JOURNAL OF APPROXIMATION THEORY 59, 316–320 (1989)

Sets of Best L_1-Approximants

DAVID A. LEGG AND DOUGLAS W. TOWNSEND

Department of Mathematical Sciences, Indiana University–Purdue University, Fort Wayne, Indiana 46805, U.S.A.

Communicated by R. Bojanic

Received October 5, 1987

1. INTRODUCTION

Let $X = L_1(\Omega, \mathscr{A}, \mu)$ and let $\mathscr{C} \subseteq X$ be an L_1-closed, convex subset. We say $g \in \mathscr{C}$ is a best L_1-approximant to $f \in X$ if $\|g - f\|_1 = \inf \|h - f\|_1$, $h \in \mathscr{C}$. For many important choices of $\mathscr{C}$, such as $\mathscr{C} = L_1(\Omega, \mathscr{B}, \mu)$, where $\mathscr{B}$ is a sub-σ-algebra of $\mathscr{A}$, or $\mathscr{C}$ the set of nondecreasing functions on $\Omega = [0, 1]$, best L_1-approximants exist to all $f \in X$. It is rare, however, that best L_1-approximants are uniquely determined. Denote by $\mu_1(f \mid \mathscr{C})$ the set of all best L_1-approximants to f by elements of $\mathscr{C}$. In this paper we study the question: If f_1 and f_2 are "close," are the sets $\mu_1(f_1 \mid \mathscr{C})$ and $\mu_1(f_2 \mid \mathscr{C})$ "close" in Hausdorff metric?

2. APPROXIMATION BY ELEMENTS OF $L_1(\Omega, \mathscr{B}, \mu)$

Let $\mathscr{B}$ be a sub-σ-algebra of $\mathscr{A}$, and let $\mathscr{C} = L_1(\Omega, \mathscr{B}, \mu)$. Shintani and Ando [4, Theorem 2] proved the existence of best L_1-approximations to $f \in X = L_1(\Omega, \mathscr{A}, \mu)$ by elements of $\mathscr{C}$. Furthermore, they characterized the set $\mu_1(f \mid \mathscr{C})$ in the following way: there exist functions $\bar{f}$ and $\underline{f}$ in $\mathscr{C}$ such that $g \in \mu_1(f \mid \mathscr{C})$ if and only if $g \in \mathscr{C}$ and $\underline{f} \leqslant g \leqslant \bar{f}$ on Ω. In particular, $\bar{f} = \sup\{g: g \in \mu_1(f \mid \mathscr{C})\}$ and $\underline{f} = \inf\{g: g \in \mu_1(f \mid \mathscr{C})\}$.

If A is a subset of a metric space M with distance d, define $\operatorname{dist}(x, A) = \inf\{d(x, a): a \in A\}$. If A and B are subsets of M, define the Hausdorff distance between them by $\operatorname{dist}(A, B) = \max\{\sup_{a \in A} \operatorname{dist}(a, B), \sup_{b \in B} \operatorname{dist}(b, A)\}$.

The most natural question at this point is: If $f_n \to f$ in L_1 as $n \to \infty$, does $\operatorname{dist}(\mu_1(f_n \mid \mathscr{C}), \mu_1(f \mid \mathscr{C})) \to 0$ as $n \to \infty$, where $d(g, h) = \|g - h\|_1$? The following example shows in general the answer is no.

