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Some results on Gaussian Besov–Lipschitz spaces and Gaussian Triebel–Lizorkin spaces

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To Professor Paul L. Butzer with deep admiration.

Abstract

In this paper we define Besov–Lipschitz and Triebel–Lizorkin spaces in the context of Gaussian harmonic analysis, the harmonic analysis of Hermite polynomial expansions. We study inclusion relations among them, some interpolation results and continuity results of some important operators (the Ornstein–Uhlenbeck and the Poisson–Hermite semigroups and the Bessel potentials) on them. We also prove that the Gaussian Sobolev spaces $L^p_\alpha(\gamma_d)$ are contained in them. The proofs are general enough to allow extensions of these results to the case of Laguerre or Jacobi expansions and even further in the general framework of diffusion semigroups.

Keywords: Hermite expansions; Fractional integrals; Fractional derivatives; Bessel potentials; Triebel–Lizorkin spaces; Besov–Lipschitz spaces

1. Introduction

Let us consider the Gaussian measure $\gamma_d(x) = \frac{e^{-|x|^2}}{\pi^{d/2}}$ with $x \in \mathbb{R}^d$ and the Ornstein–Uhlenbeck differential operator

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$$L = \frac{1}{2}\Delta_x - \langle x, \nabla_x \rangle . \tag{1}$$

Let $\beta = (\beta_1, \ldots, \beta_d) \in \mathbb{N}^d$ be a multi-index, let $\beta! = \prod_{i=1}^d \beta_i!$, $|\beta| = \sum_{i=1}^d \beta_i$, $\partial_i = \frac{\partial}{\partial x_i}$, for each $1 \le i \le d$ and $\partial^\beta = \partial_1^{\beta_1} \ldots \partial_d^{\beta_d}$.

Let us consider the normalized Hermite polynomials of order β, in d variables

$$h_\beta(x) = \frac{1}{\left(2^{|\beta|}\beta!\right)^{1/2}} \prod_{i=1}^d (-1)^{\beta_i} \mathrm{e}^{x_i^2} \frac{\partial^{\beta_i}}{\partial x_i^{\beta_i}} (\mathrm{e}^{-x_i^2}), \tag{2}$$

then, it is well known, that the Hermite polynomials are eigenfunctions of L,

$$Lh_\beta(x) = -\,|\beta|\, h_\beta(x). \tag{3}$$

Given a function $f \in L^1(\gamma_d)$ its β-Fourier–Hermite coefficient is defined by

$$\hat{f}(\beta) = \langle f, h_\beta \rangle_{\gamma_d} = \int_{\mathbb{R}^d} f(x) h_\beta(x) \gamma_d(\mathrm{d}x).$$

Let C_n be the closed subspace of $L^2(\gamma_d)$ generated by the linear combinations of $\left\{h_\beta \ : |\beta| = n\right\}$. By the orthogonality of the Hermite polynomials with respect to γ_d it is easy to see that $\{C_n\}$ is an orthogonal decomposition of $L^2(\gamma_d)$,

$$L^2(\gamma_d) = \bigoplus_{n=0}^{\infty} C_n$$

which is called the Wiener chaos.

Let J_n be the orthogonal projection of $L^2(\gamma_d)$ onto C_n, then if $f \in L^2(\gamma_d)$

$$J_n f = \sum_{|\beta|=n} \hat{f}(\beta) h_\beta .$$

Let us define the Ornstein–Uhlenbeck semigroup $\{T_t\}_{t\ge0}$ as

$$\begin{aligned} T_t f(x) &= \frac{1}{\left(1-\mathrm{e}^{-2t}\right)^{d/2}} \int_{\mathbb{R}^d} \mathrm{e}^{-\frac{\mathrm{e}^{-2t}(|x|^2+|y|^2)-2\mathrm{e}^{-t}\langle x,y\rangle}{1-\mathrm{e}^{-2t}}} f(y)\gamma_d(\mathrm{d}y) \\ &= \frac{1}{\pi^{d/2}(1-\mathrm{e}^{-2t})^{d/2}} \int_{\mathbb{R}^d} \mathrm{e}^{-\frac{|y-\mathrm{e}^{-t}x|^2}{1-\mathrm{e}^{-2t}}} f(y)\mathrm{d}y. \end{aligned} \tag{4}$$

