1} P_j \left[ f(\xi_{j+1}) - f(\xi_j) - \lambda(\xi_{j+1} - \xi_j) \right]$$

$$+ \sum_{j=k+1}^{m} \bar{P}_j \left[ \lambda(\xi_j - \xi_{j-1}) - f(\xi_j) + f(\xi_{j-1}) \right]. $$

(2.14)
Now, for strictly convex function $f$ the equality in (1.1) holds if and only if all the terms at the right-hand side of (2.14) are equal to 0, which is equivalent to the condition (2.13). Hence, our assertion is proved for the case that $\xi$ is increasing. If $\xi$ is decreasing, then we can replace $\xi$ and $p$ with $\tilde{\xi} = (\xi_m, \xi_{m-1}, \ldots, \xi_1)$ and $\tilde{p} = (p_m, p_{m-1}, \ldots, p_1)$, respectively, to get the proposed conclusions. □

3. A variant of Jensen–Steffensen’s inequality

In the following we prove a variant of Jensen–Steffensen’s inequality, which includes Theorem A as a special case.

**Theorem 2.** Let $f : I \to \mathbb{R}$, where $I$ is any interval in $\mathbb{R}$, and let $[a, b] \subseteq I$, $a < b$. Let $x = (x_1, \ldots, x_n)$ be a monotonic $n$-tuple in $[a, b]^n$ and $v = (v_1, \ldots, v_n)$ a real $n$-tuple such that $v_i \neq 0, i = 1, \ldots, n$, and $0 \leq V_j \leq V_n, j = 1, \ldots, n$, $V_n > 0$, where $V_j = \sum_{i=1}^{j} v_i$. If $f$ is convex on $I$, then

$$f\left(a + b - \frac{1}{V_n} \sum_{i=1}^{n} v_i x_i\right) \leq f(a) + f(b) - \frac{1}{V_n} \sum_{i=1}^{n} v_i f(x_i).$$

(3.1)

In case $f$ is strictly convex, the equality holds in (3.1) if and only if one of the following two cases occurs:

1. either $\bar{x} = a$ or $\bar{x} = b$,
2. there exists $l \in \{2, \ldots, n-1\}$ such that $\bar{x} = x_1 + x_n - x_l$ and

$$x_1 = a, \quad x_n = b \lor x_1 = b, \quad x_n = a,$$

$$\forall j = 2, \ldots, l, \quad \bar{V}_j = 0 \lor x_{j-1} = x_j,$$

$$\forall j = l, \ldots, n-1, \quad V_j = 0 \lor x_j = x_{j+1},$$

where $\bar{V}_j = \sum_{i=j}^{n} v_i$, $j = 1, \ldots, n$, and $\bar{\bar{x}} = (1/V_n) \sum_{i=1}^{n} v_i x_i$.

In the special case where $v_i > 0, i = 1, \ldots, n$, and $f$ is strictly convex, the equality holds in (3.1) iff $a = x_i$ or $b = x_i, i = 1, \ldots, n$.

**Proof.** First assume that $x$ is increasing. Set $m = n + 2$ and define the $m$-tuples $\xi$ and $p$ as

$$\xi_1 = a, \quad \xi_2 = x_1, \quad \xi_3 = x_2, \quad \ldots, \quad \xi_{n+1} = x_n, \quad \xi_{n+2} = b,$$

$$p_1 = 1, \quad p_2 = -\frac{v_1}{V_n}, \quad p_3 = -\frac{v_2}{V_n}, \quad \ldots, \quad p_{n+1} = -\frac{v_n}{V_n}, \quad p_{n+2} = 1.$$

Then, for $P_j = \sum_{i=1}^{j} p_i$ and $\tilde{P}_j = \sum_{i=j}^{n+2} p_i$ we get

$$P_j = \frac{\bar{V}_j}{V_n}, \quad j = 1, \ldots, n, \quad P_{n+1} = 0, \quad P_{n+2} = 1,$$

and

$$\tilde{P}_1 = 1, \quad \tilde{P}_2 = 0, \quad \tilde{P}_j = \frac{\bar{V}_{j-2}}{V_n}, \quad j = 3, \ldots, n + 2.$$
Also
\[ \bar{\xi} = \frac{1}{P_m} \sum_{i=1}^{m} p_i \xi_i = a + b - \frac{1}{V_n} \sum_{i=1}^{n} v_i x_i = a + b - \bar{x}. \]
Obviously, \( \bar{\xi} = \xi_1 \) is equivalent to \( \bar{x} = b \) and \( \bar{\xi} = \xi_{n+2} \) is equivalent to \( \bar{x} = a \). Also, the existence of some \( k \in \{3, \ldots, m-2\} \) such that \( \bar{\xi} = \xi_k \) and (2.13) hold, is equivalent to the existence of some \( l \in \{2, \ldots, n-1\} \) such that \( \bar{x} = x_1 + x_n - x_l \) and (3.2) hold. Since (1.2) holds for \( m = n + 2 \), we can apply Jensen–Steffensen’s inequality to get the desired conclusions. In case \( \mathbf{x} \) is decreasing we simply replace \( \mathbf{x} \) and \( \mathbf{v} \) with \( \tilde{\mathbf{x}} = (x_n, \ldots, x_1) \) and \( \tilde{\mathbf{v}} = (v_n, \ldots, v_1) \), respectively, and then argue in the same way. In the special case that \( v_i > 0 \) and \( V_j > 0 \), \( i = 1, \ldots, n \), and therefore according to (2.13) equality holds in (3.1) only when either \( a = x_1 = \cdots = x_n \) or \( b = x_1 = \cdots = x_n \).