0021-9045/89 $3.00

EXAMPLE 2.1. Let $\Omega=[0,1]$ with Lebesgue measure and $\mathscr{B}=\{\phi,\Omega\}$. Then g is $\mathscr{B}$-measurable if and only if g is constant. Define $f(x)$ by $f(x)=1$ on $[0,\frac{1}{2})$ and $f(x)=0$ on $[\frac{1}{2},1]$. For $n\geqslant 3$, define $f_n(x)$ by $f_n(x)=1$ on $[0,\frac{1}{2}+1/n)$ and $f_n(x)=0$ on $[\frac{1}{2}+1/n,1]$. Then clearly $f_n\to f$ in L_1, and each f_n has a unique best L_1-approximant defined by $g_n(x)=1$ on $[0,1]$. But $f(x)$ has many best L_1-approximants, defined by $g_c(x)=c$ on $[0,1]$, where $0\leqslant c\leqslant 1$. In particular, $g_0\in\mu_1(f\,|\,\mathscr{C})$ and $\operatorname{dist}(g_0,\mu_1(f_n\,|\,\mathscr{C}))=1$ for all $n\geqslant 3$. Hence $\operatorname{dist}(\mu_1(f\,|\,\mathscr{C}),\mu_1(f_n\,|\,\mathscr{C}))\geqslant 1$ for all $n\geqslant 3$. (Clearly this is an equality.)

We can, however, prove the following semi-continuity result.

THEOREM 2.2. *Let $f_n\to f$ in L_1 as $n\to\infty$ and let $\varepsilon>0$. There is an $N>0$ such that* $\operatorname{dist}(g,\mu_1(f\,|\,\mathscr{C}))<\varepsilon$ *for all $g\in\mu_1(f_n\,|\,\mathscr{C})$ with $n\geqslant N$.*

Proof. By Shintani and Ando [4, Corollary 5], we have $\bar{f}_n\vee\bar{f}\to\bar{f}$ in L_1 as $n\to\infty$ and $\underline{f}_n\wedge\underline{f}\to\underline{f}$ in L_1 as $n\to\infty$. Choose N such that $\|\bar{f}_n\vee\bar{f}-\bar{f}\|_1<\varepsilon/2$ and $\|\underline{f}_n\wedge\underline{f}-\underline{f}\|_1<\varepsilon/2$ for $n\geqslant N$. Now if $n\geqslant N$ and $g\in\mu_1(f_n\,|\,\mathscr{C})$, define $g^*=\underline{f}\vee g\wedge\bar{f}$. Then $g^*\in\mu_1(f\,|\,\mathscr{C})$. Since $\underline{f}_n\leqslant g\leqslant\bar{f}_n$, it follows that $g^*=g$ except possibly on the sets $A=\{\underline{f}_n<\underline{f}\}$ and $B=\{\bar{f}_n>\bar{f}\}$. Hence

$$\|g^*-g\|_1=\int_{A\cup B}|g^*-g|\,d\mu\leqslant\int_A|\underline{f}_n-\underline{f}|\,d\mu+\int_B|\bar{f}_n-\bar{f}|\,d\mu$$

$$=\int_A|\underline{f}_n\wedge\underline{f}-\underline{f}|\,d\mu+\int_B|\bar{f}_n\vee\bar{f}-\bar{f}|\,d\mu<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon,$$

and the theorem is proved.

If we use the uniform metric defined by $d(g,h)=\|g-h\|_\infty$, we may obtain the full continuity result.

THEOREM 2.3. *Let $f_n\to f$ uniformly as $n\to\infty$. Then* $\operatorname{dist}(\mu_1(f_n\,|\,\mathscr{C}),\mu_1(f\,|\,\mathscr{C}))\to 0$ *as $n\to\infty$.*

Proof. By Landers and Rogge [3, Theorem 18] the mappings $f\to\bar{f}$ and $f\to\underline{f}$ are monotone, which implies $\bar{f}_n\to\bar{f}$ and $\underline{f}_n\to\underline{f}$ uniformly as $n\to\infty$. If $\varepsilon>0$, choose N such that $|\bar{f}_n-\bar{f}|<\varepsilon$ and $|\underline{f}_n-\underline{f}|<\varepsilon$ on Ω for $n\geqslant N$. Then if $g\in\mu_1(f\,|\,\mathscr{C})$, define $g^*=\underline{f}\vee g\wedge\bar{f}$. Then $g^*\in\mu_1(f_n\,|\,\mathscr{C})$ and $|g^*-g|<\varepsilon$ on Ω for $n\geqslant N$. If $g\in\mu_1(f_n\,|\,\mathscr{C})$, define $g^*=\underline{f}\vee g\wedge\bar{f}$. Then $g^*\in\mu_1(f\,|\,\mathscr{C})$ and $|g^*-g|<\varepsilon$ on Ω for $n\geqslant N$. Hence $\operatorname{dist}(\mu_1(f_n\,|\,\mathscr{C}),\mu_1(f\,|\,\mathscr{C}))<\varepsilon$ for $n\geqslant N$.