The family $\{T_t\}_{t\ge0}$ is a strongly continuous Markov semigroup on $L^p(\gamma_d)$, $1 \le p < \infty$, with infinitesimal generator L. Also, by a change of variable we can write,

$$T_t f(x) = \int_{\mathbb{R}^d} f(\sqrt{1-\mathrm{e}^{-2t}}u + \mathrm{e}^{-t}x)\gamma_d(\mathrm{d}u). \tag{5}$$

Now, by Bochner subordination formula, see Stein [16], we define the Poisson–Hermite semigroup $\{P_t\}_{t\ge0}$ as

$$P_t f(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{\mathrm{e}^{-u}}{\sqrt{u}} T_{t^2/4u} f(x)\mathrm{d}u = \int_0^\infty T_s f(x) \mu_t^{(1/2)}(\mathrm{d}s), \tag{6}$$

where the measure

$$\mu_t^{(1/2)}(\mathrm{d}s) = \frac{t}{2\sqrt{\pi}} \frac{\mathrm{e}^{-t^2/4s}}{s^{3/2}} \mathrm{d}s = g(t,s)\mathrm{d}s, \tag{7}$$

is called the one-side stable measure on $(0, \infty)$ of order $1/2$.

The family $\{P_t\}_{t \geq 0}$ is also a strongly continuous semigroup on $L^p(\gamma_d)$, $1 \leq p < \infty$, with infinitesimal generator $-(-L)^{1/2}$. From (4) we obtain, after the change of variable $r = \mathrm{e}^{-t^2/4u}$,

$$\begin{aligned} P_t f(x) &= \frac{1}{2\pi^{(d+1)/2}} \int_{\mathbb{R}^d} \int_0^1 t \frac{\exp\left(t^2/4\log r\right)}{(-\log r)^{3/2}} \frac{\exp\left(\frac{-|y-rx|^2}{1-r^2}\right)}{(1-r^2)^{d/2}} \frac{\mathrm{d}r}{r} f(y)\mathrm{d}y \\ &= \int_{\mathbb{R}^d} p(t,x,y) f(y)\mathrm{d}y, \end{aligned} \tag{8}$$

with

$$p(t,x,y) = \frac{1}{2\pi^{(d+1)/2}} \int_0^1 t \frac{\exp\left(t^2/4\log r\right)}{(-\log r)^{3/2}} \frac{\exp\left(\frac{-|y-rx|^2}{1-r^2}\right)}{(1-r^2)^{d/2}} \frac{\mathrm{d}r}{r}. \tag{9}$$

In what follows, we will often going to use the notation

$$u(x,t) = P_t f(x).$$

Observe that by (3) we have that

$$T_t h_\beta(x) = \mathrm{e}^{-t|\beta|} h_\beta(x), \tag{10}$$

and

$$P_t h_\beta(x) = \mathrm{e}^{-t\sqrt{|\beta|}} h_\beta(x). \tag{11}$$

For more details on the Ornstein–Uhlembeck and Poisson–Hermite semigroups we refer to [20].

Let us observe that since $\|T_t f - f\|_{p,\gamma_d} \to 0$ and $\|P_t f - f\|_{p,\gamma_d} \to 0$ as $t \to 0$ then $\{T_t\}$ and $\{P_t\}$ play the role of "approximation of the identity" in Gaussian setting. Moreover they are, up to now, the only approximations of identity known. Therefore following H. Triebel – see [19]; section 2.6.4, Harmonic and Thermic extensions, page 152 – we are going to use them to define Gaussian Besov–Lipschitz $B^\alpha_{p,q}(\gamma_d)$ and Gaussian Triebel–Lizorkin $F^\alpha_{p,q}(\gamma_d)$ spaces. An open problem then is to find alternative definitions of those spaces and give a more explicit description about the type of regularity that they actually describe.

On the other hand, the possibility of characterize the Gaussian Besov–Lipschitz spaces in terms of modulus of smoothness, as it is done in the classical case, would be possible only if the classical translation operator $\tau_y f(x) = f(x+y)$ is replaced for a more suitable translation operator since the spaces $L^p(\gamma_d)$ are not in general closed under the action of τ_y, for instance, in the one dimensional case, let us take the function $f(x) = \mathrm{e}^{|x|^2 - |x|}$, then it is clear that $f \in L^1(\gamma_1)$ but it is easy to see that $\tau_1 f(x) = f(x+1) = \mathrm{e}^{|x+1|^2 - |x+1|} \notin L^1(\gamma_1)$. This point requires further investigations.

