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Existence of global attractors for a class of periodic Kolmogorov systems ☆

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Abstract

In this paper we use a continuation argument to prove the existence of global attractors for a class of periodic Kolmogorov systems.

1. Introduction

In this paper we consider the following periodic Kolmogorov system

$$x_i' = x_i f_i(t, x_i, \dots, x_n), \quad 1 \leqslant i \leqslant n, \tag{1.1}$$

where $f = (f_1, \dots, f_n) : \mathbb{R} \times \mathbb{R}_+^n$ is a continuous function such that:

(H$_1$) f is T-periodic in t. That is, $f(t+T, x) = f(t, x)$.
(H$_2$) The partial derivatives $\frac{\partial f_i}{\partial x_j}$ are defined and continuous in $\mathbb{R} \times \mathbb{R}_+^n$.
(H$_3$) There exist positive constants $c_1, \dots, c_n, m$, such that

$$c_i \frac{\partial f_i}{\partial x_i}(t, x) + \sum_{j \in J_i} c_j \left| \frac{\partial f_j}{\partial x_i}(t, x) \right| \leqslant -m, \quad x > 0,\ t \in \mathbb{R},\ 1 \leqslant i \leqslant n, \tag{1.2}$$

where $J_i := \{j \in \{1, \dots, n\}:\ j \neq i\}$. As usual, $\mathbb{R}_+^n$ denotes the nonnegative cone $\{x \in \mathbb{R}^n:\ x \geqslant 0\}$.

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Example. Assume that $\frac{\partial f_i}{\partial x_j}$ is constant for all i, j. In [3] we find necessary and sufficient conditions for which f satisfies the assumption (1.2).

In Theorem 1.5 of [2], it was proved that if (1.1) has a positive solution v which is defined and bounded on (t_0, ∞) for some $t_0 = t_0(v) \in \mathbb{R}$, then the system has a global attractor. In this paper we shall show that the existence of a such v is implied by (H_1)–(H_3) when (1.1) is a cooperative system or a Lotka–Volterra system. More precisely, we shall prove the following two results.

Theorem 1.1. *In addition to* (1.2) *suppose that*

$$\frac{\partial f_i}{\partial x_j} > 0 \quad \text{if } i \neq j.$$

Then, each positive solution of (1.1) *is defined on* $(t_0, +\infty)$, *for some* $t_0 \in \mathbb{R}$. *Moreover,* (1.1) *has a* T*-periodic solution* U *such that*

$$x(t) - U(t) \to 0 \quad \text{as } t \to +\infty$$

for any positive solution of the system.

Theorem 1.2. *The conclusions of Theorem* 1.1 *remain true if* (1.2) *holds and* $f_i(t, x)$ *has the form*

$$f_i(t, x) = a_i(t) - \sum_{j=1}^{n} b_{ij}(t) x_j$$

for some continuous T*-periodic functions* $a_i, b_{ij} : \mathbb{R} \to \mathbb{R}$.

In Theorem 3.1 of [3] it was “shown” that the existence of v, for Lotka–Volterra systems is a consequence of assumptions (H_1)–(H_3), however the proof of this result contains a mistake in the second line of p. 256.

The proof of our main result (Theorem 2.4 below) will be based on a continuation result applied to the following one-parameter family of ordinary differential equations:

$$x_i' = x_i\big[(1-\lambda) f_i^*(x) + \lambda f_i(t, x)\big], \quad 1 \leqslant i \leqslant n,\ \lambda \in [0, 1],$$

where $f_i^*(x) := c_i^{-1}[1 + x_1 + \cdots + x_n - (n+1)x_i]$.

2. The results

The following proposition can be obtained as a consequence of the main result in [1], but here we give a direct and very simple proof.

Proposition 2.1. *Let* $A(t) = (a_{ij}(t))$ *be a continuous* T*-periodic* $n \times n$ *matrix and suppose that there exist positive constants* $c_1, \ldots, c_n$ *such that*

$$c_i a_{ii}(t) + \sum_{j \in J_i} c_j \big|a_{ij}(t)\big| < 0, \quad t \in \mathbb{R},\ 1 \leqslant i \leqslant n.$$

If $\Phi(t)$ denotes the fundamental matrix of the system

$$x' = A(t)x \tag{2.1}$$

with $\Phi(0) =$ Identity, then the spectral radius of $\Phi(T)$ is less than one.