We have now the following result concerning two functions:

**Theorem 3.** Let \( f : I \to \mathbb{R} \) and \( g : J \to \mathbb{R} \), where \( I \) and \( J \) are intervals in \( \mathbb{R} \), be two functions such that \( f(I) \subseteq J \). Assume \( f \) to be monotonic on \( I \). Let \( \xi = (\xi_1, \ldots, \xi_m) \) be any monotonic \( m \)-tuple in \( I^m \) and \( p = (p_1, \ldots, p_m) \) any real \( m \)-tuple such that \( p_i \neq 0 \), \( i = 1, \ldots, m \), and (1.2) holds.

(i) If either \( f \) is convex on \( I \) and \( g \) is increasing and convex on \( J \), or \( f \) is concave on \( I \) and \( g \) is decreasing and convex on \( J \), then
\[
(g \circ f) \left( \frac{1}{P_m} \sum_{i=1}^{m} p_i \xi_i \right) \leq \frac{1}{P_m} \sum_{i=1}^{m} p_i f(\xi_i),
\]
(3.3)

(ii) If either \( f \) is convex on \( I \) and \( g \) is decreasing and concave on \( J \), or \( f \) is concave on \( I \) and \( g \) is increasing and concave on \( J \), then the reverse inequalities hold.

If all the assumptions on monotonicity, convexity and concavity are strengthened to the assumptions on strict monotonicity, strict convexity and strict concavity, then all the inequalities in (3.3) and its reverse are strict except in the cases described by Theorem 1, in which all the inequalities in (3.3) and its reverse become equalities.

In the special case that \( p_i > 0 \), \( i = 1, \ldots, m \) and the strict assumptions hold, all the inequalities (3.3) and their reverse are strict unless \( \xi_i = c \), \( i = 1, \ldots, m \).

**Proof.** Let us denote
\[ f(\xi) = (f(\xi_1), \ldots, f(\xi_m)). \]
Since \( \xi \) is a monotonic \( m \)-tuple in \( I^m \), \( f(I) \subseteq J \) and \( f \) is assumed to be monotonic on \( I \), \( f(\xi) \) is a monotonic \( m \)-tuple in \( J^m \). Further, in case \( f \) is convex we can apply Jensen–Steffensen’s inequality to obtain
\[
f \left( \frac{1}{P_m} \sum_{i=1}^{m} p_i \xi_i \right) \leq \frac{1}{P_m} \sum_{i=1}^{m} p_i f(\xi_i),
\]
(3.4)
while in case $f$ is concave we get
\[ f\left( \frac{1}{p_m} \sum_{i=1}^{m} p_i \xi_i \right) \geq \frac{1}{p_m} \sum_{i=1}^{m} p_i f(\xi_i). \] (3.5)

Now, the second inequality in (3.3) is a consequence of Jensen–Steffensen’s inequality for the convex function $g$ and the fact that $f(\xi)$ is monotonic $m$-tuple in $J^m$. The first inequality in (3.3) follows from (3.4) in case $g$ is increasing, and from (3.5) in case $g$ is decreasing. (ii) is proved similarly.

The assertion on the equality case in all the inequalities in (3.3) is proved by the argument similar to the one used in the proof of Theorem 1. □

(Some more general results of this type can be found in [1].)

Our next result is a variant of the inequalities in Theorem 3 analogous to those obtained in Theorem 2. The proof is omitted.

**Theorem 4.** Let $f : I \to \mathbb{R}$ and $g : J \to \mathbb{R}$, where $I$ and $J$ are intervals in $\mathbb{R}$, be two functions such that $f(I) \subseteq J$. Assume also that $f$ is monotonic on $I$. Let $[a, b] \subseteq I$, $a < b$. Let $x = (x_1, \ldots, x_n)$ be monotonic $n$-tuple in $[a, b]^n$ and $v = (v_1, \ldots, v_n)$ be a real $n$-tuple such that $v_i \neq 0$, $i = 1, \ldots, n$, and $0 \leq V_j \leq V_n$, $j = 1, \ldots, n$, $V_n > 0$, where $V_j = \sum_{i=1}^{j} v_i$.

(i) If either $f$ is convex on $I$ and $g$ is increasing and convex on $J$, or $f$ is concave on $I$ and $g$ is decreasing and convex on $J$, then
\[ (g \circ f)(a + b - \frac{1}{V_n} \sum_{i=1}^{n} v_i x_i) \leq g\left( f(a) + f(b) - \frac{1}{V_n} \sum_{i=1}^{n} v_i f(x_i) \right) \]
\[ \leq (g \circ f)(a) + (g \circ f)(b) - \frac{1}{V_n} \sum_{i=1}^{n} v_i (g \circ f)(x_i). \] (3.6)

(ii) If either $f$ is convex on $I$ and $g$ is decreasing and concave on $J$, or $f$ is concave on $I$ and $g$ is increasing and concave on $J$, then the reverse inequalities hold.