For $\alpha > 0$, the Fractional Integral or Riesz potential of order α, I_α^γ, with respect to the Gaussian measure is defined formally as

$$I_\alpha^\gamma = (-L)^{-\alpha/2}\Pi_0, \tag{12}$$

where, $\Pi_0 f = f - \int_{\mathbb{R}^d} f(y)\gamma_d(\mathrm{d}y)$, for $f \in L^2(\gamma_d)$. That means that for the Hermite polynomials $\{h_\beta\}$, for $|\beta| > 0$,

$$I_\alpha^\gamma h_\beta(x) = \frac{1}{|\beta|^{\alpha/2}} h_\beta(x), \tag{13}$$

and for $\beta = \overline{0}$, $I_\alpha^\gamma(h_{\overline{0}}) = 0$. Then by linearity can be extended to any polynomial. It is easy to see that if f is a polynomial,

$$I_\alpha^\gamma f(x) = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1}(P_t f(x) - P_\infty f(x))\mathrm{d}t. \tag{14}$$

Moreover by P. A. Meyer's multiplier theorem, see [13], I_α^γ admits a continuous extension to $L^p(\gamma_d)$, $1 < p < \infty$, and (14) can be extended for $f \in L^p(\gamma_d)$, see [15]. Also if $f \in C_B^2(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} f(y)\gamma_d(\mathrm{d}y) = 0$, then

$$I_\alpha^\gamma f = -\frac{1}{\alpha\Gamma(\alpha)} \int_0^\infty t^\alpha \frac{\partial}{\partial t} P_t f \mathrm{d}t, \tag{15}$$

see [11].

The Bessel Potential of order $\alpha > 0$, $\mathcal{J}_\alpha^\gamma$, associated to the Gaussian measure is defined formally as

$$\mathcal{J}_\alpha^\gamma = (I - L)^{-\alpha/2}, \tag{16}$$

meaning that for the Hermite polynomials we have,

$$\mathcal{J}_\alpha^\gamma h_\beta(x) = \frac{1}{(1 + |\beta|)^{\alpha/2}} h_\beta(x).$$

Again by linearity can be extended to any polynomial and Meyer's theorem allows us to extend Bessel Potentials to a continuous operator on $L^p(\gamma_d)$, $1 < p < \infty$. Additionally, it is easy to see that $\mathcal{J}_\alpha^\gamma$ is a bijection over the set of polynomials $\mathcal{P}$.The Bessel potentials can be represented as

$$\mathcal{J}_\alpha^\gamma f(x) = \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} t^\alpha \mathrm{e}^{-t} P_t f(x) \frac{\mathrm{d}t}{t}, \tag{17}$$

for more details see [4]. Moreover $\{\mathcal{J}_\alpha^\gamma\}_\alpha$ is a strongly continuous semigroup on $L^p(\gamma_d)$, $1 \le p < \infty$, with infinitesimal generator $\frac{1}{2}\log(I - L)$.

The fractional derivative of order $\alpha > 0$ with respect to the Gaussian measure D_α^γ, is defined formally as

$$D_\alpha^\gamma = (-L)^{\alpha/2}, \tag{18}$$

meaning that for the Hermite polynomials, we have

$$D_\alpha^\gamma h_\beta(x) = |\beta|^{\alpha/2} h_\beta(x), \tag{19}$$

thus by linearity can be extended to any polynomial.

The fractional derivative D_α^γ with respect to the Gaussian measure was first introduced in [11] following [7]. For more details we refer to that article. Also see [14] for improved and simpler proofs of some results contained there.

Now, if f is a polynomial, by the linearity of the operators I_α^γ and D_α^γ, (13) and (19), we get

$$\Pi_0 f = I_\alpha^\gamma (D_\alpha^\gamma f) = D_\alpha^\gamma (I_\alpha^\gamma f). \tag{20}$$

The Gaussian Sobolev spaces of order $\alpha \geq 0$, $L_\alpha^p(\gamma_d)$, $1 < p < \infty$, can be obtained, as in the classical case, as the image of $L^p(\gamma_d)$ under the Bessel potential $\mathcal{J}_\alpha^\gamma$, with the norm

$$\|f\|_{p,\alpha} := \left\| (I - L)^{\alpha/2} f \right\|_{p,\gamma_d}. \tag{21}$$

Also they can be defined as the completion of the set of polynomials $\mathcal{P}$ with respect to that norm, see [21] and therefore $\mathcal{P}$ is trivially dense there. Let us remember that it can be proved that the set of polynomials $\mathcal{P}$ is also dense in $L^p(\gamma_d)$, $1 < p < \infty$, see [1]. The fractional derivative D_α^γ can be used to characterize the Gaussian Sobolev spaces $L_\alpha^p(\gamma_d)$ see [11].