Proof. Let $w = (w_1, \dots, w_n)$ be a solution of (2.1) and define

$$r(t) = \sum_{j=1}^{n} c_j \left|w_j(t)\right|.$$

As in Theorem 1.1 of [2], there exists a countable subset N of $\mathbb{R}$ such that r is differentiable on $\mathbb{R}\backslash N$ and

$$r'(t) = \sum_{j=1}^{n} c_j \operatorname{sign}\bigl(w_j(t)\bigr) w_j'(t) = \sum_{j=1}^{n} c_j \operatorname{sign}\bigl(w_j(t)\bigr) \sum_{i=1}^{n} a_{ji}(t) w_i(t) \quad \text{if } t \notin N,$$

where $\operatorname{sign}(x)$ denotes the sign of the number real x. From this,

$$\begin{aligned} r'(t) &= \sum_{j=1}^{n} c_j a_{jj} \left|w_j(t)\right| + \sum_{i=1}^{n} \Biggl[\sum_{j\in J_i} c_j \operatorname{sign}\bigl(w_j(t)\bigr) a_{ji}(t) \Biggr] w_i(t) \\ &\leqslant \sum_{i=1}^{n} c_i a_{ii}(t) \left|w_i(t)\right| + \sum_{i=1}^{n} \Biggl[\sum_{j\in J_i} c_j \left|a_{ji}(t)\right| \Biggr] \left|w_i(t)\right|. \end{aligned}$$

Define $m_i := -\sup\{c_i a_{ii}(t) + \sum_{j\in J_i} c_j |a_{ji}(t)|\colon\ t \in \mathbb{R}\}$, $m := \min\{m_1, \dots, m_n\}$ and $c = \max\{c_1, \dots, c_n\}$. Then,

$$r'(t) \leqslant -\sum_{i=1}^{n} m_i \left|w_i(t)\right| \leqslant -m \sum_{i=1}^{n} \left|w_i(t)\right| \leqslant -mc^{-1} r(t),$$

and hence, $r(T) \leqslant \exp(-mc^{-1}T) r(0)$. That is, if we define a norm $\|\cdot\|_c$ in $\mathbb{R}^n$ by $\|x\|_c = c_i|x_i| + \dots + c_n|x_n|$, we obtain $\|\Phi(T)x\|_c \leqslant \exp(-mc^{-1}T)\|x\|_c$ and the proof is complete. □

Corollary 2.2. *Assume* (1.2) *holds and that $U = (U_1, \dots, U_n)$ is a positive T-periodic solution of* (1.1). *Then $\pi'(U(0)) - I$ is invertible, where π denotes the Poincaré map of* (1.1) *and I is the identity map.*

Proof. By the definition of π, we have $\pi'(U(0)) = \Phi(T)$, where $\Phi(t)$ is the fundamental matrix of the system

$$y_i' = f_i\bigl(t, U(t)\bigr) y_i + U_i(t) \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}\bigl(t, U(t)\bigr) y_j \tag{2.2}$$

with $\Phi(0) = I$. By the change of variables $z_i = y_i/U_i$, system (2.2) becomes

$$z_i' = \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}\bigl(t, U(t)\bigr)U_j(t)z_j \tag{2.3}$$

which satisfies the assumptions in Proposition 2.1.

Let $\Psi(t)$ be the fundamental matrix of (2.3) with $\Psi(0) = I$. By Proposition 2.1, the spectral radius of $\Psi(T)$ is less than one and the proof follows easily since $\Phi(T)$ and $\Psi(T)$ are similar linear maps. $\square$

The proof of our main result requires the following consequence of the Implicit Function Theorem.