If all the assumptions on monotonicity, convexity and concavity are strengthened to the assumptions on strict monotonicity, strict convexity and strict concavity, then all the inequalities in (3.6) are strict except in the cases (1) and (2) described in Theorem 2, in which all the inequalities in (3.6) become equalities.

In case $v$ is also positive, the $n$-tuple $x$ need not be monotonic and the inequalities proposed above are still valid. Namely, in this case we can simply replace $x$ with $\tilde{x} = (x_{i_1}, \ldots, x_{i_n})$ and $v$ with $\tilde{v} = (v_{i_1}, \ldots, v_{i_n})$, where $(i_1, i_2, \ldots, i_n)$ is a permutation of $(1, 2, \ldots, n)$ such that $\tilde{x}$ is increasing.
It is interesting that these results concerning two functions, each of which is either convex or concave, can be generalized by induction to any set of functions satisfying certain conditions.

Consider a set of \( r + 1, r \geq 1 \), functions

\[
f : I \to \mathbb{R}, \quad g_1 : I_1 \to \mathbb{R}, \quad \ldots, \quad g_r : I_r \to \mathbb{R},
\]

where \( I, I_1, \ldots, I_r \) are intervals in \( \mathbb{R} \) such that

\[
f(I) \subseteq I_1, \quad g_k(I_k) \subseteq I_{k+1}, \quad k = 1, \ldots, r - 1.
\]

Then the following two sets of auxiliary functions \( F_k, G_k, k = 1, \ldots, r \), such that

\[
F_k : I \to \mathbb{R}, \quad F_k = g_k \circ g_{k-1} \circ \cdots \circ g_1 \circ f,
\]

\[
G_k : I_k \to \mathbb{R}, \quad G_k = g_r \circ g_{r-1} \circ \cdots \circ g_k.
\]

Furthermore, for any given monotonic \( \xi = (\xi_1, \ldots, \xi_m) \) in \( I^m \) and real \( p = (p_1, \ldots, p_m) \) satisfying (1.2) we define the value \( \bar{\xi} \in I \) as

\[
\bar{\xi} = \frac{1}{P_m} \sum_{i=1}^{m} p_i \xi_i.
\]

Also, we define \( f(\xi) \) and \( F_k(\xi), k = 1, \ldots, r \), and the values \( f(\bar{\xi}) \) and \( F_k(\bar{\xi}) \) by

\[
f(\xi) = (f(\xi_1), \ldots, f(\xi_m)), \quad F_k(\xi) = (F_k(\xi_1), \ldots, F_k(\xi_m)),
\]

\[
f(\bar{\xi}) = \frac{1}{P_m} \sum_{i=1}^{m} p_i f(\xi_i), \quad F_k(\bar{\xi}) = \frac{1}{P_m} \sum_{i=1}^{m} p_i F_k(\xi_i).
\]

Now, we assume that each of the considered functions of the set \( f, g_1, \ldots, g_r \) is either convex or concave and that the following monotonicity condition is fulfilled.

**Monotonicity condition.** Denote \( g_0 = f \). We say that a set of functions \( g_0, g_1, \ldots, g_r \) satisfies the Monotonicity Condition (MC), if all \( k \in \{0, 1, r - 1\} \) and all pairs \( (g_k, g_{k+1}) \) satisfy the following:

(i) when both functions \( g_k \) and \( g_{k+1} \) are either convex or concave, then \( g_{k+1} \) is increasing;

(ii) when either \( g_k \) is convex and \( g_{k+1} \) is concave, or \( g_k \) is concave and \( g_{k+1} \) is convex, then \( g_{k+1} \) is decreasing.

Note that when the functions \( f, g_1, \ldots, g_r \) satisfy the above stated MC, then all of them except possibly \( g_0 = f \) are monotonic. Also, we have:

**Proposition 2.** Let the functions \( f, g_1, \ldots, g_r \) be as above and satisfy MC. Let \( F_k, k = 1, \ldots, r \), be defined by (3.8). Then for all \( k \in \{1, \ldots, r\} \) we have

(i) if \( g_k \) is convex on \( I \), then \( F_k \) is convex on \( I \);

(ii) if \( g_k \) is concave on \( I \), then \( F_k \) is concave on \( I \).
Proof. For $k = 1$ the proposed conclusions follow directly from definitions of convexity and concavity, while for $k > 1$ the proposed conclusions easily follow by induction. The details are left to the reader.

It is now clear that we can extend Theorem 4 to the following general result:

**Theorem 5.** Let $f : I \to \mathbb{R}$ and $g_k : I_k \to \mathbb{R}$, $k = 1, \ldots, r$, be either convex or concave, where $I$ and $I_k$, $k = 1, \ldots, r$, are intervals in $\mathbb{R}$ satisfying (3.7). Define the auxiliary functions $F_k$ and $G_k$ by (3.8) and (3.9), respectively. Assume that $f$ and $g_k$, $k = 1, \ldots, r$, satisfy MC and additionally assume $f$ to be monotonic. Then for any monotonic $m$-tuple $\xi \in I^m$ and real $m$-tuple $p$ with $p_i \neq 0$, $i = 1, \ldots, m$, satisfying (1.2), we have:

(i) if $g_r$ is convex on $I$, then
\[
F_r(\xi) \leq G_1(f(\xi)) \leq G_2(F_1(\xi)) \leq \cdots \leq G_r(F_{r-1}(\xi)) \leq F_r(\xi),
\]

(ii) if $g_r$ is concave on $I$, then the reverse of (3.12) holds, where the value $\bar{\xi}$ is defined by (3.10) and values $\bar{f}(\bar{\xi})$ and $\bar{F}_k(\bar{\xi})$ are defined by (3.11).