As usual in what follows C represents a constant that is not necessarily the same in each occurrence.

2. The main results

As it was already mentioned in the introduction, the main objective of this paper is to introduce the Gaussian Besov–Lipschitz $B_{p,q}^\alpha(\gamma_d)$ and the Gaussian Triebel–Lizorkin $F_{p,q}^\alpha(\gamma_d)$ spaces, for any $\alpha \geq 0$. We will follow Stein [16] scheme to define and study the $B_{p,q}^\alpha(\gamma_d)$ spaces, but since the Poisson–Hermite semigroup is not a convolution semigroup the proofs of the results will be totally different to the ones in Stein's book. We will use, in an essential way, the representation of the Poisson–Hermite semigroup (6) using the one-side stable measure, $\mu_t^{(1/2)}$ defined in (7). From that fact, it is then clear that similar constructions are possible for the harmonic analysis of Laguerre or Jacobi polynomial expansions and even further in the framework of general diffusion semigroups but we are not going to consider those cases here. Let us point out that Hermite, Laguerre and Jacobi are the only cases of diffusion semigroups associated to orthogonal polynomials, see Mazet [12].

On the other hand, Besov–Lipschitz spaces can be also obtained as interpolated spaces using interpolation theory for semigroups defined on a Banach space, see for instance Chapter 3 of [2] or [17].

We will need some technical results for the measure $\mu_t^{(1/2)}$. First, in what follows since $\mu_t^{(1/2)}(\mathrm{d}s) = \frac{t}{2\sqrt{\pi}} \frac{\mathrm{e}^{-t^2/4s}}{s^{3/2}} \mathrm{d}s = g(t,s)\mathrm{d}s$, for any $k \in \mathbb{N}$, the notation $\frac{\partial^k}{\partial t^k} \mu_t^{(1/2)}(\mathrm{d}s)$ will denote

$$\frac{\partial^k}{\partial t^k} \mu_t^{(1/2)}(\mathrm{d}s) := \frac{\partial^k g(t,s)}{\partial t^k} \mathrm{d}s. \tag{22}$$

Then by induction it can be seen that

$$\frac{\partial^k \mu_t^{(1/2)}}{\partial t^k}(\mathrm{d}s) = \left(\sum_{\substack{i \in \mathbb{Z}, j \in \mathbb{N}, \\ 0 \leq j \leq k, 2j - i = k}} a_{i,j} \frac{t^i}{s^j} \right) \mu_t^{(1/2)}(\mathrm{d}s) \tag{23}$$

where $\{a_{i,j}\}$ is a (finite) set of constants.

Moreover, using the change of variable $u = \frac{t^2}{4s}$, it is easy to see that given $k \in \mathbb{N}$ and $t > 0$

$$\int_0^{+\infty} \frac{1}{s^k} \mu_t^{\frac{1}{2}}(\mathrm{d}s) = \frac{C_k}{t^{2k}}, \tag{24}$$

where $C_k = \frac{2^{2k}\Gamma(k+\frac{1}{2})}{\pi^{\frac{1}{2}}}$. Finally, using the two previous results we get that if $k \in \mathbb{N}$ and $t > 0$, then

$$\int_0^{+\infty} \left| \frac{\partial^k}{\partial t^k} \mu_t^{(1/2)} \right| (\mathrm{d}s) \leq \frac{C_k}{t^k}. \tag{25}$$

Now, considering the maximal function of the Ornstein–Uhlenbeck semigroup,

$$T^* f(x) = \sup_{t>0} |T_t f(x)|,$$

we have the following inequality that will be used later,

Lemma 2.1.

$$\left| \frac{\partial^k P_t f(x)}{\partial t^k} \right| \leq C_k T^* f(x) t^{-k}.$$