Lemma 2.3. *Let $H : D \to \mathbb{R}^n$ be a continuously differentiable function defined in an open subset D of $[0,1] \times \mathbb{R}^n$ and suppose that*:

(i) $H^{-1}(0)$ *is a compact set.*
(ii) $H^{-1}(0) \cap (\{0\} \times \mathbb{R}^n)$ *is a single set.*
(iii) *The partial derivative $H_x(\lambda, x)$ is invertible for each $(\lambda, x) \in H^{-1}(0)$. Then $H^{-1}(0) \cap (\{1\} \times \mathbb{R}^n)$ is a single set.*

Proof. For each $\lambda \in [0,1]$, let us write $P_\lambda = H^{-1}(0) \cap (\{\lambda\} \times \mathbb{R}^n)$ and note that, by the Implicit Function Theorem, there exists $\mu \in (0,1]$ such that $P_\lambda \neq \emptyset$ for any $\lambda \in [0, \mu)$.

Claim 1. There exists $\varepsilon \in (0,1]$ such that P_λ is a single set for all $\lambda \in [0, \varepsilon)$. To show this, assume on the contrary the existence of two sequences $((\lambda_k, x^k))$, $((\lambda_k, y^k))$ in $H^{-1}(0)$ such that $\lambda_k \to 0$ and $x^k \neq y^k$. Using our assumptions (i) and (ii), it is easy to show that $x^k \to x^*$ and $y^k \to x^*$, where $\{(0, x^*)\} = P_0$.

Let w be a limit point of the sequence $(w^k = \|x^k - y^k\|^{-1}(x^k - y^k))$. Then,

$$\begin{aligned} 0 &= \bigl[H\bigl(\lambda_k, x^k\bigr) - H\bigl(\lambda_k, y^k\bigr)\bigr]\bigl\|x^k - y^k\bigr\|^{-1} \\ &= \left(\int_0^1 H_x\bigl(\lambda_k, (1-s)y^k + sx^k\bigr)\,ds\right)(w^k) \to H_x(0, x^*)w, \end{aligned}$$

which contradicts our assumption (iii) and proves the claim.

Let ε be given by the above claim. Using (iii), it is easy to show that P_ε is a nonempty set. In fact,

Claim 2. P_ε is a single set. To show this assume by contradiction the existence of $(\varepsilon, x^i) \in P_\varepsilon$; $i = 0, 1$; such that $x^0 \neq x^1$. By (iii) and the Implicit Function Theorem, there exist continuous functions $\psi_i : [\delta, \varepsilon] \to \mathbb{R}^n$, for some $\delta \in (0, \varepsilon)$, such that $\psi_i(\varepsilon) = x^i$ and $H(\lambda, \psi_i(\lambda)) = 0$ in $[\delta, \varepsilon]$. This contradicts Claim 1 and the proof of Claim 2 is complete.

The proof follows now from Claims 1, 2 and a well-known continuation argument. $\square$

Theorem 2.4. *In addition to* (1.2) *suppose that*

$$\int_0^T f_i(t, 0)\,dt > 0 \quad \forall i \tag{2.4}$$

and that

$$\frac{\partial f_i}{\partial x_j} > 0 \quad \text{if } i \neq j. \tag{2.5}$$

Then (1.1) *has a positive T-periodic solution.*

Proof. Note first that condition (1.2) becomes

$$\sum_{j=1}^{n} c_j \frac{\partial f_j}{\partial x_i} \leqslant -m.$$

Given $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, we define $S(x) = x_1 + \dots + x_n$ and

$$f_i^*(x) = c_i^{-1}\big[1 + S(x) - (n+1)x_i\big], \quad 1 \leqslant i \leqslant n.$$

Obviously, $f^* := (f_1^*, \dots, f_n^*)$ satisfies assumptions (1.2) and (2.5). In fact,

$$\sum_{j=1}^{n} c_j \frac{\partial f_j^*}{\partial x_i} \equiv -1 \quad \forall i.$$

Note also that $f^*(x^*) = 0$, where $x^* = (x_1^*, \dots, x_n^*)$ and $x_i^* = 1$ for all i.