3. Approximation by Nondecreasing Functions

Let $\mathcal{N}$ be the set of nondecreasing functions on $[0, 1]$, and suppose $f \in L_1[0, 1]$. The set $\mu_1(f \mid \mathcal{N})$ of all best L_1-approximations to f by elements of $\mathcal{N}$ is characterized in [1, 2] as follows. Define $\bar{f}$ and $\underline{f}$ by $\bar{f}(x) = \sup\{q(x): q \in \mu_1(f \mid \mathcal{N})\}$ and $\underline{f}(x) = \inf\{q(x): q \in \mu_1(f \mid \mathcal{N})\}$. It is shown in [3, Theorem 14] that $\bar{f}$ and $\underline{f}$ are in $\mu_1(f \mid \mathcal{N})$. Let $U = \bigcup U_i$, where U_i is a maximal open interval on which both $\bar{f}$ and $\underline{f}$ are constant and $\bar{f} \neq \underline{f}$. Define $h_f: U \to R$ by

$$h_f(x) = \begin{cases} 1 & \text{if } f(x) \geqslant \bar{f}(x) \\ -1 & \text{if } f(x) \leqslant \underline{f}(x) \\ 0 & \text{if } \underline{f}(x) < f(x) < \bar{f}(x), \end{cases}$$

and, if $x \in U_i = (u_i, v_i)$, define k_f by

$$k_f(x) = \int_{u_i}^{x} h_f(t)\, dt.$$

Then for any $q \in \mathcal{N}$, we have $q \in \mu_1(f \mid \mathcal{N})$ if and only if

(i) $\underline{f} \leqslant g \leqslant \bar{f}$ on $[0, 1]$, and

(ii) q is constant on components of $\{[k_f \neq 0] \cap U_i\}$, $i \geqslant 1$.

We use the notation $\mu(A; [a, b])$ to denote $\mu(A)/(b-a)$, the relative measure of A in $[a, b]$. The following lemma was proved in [2] and will be used later in this paper.

Lemma 3.1. *If $q \in \mu_1(f \mid \mathcal{N})$ and q is not constant at $s \in [0, 1]$, then*

(1) $\mu([f \geqslant q]; [s, t]) \geqslant \frac{1}{2}$ *for* $s < t \leqslant 1$, *and*

(2) $\mu([f \leqslant q]; [t, s]) \geqslant \frac{1}{2}$ *for* $0 \leqslant t < s$.

The main result of this section is an easy consequence of the following lemma.

Lemma 3.2. *Suppose $\varepsilon > 0$ and $f, g \in L_1[0, 1]$. If $|f(x) - g(x)| < \varepsilon$ for all $0 \leqslant x \leqslant 1$, then for any $f^* \in \mu_1(f \mid \mathcal{N})$ there is a $g^* \in \mu_1(g \mid \mathcal{N})$ so that $|f^*(x) - g^*(x)| < 8\varepsilon$ for all $0 \leqslant x \leqslant 1$.*

Proof. We have by [3, Theorem 18] that $|\bar{g}(x) - \bar{f}(x)| < \varepsilon$ and $|\underline{g}(x) - \underline{f}(x)| < \varepsilon$ for all $0 \leqslant x \leqslant 1$.