Proof. Using (25) and the dominated convergence theorem, we have

$$\begin{aligned} \left| \frac{\partial^k P_t f(x)}{\partial t^k} \right| &= \left| \int_0^{+\infty} T_s f(x) \frac{\partial^k}{\partial t^k} \mu_t^{(1/2)}(\mathrm{d}s) \right| \\ &\leq \int_0^{+\infty} |T_s f(x)| \left| \frac{\partial^k}{\partial t^k} \mu_t^{(1/2)}(\mathrm{d}s) \right| \\ &\leq \int_0^{+\infty} T^* f(x) \left| \frac{\partial^k}{\partial t^k} \mu_t^{(1/2)}(\mathrm{d}s) \right| \leq C_k\, T^* f(x) t^{-k}. \quad \square \end{aligned}$$

Lemma 2.2. *Given $f \in L^p(\gamma_d)$, $\alpha \geq 0$ and k, l integers greater than α, then*

$$\left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{p,\gamma_d} \leq A_k t^{-k+\alpha} \text{ if and only if } \left\| \frac{\partial^l P_t f}{\partial t^l} \right\|_{p,\gamma_d} \leq A_l t^{-l+\alpha}.$$

Moreover, if $A_k(f)$, $A_l(f)$ are the smallest constants appearing in the above inequalities then there exist constants $A_{k,l,\alpha}$ and $D_{k,l,\alpha}$ such that

$$A_{k,l,\alpha} A_k(f) \leq A_l(f) \leq D_{k,l,\alpha} A_k(f),$$

for all $f \in L^p(\gamma_d)$.

Proof. Let us suppose, without loss of generality, that $k \geq l$. We will prove first the direct implication. For this, we use the representation of the Poisson–Hermite semigroup (6),

$$P_t f(x) = \int_0^{+\infty} T_s f(x) \mu_t^{(1/2)}(\mathrm{d}s),$$

then differentiating k times with respect to t,

$$\frac{\partial^k P_t f(x)}{\partial t^k} = \int_0^{+\infty} T_s f(x) \frac{\partial^k}{\partial t^k} \mu_t^{(1/2)}(\mathrm{d}s).$$

Using the identity (23) it is easy to prove that for all $m \in \mathbb{N}$

$$\lim_{t \to +\infty} \frac{\partial^m P_t f(x)}{\partial t^m} = 0,$$

and therefore given $n \in \mathbb{N}$, $n > \alpha$

$$\frac{\partial^n P_t f(x)}{\partial t^n} = -\int_t^{+\infty} \frac{\partial^{n+1} P_s f(x)}{\partial s^{n+1}} \mathrm{d}s.$$

Thus,

$$\begin{aligned}
\left\| \frac{\partial^n P_t f}{\partial t^n} \right\|_{p,\gamma_d} &\leq \int_t^{+\infty} \left\| \frac{\partial^{n+1} P_s f}{\partial s^{n+1}} \right\|_{p,\gamma_d} \mathrm{d}s \\
&\leq \int_t^{+\infty} A_{n+1}(f) s^{-(n+1)+\alpha} \mathrm{d}s \\
&= \frac{A_{n+1}(f)}{n-\alpha} t^{-n+\alpha}.
\end{aligned}$$

Then

$$A_n(f) \leq \frac{A_{n+1}(f)}{n-\alpha},$$

and as $n > \alpha$ is arbitrary, then by using the above result $k-l$ times, we get

$$\begin{aligned}
A_l(f) &\leq \frac{A_{l+1}(f)}{l-\alpha} \leq \frac{A_{l+2}}{(l-\alpha)(l+1-\alpha)} \leq \cdots \leq \frac{A_k(f)}{(l-\alpha)(l+1-\alpha)\cdots(k-1-\alpha)} \\
&= D_{k,l,\alpha} A_k(f).
\end{aligned}$$

To prove the converse implication, using again the representation (6), we get,

$$u(x, t_1 + t_2) = P_{t_1}(P_{t_2} f)(x) = \int_0^{+\infty} T_s(P_{t_2} f)(x) \mu_{t_1}^{(1/2)}(\mathrm{d}s).$$

Therefore, taking $t = t_1 + t_2$ and differentiating l times with respect to t_2 and $k-l$ times with respect to t_1 we get

$$\frac{\partial^k u(x,t)}{\partial t^k} = \int_0^{+\infty} T_s \left(\frac{\partial^l P_{t_2} f(x)}{\partial t_2^l} \right) \frac{\partial^{k-l}}{\partial t_1^{k-l}} \mu_{t_1}^{(1/2)}(\mathrm{d}s). \tag{26}$$