For each $\lambda \in [0,1]$, let us define $f^\lambda = (f_1^\lambda, \dots, f_n^\lambda) : \mathbb{R} \times \mathbb{R}^n_{++} \to \mathbb{R}^n$ by $f_i^\lambda(t,x) = (1-\lambda) f_i^*(x) + \lambda f_i(t,x)$, where $\mathbb{R}^n_{++} := \{x \in \mathbb{R}^n\colon\ x > 0\}$, and note that

$$\sum_{j=1}^{n} c_j \frac{\partial f_j^\lambda}{\partial x_i}(t,x) \leqslant -\overline{m} := -\min\{1, m\}, \quad 1 \leqslant i \leqslant n. \tag{2.6}$$

Let $\pi_\lambda : D_\lambda \to \mathbb{R}^n$ be the Poincaré map of the system

$$x_i' = x_i f_i^\lambda(t, x_1, \dots, x_n), \quad 1 \leqslant i \leqslant n. \tag{2.7}$$

It is well known that D_λ is a (possibly empty) open subset of $\mathbb{R}^n$ and by the continuous dependence in the parameters, $D := \{(\lambda, x) \in [0,1] \times \mathbb{R}^n\colon\ x \in D_\lambda\}$ is an open subset of $[0,1] \times \mathbb{R}^n$ and $H : D \to \mathbb{R}^n$; $H(\lambda, x) := \pi_\lambda(x) - x$; is a continuously differentiable function. Note that D is nonempty since $(0, x^*) \in D$. Notice also that by (2.6) and Corollary 2.2, the partial derivative $H_x(\lambda, x)$ is invertible for all $(\lambda, x) \in H^{-1}(0)$. Thus, H satisfies assumption (iii) of Lemma 2.3.

On the other hand, $H(0, x^*) = 0$ and by Corollary 1.7 of [2], $H^{-1}(0) \cap (\{0\} \times \mathbb{R}^n) = \{(0, x^*)\}$. Finally, let $((\lambda_k, x^k))$ be a sequence in $H^{-1}(0)$ and let $u^k = (u_1^k, \dots, u_n^k)$ be the positive T-periodic solution of the equation

$$x_i' = x_i\big[(1-\lambda_k) f_i^*(x) + \lambda_k f_i(t,x)\big], \quad 1 \leqslant i \leqslant n,$$

determined by the condition $u^k(0) = x^k$.

Claim. The sequence (u^k) is uniformly bounded. That is, there exists $M > 0$ such that $\|u^k(t)\| \leqslant M$ for all $t \in \mathbb{R}$ and $k \in \mathbb{N}$. To show this let us define

$$W_k(t) = \sum_{j=1}^{n} c_j \ln\big(u_j^k(t)\big) \quad \text{and} \quad \phi_k(t,x) = \sum_{j=1}^{n} c_j f_j^{\lambda_k}(t,x).$$

It is easy to show that

$$\frac{\partial \phi_k}{\partial x_i}(t,x) \leqslant -\overline{m} \quad \forall i,k,t,x,$$

and hence there exists a constant $K > 0$ such that

$$\phi_k(t,x) \leqslant \phi_k(t,0) - \overline{m}S(x) \leqslant K - \overline{m}S(x).$$

On the other hand, $W_k'(t) = \phi_k(t, u^k(t)) \leqslant K - \overline{m}S(u^k(t)) \leqslant K - \overline{m}d^{-1}W_k(t)$, where $d := \max\{c_1,\dots,c_n\}$ and thus, $W_k(t) \leqslant dK/\overline{m}$ $\forall t \in \mathbb{R}$; $k \in \mathbb{N}$, since W_k is T-periodic for all k. That is, there exists a constant $M > 0$ such that

$$u_1^k(t)^{c_1}\cdots u_n^k(t)^{c_n} \leqslant M \quad \forall t \in \mathbb{R},\ k \in \mathbb{N}. \tag{2.8}$$

By (2.6) and (2.5),

$$\frac{\partial f_i^\lambda}{\partial x_i} \leqslant -\frac{\overline{m}}{c_i} < 0,$$

and by (2.4),

$$\int_0^T f_i^\lambda(t,0)\,dt > 0 \quad \forall \lambda \in [0,1],\ 1 \leqslant i \leqslant n.$$