If all the assumptions on monotonicity, convexity and concavity are strengthened to the assumptions on strict monotonicity, strict convexity and strict concavity, then all the inequalities in (i) and (ii) are strict except in the cases described by Theorem 2, in which all the inequalities in (i) and (ii) become equalities.

Proof. Under the given assumptions, the monotonicity of $f$ ensures that all $f(\xi) \in I^m_k$ and $F_k(\xi) \in I^m_k$, $k = 1, \ldots, r - 1$, are monotonic. Now, for $r = 1$ we set $g = g_1$ and it is easily seen that the proposed statement is in fact the statement of Theorem 3. The general case then easily follows by induction. The details are left to the reader.

**Corollary 1.** Let all the assumptions of Theorem 5 be satisfied. Let $[a, b] \subseteq I$, $a < b$. Let $x = (x_1, \ldots, x_n)$ be monotonic $n$-tuple in $[a, b]^n$ and $v = (v_1, \ldots, v_n)$ be real $n$-tuple such that $v_i \neq 0$, $i = 1, \ldots, n$, and $0 \leq V_j \leq V_n$, $j = 1, \ldots, n$, $V_n > 0$, where $V_j = \sum_{i=1}^n v_i$. Then

(i) if $g_r$ is convex on $I$, then (3.12) holds,

(ii) if $g_r$ is concave on $I$, then the reverse of (3.12) holds,

where
\[
\bar{\xi} = a + b - \frac{1}{V_n} \sum_{i=1}^n v_i x_i, \quad \bar{F}_k(\bar{\xi}) = F_k(a) + F_k(b) - \frac{1}{V_n} \sum_{i=1}^n v_i F_k(x_i).
\]

If all the assumptions on monotonicity, convexity and concavity are strengthened to the assumptions on strict monotonicity, strict convexity and strict concavity, then all the inequalities in (i) and (ii) are strict except in the cases (1) and (2) described in Theorem 2, in which all the inequalities in (i) and (ii) become equalities.
Proof. Define $\xi$ and $p$ exactly as in the proof of Theorem 2 and then apply Theorem 5. □

4. Jensen–Steffensen’s inequality and quasi-arithmetic means

Let $f : I \to \mathbb{R}$ be strictly monotonic and continuous function, where $I$ is an interval in $\mathbb{R}$. Then for a given $m$-tuple $\xi = (\xi_1, \ldots, \xi_m)$ in $I^m$ and nonnegative $m$-tuple $p = (p_1, \ldots, p_m)$ with $P_m = \sum_{i=1}^{m} p_i > 0$, the value

$$M_f (\xi, p) = f^{-1} \left( \frac{1}{P_m} \sum_{i=1}^{m} p_i f(\xi_i) \right)$$

is well defined and is called quasi-arithmetic $f$-mean of $\xi$ with weights $p$ (see [2, p. 215]). If $\xi$ is assumed to be monotonic $m$-tuple in $I^m$ and $p$ any real $m$-tuple satisfying (1.2), then $M_f (\xi, p)$ is still well defined. Moreover, the following result is true.

Theorem 6. Let $f$ and $g$ be two continuous strictly monotonic functions on an interval $I$ and let $m \geq 2$. The inequality

$$f^{-1} \left( \frac{1}{P_m} \sum_{i=1}^{m} p_i f(\xi_i) \right) \leq g^{-1} \left( \frac{1}{P_m} \sum_{i=1}^{m} p_i g(\xi_i) \right)$$

(4.1)

holds for all monotonic $m$-tuples $\xi$ in $I^m$ and real $m$-tuples $p$ with $p_i \neq 0$, $i = 1, \ldots, m$, satisfying (1.2), if and only if either $g \circ f^{-1}$ is convex and $g$ is strictly increasing, or $g \circ f^{-1}$ is concave and $g$ is strictly decreasing. The reverse inequality holds for all monotonic $m$-tuples $\xi$ in $I^m$ and real $m$-tuples $p$ with $p_i \neq 0$, $i = 1, \ldots, m$, satisfying (1.2), if and only if either $g \circ f^{-1}$ is concave and $g$ is strictly increasing, or $g \circ f^{-1}$ is convex and $g$ is strictly decreasing.

In case the function $g \circ f^{-1}$ is strictly convex (strictly concave), the inequality (4.1) (reverse of (4.1)) becomes equality if and only if one of the following three cases occurs:

(i) $m = 2$ and $\xi_1 = \bar{\xi}_2$,
(ii) $m \geq 3$ and either $\bar{\xi} = \xi_1$ or $\bar{\xi} = \xi_m$,
(iii) $m \geq 3$ and there exists $k \in \{3, \ldots, m - 2\}$ such that $\bar{\xi} = \xi_k$ and

$$\begin{align*}
\forall j = 1, \ldots, k - 1, \quad P_j &= 0 \lor \xi_j = \xi_{j+1}, \\
\forall j = k + 1, \ldots, m, \quad \hat{p}_j &= 0 \lor \xi_j = \xi_{j-1},
\end{align*}$$

(4.2)

where $\bar{\xi} = (1/P_m) \sum_{i=1}^{m} p_i \xi_i$.