Thus, using the inequality (25) and the fact that the Ornstein–Uhlenbeck semigroup is a contraction semigroup, we get

$$\begin{aligned}
\left\| \frac{\partial^k u(\cdot,t)}{\partial t^k} \right\|_{p,\gamma_d} &\leq \int_0^{+\infty} \left\| T_s \left(\frac{\partial^l P_{t_2} f}{\partial t_2^l} \right) \right\|_{p,\gamma_d} \left| \frac{\partial^{k-l} \mu_{t_1}^{(1/2)}}{\partial t_1^{k-l}} (\mathrm{d}s) \right| \\
&\leq \left\| \frac{\partial^l P_{t_2} f}{\partial t_2^l} \right\|_{p,\gamma_d} \int_0^{+\infty} \left| \frac{\partial^{k-l}}{\partial t_1^{k-l}} \mu_{t_1}^{(1/2)}(\mathrm{d}s) \right|
\end{aligned}$$

$$\leq C_{k-l} \left\| \frac{\partial^l}{\partial t_2^l} P_{t_2} f \right\|_{p,\gamma_d} t_1^{l-k}$$
$$\leq C_{k-l} A_l(f) t_2^{-l+\alpha} t_1^{l-k}.$$

Therefore, taking $t_1 = t_2 = \frac{t}{2}$,

$$\left\| \frac{\partial^k u(\cdot,t)}{\partial t^k} \right\|_{p,\gamma_d} \leq C_{k-l} A_l(f) \left(\frac{t}{2}\right)^{-k+\alpha},$$

and then,

$$A_k(f) \leq \frac{C_{k-l}}{2^{-k+\alpha}} A_l(f). \quad \square$$

The following technical result will be the key to defining Gaussian Besov–Lipschitz spaces,

Lemma 2.3. *Given $\alpha \geq 0$ and k, l integers greater than α. Then*

$$\left(\int_0^{+\infty} \left(t^{k-\alpha} \left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{p,\gamma_d} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty$$

if and only if

$$\left(\int_0^{+\infty} \left(t^{l-\alpha} \left\| \frac{\partial^l P_t f}{\partial t^l} \right\|_{p,\gamma_d} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty.$$

Moreover, there exist constants $A_{k,l,\alpha}$, $D_{k,l,\alpha}$ such that

$$D_{k,l,\alpha} \left(\int_0^{+\infty} \left(t^{l-\alpha} \left\| \frac{\partial^l P_t f}{\partial t^l} \right\|_{p,\gamma_d} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}}$$
$$\leq \left(\int_0^{+\infty} \left(t^{k-\alpha} \left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{p,\gamma_d} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}}$$
$$\leq A_{k,l,\alpha} \left(\int_0^{+\infty} \left(t^{l-\alpha} \left\| \frac{\partial^l P_t f}{\partial t^l} \right\|_{p,\gamma_d} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}}$$

Proof. Let us suppose, without loss of generality, that $k \geq l$. We will prove first the converse implication; from Lemma 2.2, we have,

$$\left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{p,\gamma_d} \leq C_{k-l} \left\| \frac{\partial^l P_{\frac{t}{2}} f}{\partial (\frac{t}{2})^l} \right\|_{p,\gamma_d} \left(\frac{t}{2}\right)^{l-k}.$$

Thus,

$$\left(\int_0^{+\infty} \left(t^{k-\alpha} \left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{p,\gamma_d} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}}$$

$$\leq \frac{C_{k-l}}{2^{l-k}} \left(\int_0^{+\infty} \left(t^{l-\alpha} \left\| \frac{\partial^l P_{t/2} f}{\partial \left(\frac{t}{2}\right)^l} \right\|_{p,\gamma_d} \right)^q \frac{\mathrm{d}t}{t} \right)^{\frac{1}{q}}$$

$$= A_{k,l,\alpha} \left(\int_0^{+\infty} \left(s^{l-\alpha} \left\| \frac{\partial^l P_s f}{\partial s^l} \right\|_{p,\gamma_d} \right)^q \frac{\mathrm{d}s}{s} \right)^{\frac{1}{q}}$$

with $A_{k,l,\alpha} = C_{k-l} 2^{k-\alpha}$.