From this and Theorems 2.3 and 2.7 of [4], the logistic equation $x' = x f_i^\lambda(t, xe_i)$ has a positive T-periodic solution $U_i(t,\lambda)$ which is jointly continuous in $(t,\lambda) \in \mathbb{R} \times [0,1]$. (Here and henceforth, $e_1,\dots,e_n$ denotes the canonical vector basis of $\mathbb{R}^n$.) In particular, there exists a constant $\alpha > 0$ such that $U_i(t,\lambda) > \alpha$ $\forall i,t,\lambda$.

On the other hand, $(u_i^k)'(t) > u_i^k(t) f_i^{\lambda_k}(t, u_i^k(t)e_i)$ and hence $u_i^k(t) > U_i(t,\lambda_k) > \alpha$ $\forall i,k,t$. See proof of Proposition 2.1 of [4]. From this and (2.8), $u_i^k(t)^{c_i} \leqslant M\alpha^{c_i - S(c)}$ and the proof of the claim is complete.

By the above claim and (2.7) we conclude that the sequence of derivatives $((u^k)')$ is uniformly bounded, and by Ascoli's theorem we can suppose, without loss of generality that (u^k) converges uniformly to a T-periodic continuous function $v = (v_1,\dots,v_n): \mathbb{R} \to \mathbb{R}^n$. On the other hand, we can also assume that (λ_k) converges to a point $\mu \in [0,1]$ and now it is easy to show that v is a solution of the system

$$x_i' = x_i f_i^\mu(t,x), \quad 1 \leqslant i \leqslant n.$$

Note also that $v_i(t) \geqslant \alpha$ $\forall t \in \mathbb{R}$; $1 \leqslant i \leqslant n$.

Finally, $x^k = u^k(0) \to v(0)$ and hence, $H^{-1}(0)$ is a compact subset of $[0,1] \times \mathbb{R}^n$. The proof follows now from Lemma 2.3. □

Proof of Theorem 1.1. Let us fix $p > 0$ in $\mathbb{R}^n$ such that $p + f(t,0) > 0$ for all $t \in \mathbb{R}$, and define $g(t,x) = p + f(t,x)$. By Theorem 2.4 above and Theorem 1.5 of [2], it follows that each positive solution of the system

$$x_i' = x_i g_i(t,x), \quad 1 \leqslant i \leqslant n, \tag{2.9}$$

is defined and bounded on a terminal interval of $\mathbb{R}$.

Let u be a solution of (1.1) such that $u(0) > 0$ and let v be the solution of (2.9) determined by the initial condition $v(0) = u(0)$. Since $g(t,x) > f(t,x)$, it follows from Kamke's theorem that $u(t) \leqslant v(t)$ for all $t \geqslant 0$ in the domain of u. From this u is defined and bounded on a terminal interval of $\mathbb{R}$ and the proof follows from Theorem 1.5 of [2]. □

Proof of Theorem 1.2. Since

$$c_i b_{ii}(t) + \sum_{j \in J_i} c_j \left|b_{ji}(t)\right| \leqslant -m, \quad t \in \mathbb{R},\ 1 \leqslant i \leqslant n,$$

there exists $\varepsilon > 0$ such that

$$c_i b_{ii}(t) + \sum_{j \in J_i} c_j \left[\varepsilon + \left|b_{ji}(t)\right|\right] \leqslant -m/2, \quad t \in \mathbb{R},\ 1 \leqslant i \leqslant n.$$

Define $\beta_{ii} = b_{ii}$, $\beta_{ij}(t) = -|b_{ij}(t)| - \varepsilon$, and

$$g_i(t,x) = a_i(t) - \sum_{j=1}^{n} \beta_{ij}(t) x_j, \quad 1 \leqslant i \leqslant n,$$

then g satisfies the assumptions in Theorem 1.1 and $f(t,x) \leqslant g(t,x)$. The proof follows now as in Theorem 1.1. □

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