Let U and U_i, for $i \geqslant 1$, be defined as above for f, and let V and V_i, for $i \geqslant 1$, be defined as above for g. Let $U(\varepsilon) = \bigcup_{i \in I(\varepsilon)} U_i$, where $I(\varepsilon)$ is the set of indices such that $\bar{f}(x) - \underline{f}(x) > 6\varepsilon$ for $x \in U_i$. Since $\bar{f}$ is continuous from the right and $\underline{f}$ is continuous from the left, it follows that $\bar{f}(x)-\underline{f}(x)>6\varepsilon$ for $x\in\bar{U}_i$. For $f^*\in\mu_1(f\mid\mathcal{N})$ we define g^* as follows: if $x\in U(\varepsilon)$, then $g^*(x)=\underline{g}(x)\vee f^*(x)\wedge\bar{g}(x)$; if $x<y$ for all $y\in U(\varepsilon)$, then $g^*(x)=\underline{g}(x)$; and otherwise,

$$g^*(x)=(\sup_{\substack{y<x\\ y\in U(\varepsilon)}} g^*(y))\vee\underline{g}(x).$$

It is clear from the definition of g^* that $\underline{g}(x)\leqslant g^*(x)\leqslant\bar{g}(x)$ for all $0\leqslant x\leqslant 1$. Thus $g^*(x)$ will be in $\mu_1(g\mid\mathcal{N})$ provided

$$g^* \text{ is constant on components of } \{V_i\cap[k_g\neq 0]\},\ i\geqslant 1. \tag{1}$$

Suppose (1) is not true. Then there is an $x_0\in V_j$ for some j so that $k_g(x_0)\neq 0$ and g^* is not constant at x_0. Since g^* is constant on maximal components of the complement of $\overline{U(\varepsilon)}$, where $\overline{U(\varepsilon)}$ is equal to either $U(\varepsilon)$ or $U(\varepsilon)\cup\{1\}$, we have $x_0\in U(\varepsilon)$ and, from the definition of g^*, f^* is not constant at x_0. It follows from Corollary 2 and Theorem 5 of [2] that either f^* has a jump discontinuity at x_0 or $\bar{f}(x)=\underline{f}(x)=f(x)$ almost everywhere in an interval containing x_0. Since $\bar{f}(x)\neq\underline{f}(x)$ for all $x\in U(\varepsilon)$, we have that f^*, and hence g^*, has a jump discontinuity at x_0. Clearly since $\bar{f}(x_0)-\underline{f}(x_0)>6\varepsilon$, we have that $\bar{g}(x_0)-\underline{g}(x_0)>4\varepsilon$ and hence, $\bar{g}(x)-\underline{g}(x)>4\varepsilon$ for all $x\in V_j$. It follows that $\bar{f}(x)-\underline{f}(x)>2\varepsilon$ for all $x\in V_j$, and thus $\mu(V_j-U)=0$. Also, it is shown in [2] that

$$\mu([\underline{f}<f<\bar{f}]\cap V_j)=0 \qquad \text{and} \qquad \mu([\underline{g}<g<\bar{g}]\cap V_j)=0. \tag{2}$$

We now show that for almost all $x\in V_j\cap U$ we have

$$h_g(x)=h_f(x). \tag{3}$$

If $h_g(x)=1$, then $f(x)>g(x)-\varepsilon\geqslant\bar{g}(x)-\varepsilon\geqslant\bar{f}(x)-2\varepsilon>\underline{f}(x)$. In view of (2) we have $f(x)\geqslant\bar{f}(x)$ for almost all such x, implying $h_f(x)=1$. On the other hand, if $h_f(x)=1$, then $g(x)>f(x)-\varepsilon\geqslant\bar{f}(x)-\varepsilon>\bar{g}(x)-2\varepsilon>\underline{g}(x)$. In view of (2) we have $g(x)\geqslant\bar{g}(x)$ for almost all such x, implying $h_g(x)=1$. The proof that $h_f(x)=-1$ if and only if $h_g(x)=-1$ for almost all x for which $h_f(x)=-1$ or $h_g(x)=-1$ is similar, and (3) follows.