Proof. Since $f$ and $g$ are continuous and strictly monotone, we know that $f(I) = I_1$ and $g(I) = I_2$ are intervals in $\mathbb{R}$. Now take any monotonic $m$-tuple $\xi$ in $I^m$ and any real $m$-tuple $p$ satisfying condition (1.2). Since $f$ and $g$ are strictly monotonic, then

$$f(\xi) = (f(\xi_1), \ldots, f(\xi_m)) \in I_1^m, \quad g(\xi) = (g(\xi_1), \ldots, g(\xi_m)) \in I_2^m$$
are both monotonic and consequently we have

\[ \frac{1}{P_m} \sum_{i=1}^{m} p_i f(\xi_i) \in J_1, \quad \frac{1}{P_m} \sum_{i=1}^{m} p_i g(\xi_i) \in J_2. \]

Hence, both sides of (4.1) and its reverse are well defined.

If the function \( g \circ f^{-1} \) is convex on \( J_1 \), then for all \( m \)-tuples \( \xi \) and \( p \) satisfying the above conditions, Jensen–Steffensen’s inequality (1.1) gives

\[ (g \circ f^{-1}) \left( \frac{1}{P_m} \sum_{i=1}^{m} p_i f(\xi_i) \right) \leq \frac{1}{P_m} \sum_{i=1}^{m} p_i (g \circ f^{-1})(f(\xi_i)), \]

which can be rewritten as

\[ g \left( f^{-1} \left( \frac{1}{P_m} \sum_{i=1}^{m} p_i f(\xi_i) \right) \right) \leq \frac{1}{P_m} \sum_{i=1}^{m} p_i g(\xi_i). \]  \hspace{1cm} (4.3)

In case \( g \circ f^{-1} \) is concave, we obtain the reverse of the above inequality.

If the function \( g \) is strictly increasing, then the inverse function \( g^{-1} \) is also strictly increasing, so that (4.3) implies (4.1). If \( g \) is strictly decreasing, then \( g^{-1} \) is strictly decreasing too, so that in this case the reverse of (4.3) implies (4.1). On the other hand, analogous arguments give the reverse of (4.1), that is, we get the reverse of (4.1) in the cases when \( g \circ f^{-1} \) is convex and \( g \) is strictly decreasing, or \( g \circ f^{-1} \) is concave and \( g \) is strictly increasing.

Now, if \( g \circ f^{-1} \) is strictly convex (strictly concave), then by Theorem 1 the inequality in (4.3) (its reverse), or equivalently the inequality in (4.1) (its reverse), becomes equality exactly when one of the following three cases occurs:

(i) \( m = 2 \) and \( f(\xi_1) = f(\xi_2) \), that is \( \xi_1 = \xi_2 \), since \( f \) is one-to-one function.

(ii) \( m \geq 3 \) and either \( f(\bar{\xi}) = f(\xi_1) \) or \( f(\bar{\xi}) = f(\xi_m) \), where we denote \( \bar{\xi} = (1/P_m) \sum_{i=1}^{m} p_i f(\xi_i) \). Now using the identities (2.8) and (2.9) and the fact that \( f \) is one-to-one function, it is easy to see that \( f(\bar{\xi}) = f(\xi_1) \) is equivalent to \( \bar{\xi} = \xi_1 \) (by (2.6)) and that \( f(\bar{\xi}) = f(\xi_m) \) is equivalent to \( \bar{\xi} = \xi_m \) (by (2.7)).

(iii) \( m \geq 3 \) and there exists \( k \in \{3, \ldots, m-2\} \) such that \( f(\bar{\xi}) = f(\xi_k) \) and

\[ \forall j = 1, \ldots, k-1, \quad P_j = 0 \lor f(\xi_j) = f(\xi_{j+1}), \]
\[ \forall j = k+1, \ldots, m, \quad \bar{P}_j = 0 \lor f(\xi_j) = f(\xi_{j-1}). \]  \hspace{1cm} (4.4)

Again, using the fact that \( f \) is one-to-one function and the identities

\[ P_m \xi_k - \sum_{i=1}^{m} p_i \xi_i = \sum_{j=1}^{k-1} P_j (\xi_j - \xi_{j+1}) + \sum_{j=k+1}^{m} \bar{P}_j (\xi_j - \xi_{j-1}) \]

and

\[ P_m f(\xi_k) - \sum_{i=1}^{m} p_i f(\xi_i) = \sum_{j=1}^{k-1} P_j [f(\xi_j) - f(\xi_{j+1})] + \sum_{j=k+1}^{m} \bar{P}_j [f(\xi_j) - f(\xi_{j-1})], \]
it is easy to see that \( \bar{f}(\xi_k) = f(\xi_k) \) and (4.4) are equivalent to \( \bar{\xi} = \xi_k \) and (4.2).

Theorem 6 can be extended in the following way.