For the direct implication, given $n \in \mathbb{N}$, $n > \alpha$, again using the previous lemma

$$\left\| \frac{\partial^n P_t f}{\partial t^n} \right\|_{p,\gamma_d} \leq \int_t^{+\infty} \left\| \frac{\partial^{n+1} P_s f}{\partial s^{n+1}} \right\|_{p,\gamma_d} \mathrm{d}s.$$

Therefore, by using the Hardy inequality [16]

$$\left(\int_0^{+\infty} \left(t^{n-\alpha} \left\| \frac{\partial^n P_t f}{\partial t^n} \right\|_{p,\gamma_d} \right)^q \frac{\mathrm{d}t}{t} \right)^{\frac{1}{q}}$$

$$\leq \left(\int_0^{+\infty} \left(t^{n-\alpha} \int_t^{+\infty} \left\| \frac{\partial^{n+1} P_s f}{\partial s^{n+1}} \right\|_{p,\gamma_d} \mathrm{d}s \right)^q \frac{\mathrm{d}t}{t} \right)^{\frac{1}{q}}$$

$$= \left(\int_0^{+\infty} \left(\int_t^{+\infty} \left\| \frac{\partial^{n+1} P_s f}{\partial s^{n+1}} \right\|_{p,\gamma_d} \mathrm{d}s \right)^q t^{(n-\alpha)q-1} \mathrm{d}t \right)^{\frac{1}{q}}$$

$$\leq \frac{1}{n-\alpha} \left(\int_0^{+\infty} \left(s^{n+1-\alpha} \left\| \frac{\partial^{n+1} P_s f}{\partial s^{n+1}} \right\|_{p,\gamma_d} \right)^q \frac{\mathrm{d}s}{s} \right)^{\frac{1}{q}}.$$

Now, as $n > \alpha$ is arbitrary, using the above result $k - l$ times

$$\left(\int_0^{+\infty} \left(t^{l-\alpha} \left\| \frac{\partial^l P_t f}{\partial t^l} \right\|_{p,\gamma_d} \right)^q \frac{\mathrm{d}t}{t} \right)^{\frac{1}{q}}$$

$$\leq \frac{1}{l-\alpha} \left(\int_0^{+\infty} \left(t^{l+1-\alpha} \left\| \frac{\partial^{l+1} P_t f}{\partial t^{l+1}} \right\|_{p,\gamma_d} \right)^q \frac{\mathrm{d}t}{t} \right)^{\frac{1}{q}}$$

$$\leq \frac{1}{(l-\alpha).(l+1-\alpha)} \left(\int_0^{+\infty} \left(t^{l+2-\alpha} \left\| \frac{\partial^{l+2} P_t f}{\partial t^{l+2}} \right\|_{p,\gamma_d} \right)^q \frac{\mathrm{d}t}{t} \right)^{\frac{1}{q}}$$

$$\cdots$$

$$\leq D_{k,l,\alpha} \left(\int_0^{+\infty} \left(t^{k-\alpha} \left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{p,\gamma_d} \right)^q \frac{\mathrm{d}t}{t} \right)^{\frac{1}{q}}$$

where $D_{k,l,\alpha} = \frac{1}{(l-\alpha).(l+1-\alpha)\cdots(k-1-\alpha)}$. □

Now, following the classical case, see for instance [6,16,18,19], we are going to define the Gaussian Besov–Lipschitz $B^{\alpha}_{p,q}(\gamma_d)$ spaces or Besov–Lipschitz spaces for Hermite polynomial expansions,

Definition 2.1. Let $\alpha \geq 0$, k be the smallest integer greater than α, and $1 \leq p, q \leq \infty$. For $1 \leq q < \infty$ the Gaussian Besov–Lipschitz space $B^{\alpha}_{p,q}(\gamma_d)$ is defined as the set of functions $f \in L^p(\gamma_d)$ for which

$$\left(\int_0^{\infty} \left(t^{k-\alpha} \left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{p,\gamma_d} \right)^q \frac{\mathrm{d}t}{t} \right)^{1/q} < \infty. \tag{27}$$

The norm of $f \in B^{\alpha}_{p,q}(\gamma_d)$ is defined as

$$\|f\|_{B^{\alpha}_{p,q}} := \|f\|_{p,\gamma_d} + \left(\int_0^{\infty} \left(t^{k-\alpha} \left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{p,\gamma_d} \right)^q \frac{\mathrm{d}t}{t} \right)^{1/q}. \tag{28}$$