Now if $V_j=(w,z)$ then $G=\frac{1}{2}(\bar{g}+\underline{g})$ is not constant at w. From Lemma 3.1 we have $\mu([g\geqslant G];[w,x_0])\geqslant\frac{1}{2}$, and in view of (2), $\mu([g\geqslant\bar{g}];[w,x_0])\geqslant\frac{1}{2}$, implying

$$\int_w^{x_0} h_g(t)\,dt\geqslant 0. \tag{4}$$

Also since f^* is not constant at x_0, we have from Lemma 3.1 that $\mu([f \leqslant f^*]; [w, x_0]) \geqslant \frac{1}{2}$. In view of (2), $\mu([f \leqslant \underline{f}]; [w, x_0]) \geqslant \frac{1}{2}$, implying $\int_{[w, x_0] \cap U} h_f(t)\, dt \leqslant 0$. It follows from (3) and the fact that $(V_j - U) = 0$ that

$$\int_w^{x_0} h_g(t)\, dt \leqslant 0. \tag{5}$$

From (4) and (5) we see that $\int_w^{x_0} h_g(t)\, dt = 0$, implying that $k_g(x_0) = 0$, a contradiction. Thus (1) is proved and $g^* \in \mu_1(g \mid \mathcal{N})$.

We now show that $|g^* - f^*| < 8\varepsilon$ for all $x \in [0, 1]$. We have that $|\bar{f}(x) - \bar{g}(x)| < \varepsilon$ and $|\underline{f}(x) - \underline{g}(x)| < \varepsilon$ for all $x \in [0, 1]$. If $x \in U(\varepsilon)$, then $g^*(x)$ equals $f^*(x)$, $\bar{g}(x)$, or $\underline{g}(x)$. If $g^*(x) = \bar{g}(x)$ then from the definition of g^* we have $\bar{g}(x) \leqslant f^*(x) \leqslant \bar{f}(x)$. Thus $|g^*(x) - f^*(x)| = |\bar{g}(x) - f^*(x)| \leqslant |\bar{g}(x) - \bar{f}(x)| < \varepsilon$. If $g^*(x) = \underline{g}(x)$, then again from the definition of g^* we have $\underline{f}(x) \leqslant f^*(x) \leqslant \underline{g}(x)$. Thus $|g^*(x) - f^*(x)| = |\underline{g}(x) - f^*(x)| \leqslant |\underline{g}(x) - \underline{f}(x)| < \varepsilon$. On the other hand, if $x \in U(\varepsilon)$, then $|\bar{f}(x) - \underline{f}(x)| \leqslant 6\varepsilon$. Thus $|f^*(x) - g^*(x)| \leqslant \max(\bar{f}(x), \bar{g}(x)) - \min(\underline{f}(x), \underline{g}(x)) \leqslant 8\varepsilon$, and the lemma is proved.

The following theorem is an easy consequence of Lemma 3.2.

THEOREM 3.3. *For any* $\varepsilon > 0$, *if* $f, g \in L_1\ [0, 1]$ *satisfy* $|f(x) - g(x)| < \varepsilon/8$ *for all* $0 \leqslant x \leqslant 1$, *then* $\operatorname{dist}(\mu_1(f \mid \mathcal{N}), \mu_1(g \mid \mathcal{N})) < \varepsilon$ *in the uniform metric.*

REFERENCES

1. R. HUOTARI, D. LEGG, A. MEYEROWITZ, AND D. TOWNSEND, The natural best L_1-approximation by nondecreasing functions, *J. Approx. Theory* **52** (1988), 132–140.
2. R. HUOTARI, A. MEYEROWITZ, AND M. SHEARD, Best monotone approximants in L_1 [0, 1], *J. Approx. Theory* **47** (1986), 85–91.
3. D. LANDERS AND L. ROGGE, Best approximants in L_φ-spaces, *Z. Wahrsch. Verw. Gebiete* **51** (1980), 215–237.
4. T. SHINTANI AND T. ANDO, Best approximants in L_1-space, *Z. Wahrsch. Verw. Gebiete* **33** (1975), 33–39.