**Theorem 7.** Let \( F_k : I \rightarrow \mathbb{R}, k = 0, 1, \ldots, r \), be continuous and strictly monotonic functions on an interval \( I \) in \( \mathbb{R} \). Let the functions \( g_1, g_2, \ldots, g_r \) be defined as

\[
g_k = F_k \circ F_{k-1}^{-1} : I_k \rightarrow I_{k+1}, \quad k = 1, \ldots, r,
\]

where \( I_j = F_j^{-1}(I), \ j = 1, \ldots, r + 1 \), are intervals in \( \mathbb{R} \). Assume that each function \( g_k, k = 1, \ldots, r, \) is either convex or concave on \( I_k \) and that the set of functions \( g_1, g_2, \ldots, g_r \) is satisfying MC. Then for all monotonic \( m \)-tuples \( \xi \) in \( I^m \) and all real \( m \)-tuples \( p \) with \( p_i \neq 0, i = 1, \ldots, m \), satisfying (1.2),

1. if either \( F_1 \circ F_0^{-1} \) is convex and \( F_1 \) is strictly increasing, or \( F_1 \circ F_0^{-1} \) is concave and \( F_1 \) is strictly decreasing, then

\[
M_{F_0}(\xi; p) \leq M_{F_1}(\xi; p) \leq M_{F_2}(\xi; p) \leq \cdots \leq M_{F_r}(\xi; p), \quad (4.5)
\]

2. if either \( F_1 \circ F_0^{-1} \) is concave and \( F_1 \) is strictly increasing, or \( F_1 \circ F_0^{-1} \) is convex and \( F_1 \) is strictly decreasing, then the reverse inequalities hold,

where

\[
M_{F_k}(\xi; p) = F_k^{-1}\left(\frac{1}{p_m} \sum_{i=1}^{m} p_i F_k(\xi_i)\right), \quad k = 0, 1, \ldots, r.
\]

In case each of the functions \( g_k, k = 1, \ldots, r, \) is either strictly convex or strictly concave on \( I_k \), all the inequalities in (4.5) and its reverse are strict unless one of the cases (i)–(iii) from Theorem 6 occurs. In those three cases all the inequalities in (4.5) and its reverse become equalities.

**Proof.** The first inequality in (4.5) as well as the first inequality in its reverse follows directly by Theorem 6. The other inequalities in (4.5) and in its reverse are consequences of Theorem 6 and the fact that the set \( g_1, g_2, \ldots, g_r \) is satisfying MC. For example, assume that \( F_1 \circ F_0^{-1} \) is convex and \( F_1 \) is strictly increasing. Then for \( F_2 \circ F_1^{-1} \) we have two possibilities: either it is convex or it is concave. If \( F_2 \circ F_1^{-1} \) is convex, then it must be strictly increasing by MC, and \( F_2 = (F_2 \circ F_1^{-1}) \circ F_1 \) is strictly increasing too, while in case \( F_2 \circ F_1^{-1} \) is concave, it must be strictly decreasing by MC and \( F_2 = (F_2 \circ F_1^{-1}) \circ F_1 \) is strictly decreasing too. Hence, in both cases applying Theorem 6 with \( f = F_1 \) and \( g = F_2 \) we obtain the second inequality in (4.5). In all other cases we argue similarly. Finally, the proof of the assertion on the equalities in (4.5) and its reverse is analogous to the proof in Theorem 6. \( \square \)

**Corollary 2.** Let all the assumptions of Theorem 7 be satisfied. Let \([a, b] \subseteq I, a < b\). Let \( x = (x_1, \ldots, x_n) \) be a monotonic \( n \)-tuple in \([a, b]^n\) and \( v = (v_1, \ldots, v_n) \) be a real \( n \)-tuple such that \( v_i \neq 0, i = 1, \ldots, n, \) and \( 0 \leq V_j \leq V_n, j = 1, \ldots, n, V_n > 0, \) where \( V_j = \sum_{i=1}^{j} v_i \).
(i) If either $F_1 \circ F_0^{-1}$ is convex and $F_1$ is strictly increasing, or $F_1 \circ F_0^{-1}$ is concave and $F_1$ is strictly decreasing, then
\[ \tilde{M}_{F_1}(a, b, x, v) \leq \tilde{M}_{F_1}(a, b, x, v) \leq \tilde{M}_{F_1}(a, b, x, v) \leq \cdots \leq \tilde{M}_{F_1}(a, b, x, v). \]

(ii) If either $F_1 \circ F_0^{-1}$ is concave and $F_1$ is strictly increasing, or $F_1 \circ F_0^{-1}$ is convex and $F_1$ is strictly decreasing, then the reverse inequalities hold,

where
\[ \tilde{M}_{F_k}(a, b, x, v) = F_k^{-1}\left(F_k(a) + F_k(b) - \frac{1}{V_n} \sum_{i=1}^{n} v_i F_k(x_i)\right), \quad k = 0, 1, \ldots, r. \]

In case each of the functions $g_k$, $k = 1, \ldots, r$, is either strictly convex or strictly concave on $I_k$, all the proposed inequalities are strict unless one of the cases (1) and (2) in Theorem 2 occurs. In those two cases all the proposed inequalities become equalities.

**Proof.** Define $\xi$ and $p$ exactly as in the proof of Theorem 2 and then apply Theorem 7. The assertion on the equality case follows by the fact that (i)–(iii) from Theorem 6 reduce to (1), (2) from Theorem 2.

**Remark 1.** In case $v$ is also positive, the $n$-tuple $x$ need not be monotonic and the inequalities proposed in Corollary 2 are still valid since in this case $x$ and $v$ can be replaced with $\tilde{x} = (x_{i_1}, \ldots, x_{i_n})$ and $\tilde{v} = (v_{i_1}, \ldots, v_{i_n})$, respectively, where $(i_1, i_2, \ldots, i_n)$ is a permutation of $(1, 2, \ldots, n)$ such that $\tilde{x}$ is increasing.

5. Examples

We show now how some monotonicity properties of power means proved in [3] can be obtained as a special case of Theorem 7.

**Example 1.** Let $r, s, \rho, \sigma \in \mathbb{R}$ be arbitrarily chosen real numbers satisfying
\[ r < s < 0 < \rho < \sigma. \]
Let $I = (0, \infty)$ and $F_k : I \to \mathbb{R}$, $k = 0, 1, 2, 3, 4$, be defined as
\[ F_0(x) = x^r, \quad F_1(x) = x^s, \quad F_2(x) = \ln x, \quad F_3(x) = x^\rho, \quad F_4(x) = x^\sigma. \]
Then $F_0$ and $F_1$ are strictly decreasing, while $F_2$, $F_3$ and $F_4$ are strictly increasing and all of them are continuous on $I$. Also, $F_2(I) = \mathbb{R}$ and $F_k(I) = I$ for $k \in \{0, 1, 3, 4\}$. Furthermore, the functions $g_k$, $k = 1, 2, 3, 4$, defined in Theorem 7 are in this case given by
\[ g_1(x) = x^{s/r}, \quad g_2(x) = \frac{1}{s} \ln x, \quad g_3(x) = e^\rho x, \quad g_4(x) = x^{\sigma/\rho}. \]

The function $g_1$ is strictly concave and strictly increasing, while $g_2$, $g_3$ and $g_4$ are strictly convex, $g_2$ is strictly decreasing, $g_3$ and $g_4$ are strictly increasing. Hence all the assumptions of Theorem 7(i) are satisfied so that
\[ M_{F_0}(\xi, p) \leq M_{F_1}(\xi, p) \leq M_{F_2}(\xi, p) \leq M_{F_3}(\xi, p) \leq M_{F_4}(\xi, p) \] (5.1)
hold for all monotonic \( m \)-tuples \( \xi \) in \( I^m \) and all real \( m \)-tuples \( p \) with \( p_i \neq 0 \), \( i = 1, \ldots, m \) satisfying (1.2). Moreover, equalities in (5.1) hold if and only if one of the cases (i)--(iii) in Theorem 6 occurs.

Now, for a given monotonic \( m \)-tuple \( \xi \) in \( (0, \infty)^m \) and a real \( m \)-tuple \( p \) with \( p_i \neq 0 \), \( i = 1, \ldots, m \), satisfying (1.2) we define \( t \)-mean \( M_{[t]}(\xi, p) \) of \( \xi \) with quasi-weights \( p \) as

\[
M_{[t]}(\xi, p) = \left( \frac{1}{P_m} \sum_{i=1}^{m} p_i \xi_i^t \right)^{1/t}, \quad t \in \mathbb{R} \setminus \{0\},
\]

\[
M_{[0]}(\xi, p) = \left( \prod_{i=1}^{m} \xi_i^{P_i} \right)^{1/P_m}.
\]

Obviously for any \( t \in \mathbb{R} \) we have

\[
\min(\xi_1, \ldots, \xi_m) \leq M_{[t]}(\xi, p) \leq \max(\xi_1, \ldots, \xi_m).
\]

Also from (5.1) we get that

\[
M_{[r]}(\xi, p) \leq M_{[s]}(\xi, p)
\]

holds for any \( r, s \in \mathbb{R} \) such that \( r < s \) and equality holds if and only if one of the cases (i)--(iii) in Theorem 6 occurs. Furthermore, by Corollary 2, if \([a, b] \subseteq (0, \infty), a < b, x = (x_1, \ldots, x_n)\) is a monotonic \( n \)-tuple in \([a, b]^n\) and \( v = (v_1, \ldots, v_n)\), is a real \( n \)-tuple such that \( v_i \neq 0, i = 1, \ldots, n, 0 \leq V_j \leq V_n, j = 1, \ldots, n, V_n > 0 \), where \( V_j = \sum_{i=1}^{j} v_i \), then

\[
a \leq \tilde{M}_{[t]}(a, b, x, v) \leq \tilde{M}_{[1]}(a, b, x, v) \leq b
\]

holds for any \( r, s \in \mathbb{R} \) such that \( r < s \). Here we denote

\[
\tilde{M}_{[t]}(a, b, x, v) = \left( a^t + b^t - \frac{1}{V_n} \sum_{i=1}^{n} v_i x_i^t \right)^{1/t}, \quad t \in \mathbb{R} \setminus \{0\},
\]

\[
\tilde{M}_{[0]}(a, b, x, v) = \frac{ab}{\left( \prod_{i=1}^{n} x_i^{1/v_i} \right)^{1/V_n}}.
\]

The equality \( \tilde{M}_{[t]}(a, b, x, v) = \tilde{M}_{[1]}(a, b, x, v) \) holds if and only if one of the cases (1) and (2) in Theorem 2 occurs. The equality \( a = \tilde{M}_{[t]}(a, b, x, v) \) occurs only in the case \( x = b \), while the equality \( \tilde{M}_{[t]}(a, b, x, v) = b \) occurs only in the case \( x = a \), where \( \tilde{x} = (1/V_n) \sum_{i=1}^{n} v_i x_i \). Especially, if \( v_i = w_i > 0, i = 1, \ldots, n, \sum_{i=1}^{n} w_i = 1 \) and if at least one of the \( x_k \) satisfies \( a < x_k < b \), then it is easy to see that \( a < \tilde{x} < b \) and that the case (2) in Theorem 2 is impossible. Therefore, all the inequalities in (5.2) are strict in this special case. In fact, (5.2) reduces to the Mercer’s result [3, Theorem 2.1]
where \( Q_t(a, b, x) = \tilde{M}_t(a, b, x, v) \).

**Example 2.** Choosing \( r = -1 \) and \( \sigma = 1 \) and applying (5.1) we obtain the following extension of classical HGA inequalities:

\[
H(\xi, p) \leq G(\xi, p) \leq A(\xi, p),
\]

where

\[
H(\xi, p) = \left( \frac{1}{P_m} \sum_{i=1}^{m} p_i \xi_i \right)^{-1}, \quad G(\xi, p) = \left( \prod_{i=1}^{m} \xi_i^{p_i} \right)^{1/P_m},
\]

\[
A(\xi, p) = \frac{1}{P_m} \sum_{i=1}^{m} p_i \xi_i
\]

are generalized harmonic, geometric and arithmetic mean of \( \xi \) with quasi-weights \( p \). The generalized HGA inequalities (5.3) hold under the same assumptions on \( \xi \) and \( p \) and the conditions for equality case are the same as for (5.1). However, the ‘sequence’ of the inequalities in (5.1) can be regarded as an interpolating ‘sequence’ for HGA inequalities (5.3).

As a special case of the HGA inequalities in (5.3), the following variant of HGA inequalities is obtained from (5.2):

\[
\left( a^{-1} + b^{-1} - \frac{1}{V_n} \sum_{i=1}^{n} v_i x_i \right)^{-1} \leq \frac{ab}{\left( \prod_{i=1}^{n} x_i^{v_i} \right)^{1/V_n}} \leq a + b - \frac{1}{V_n} \sum_{i=1}^{n} v_i x_i.
\]

However, the assumptions on \( x \) and \( v \) and the conditions for equality case for (5.4) are the same as those for (5.2).

**Remark 2.** In case \( v \) is positive, i.e., \( v_i > 0, i = 1, \ldots, n \), the monotonicity condition on \( x \) can be omitted and (5.2) and (5.4) will still be valid. The argument for that is the same as the one given in Remark 1.

**Example 3.** Let \( 0 < r \leq 1 \) and \( f : (0, 1) \rightarrow \mathbb{R}, g : \mathbb{R} \rightarrow \mathbb{R} \) and \( g \circ f : (0, 1) \rightarrow \mathbb{R} \) be defined by

\[
f(x) = \ln \frac{x}{1-x}, \quad g(x) = e^{-rx}, \quad (g \circ f)(x) = \left( \frac{1-x}{x} \right)^r.
\]

Then \( f \) is strictly concave and strictly increasing on \( I = (0, \frac{1}{2}] \), while \( g \) is strictly convex and strictly decreasing on \( J = \mathbb{R} \), so that the inequalities (3.3) can be applied to obtain the following inequalities:

\[
\left( \frac{1}{P_m} \sum_{i=1}^{m} p_i \xi_i \right)^{r/P_m} = \left( \frac{1}{P_m} \sum_{i=1}^{m} p_i \xi_i \right)^{r/P_m} \leq \left( \prod_{i=1}^{m} \left( 1 - \frac{\xi_i}{\xi_i} \right)^{p_i} \right)^{1/P_m} \leq \frac{1}{P_m} \sum_{i=1}^{m} p_i \left( \frac{1 - \xi_i}{\xi_i} \right)^r.
\]
Since $\phi(t) = t^{-1/r}$, $t > 0$, is strictly decreasing, the above inequalities can be rewritten in the equivalent form

$$
\left[ \frac{1}{P_m} \sum_{i=1}^{m} p_i \left( \frac{1 - \xi_i}{\xi_i} \right)^{r} \right]^{-1/r} \leq \frac{G(\xi, p)}{G'(\xi, p)} \leq \frac{A(\xi, p)}{A'(\xi, p)},
$$

(5.5)

where $G(\xi, p)$, $A(\xi, p)$ are defined as in Example 2 and

$$
G'(\xi, p) = \left[ \prod_{i=1}^{m} (1 - \xi_i)^{p_i} \right]^{1/P_m}, \quad A'(\xi, p) = 1 - A(\xi, p) = \frac{1}{P_m} \sum_{i=1}^{m} p_i (1 - \xi_i).
$$

In case $r = 1$ the left-hand side of the first inequality in (5.5) can be rewritten as $H(\xi, p)/(1 - H(\xi, p))$, where $H(\xi, p)$ is defined as in Example 2. (5.5) holds for any monotonic $m$-tuple $\xi$ in $(0, \frac{1}{2})^m$ and any real $m$-tuple $p$ with $p_i \neq 0$, $i = 1, \ldots, m$, satisfying (1.2). Also, the conditions for equality are the same as for (5.1). The second inequality in (5.5) is obviously a generalization of weighted Ky Fan’s inequality (see, for example, [5. pp. 25–28]).

References