For $q = \infty$ the Gaussian Besov–Lipschitz space $B^{\alpha}_{p,\infty}(\gamma_d)$ is defined as the set of functions $f \in L^p(\gamma_d)$ for which exists a constant A such that

$$\left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{p,\gamma_d} \leq A t^{-k+\alpha}$$

and then the norm of $f \in B^{\alpha}_{p,\infty}(\gamma_d)$ is defined as

$$\|f\|_{B^{\alpha}_{p,\infty}} := \|f\|_{p,\gamma_d} + A_k(f), \tag{29}$$

where $A_k(f)$ is the smallest constant A appearing in the above inequality.

In particular, the space $B^{\alpha}_{\infty,\infty}(\gamma_d)$ is the Gaussian Lipschitz space $Lip_{\alpha}(\gamma_d)$.

Lemma 2.3 show us that we could have replaced k by any other integer l greater than α and the resulting norms are equivalent.

In what follows, we need the following technical result about $L^p(\gamma_d)$-norms of the derivatives of the Poisson–Hermite semigroup,

Lemma 2.4. *Suppose $f \in L^p(\gamma_d)$, then for any integer k the function $\| \frac{\partial^k P_t f}{\partial t^k} \|_{p,\gamma_d}$ is a non-increasing function of t, for $0 < t < +\infty$. Moreover,*

$$\left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{p,\gamma_d} \leq C \|f\|_{p,\gamma_d} t^{-k}, \quad t > 0. \tag{30}$$

Proof. Let us consider first the case $k = 0$. Let us fix $t_1, t_2 > 0$, by using the semigroup property we get

$$u(x, t_1 + t_2) = P_{t_1+t_2} f(x) = P_{t_1}(P_{t_2} f(x)) = P_{t_1}(u(x, t_2)).$$

Therefore, by definition of P_t, Jensen's inequality and the invariance of γ_d

$$\begin{aligned}
\int_{\mathbb{R}^d} |u(x, t_1 + t_2)|^p \gamma_d(\mathrm{d}x) &= \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} p(t_1, x, y) u(y, t_2) \mathrm{d}y \right|^p \gamma_d(\mathrm{d}x) \\
&\leq \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} p(t_1, x, y) |u(y, t_2)|^p \mathrm{d}y \right) \gamma_d(\mathrm{d}x) \\
&= \int_{\mathbb{R}^d} P_{t_1}(|u(x, t_2)|^p) \gamma_d(\mathrm{d}x) \\
&= \int_{\mathbb{R}^d} |u(x, t_2)|^p \gamma_d(\mathrm{d}x).
\end{aligned}$$

Thus

$$\|P_{t_1+t_2} f\|_{p,\gamma_d} \leq \|P_{t_2} f\|_{p,\gamma_d}.$$

Now to prove the general case, $k > 0$. Differentiating the identity $u(x, t_1+t_2) = P_{t_1}(u(x, t_2))$, k times with respect to t_2 to get

$$\frac{\partial^k u(x, t_1 + t_2)}{\partial (t_1 + t_2)^k} = P_{t_1}\left(\frac{\partial^k u(x, t_2)}{\partial t_2^k}\right)$$

and then use a analogous argument to the one above.

In order to prove (30) we use again the representation (6) of the Poisson–Hermite semigroup and differentiating it k times with respect to t we get

$$\frac{\partial^k P_t f(x)}{\partial t^k} = \int_0^{+\infty} T_s f(x) \frac{\partial^k}{\partial t^k} \mu_t^{(1/2)}(\mathrm{d}s),$$

thus, by Minkowski's integral inequality, the contractive property of the Ornstein–Uhlenbeck semigroup and inequality (25), we get for $t > 0$

$$\begin{aligned}
\left\|\frac{\partial^k P_t f}{\partial t^k}\right\|_{p,\gamma_d} &\leq \int_0^{+\infty} \left\| T_s f \frac{\partial^k}{\partial t^k} \mu_t^{(1/2)}(\mathrm{d}s)\right\|_{p,\gamma_d} \\
&= \int_0^{+\infty} \|T_s f\|_{p,\gamma_d} \left|\frac{\partial^k}{\partial t^k} \mu_t^{(1/2)}(\mathrm{d}s)\right| \\
&\leq \|f\|_{p,\gamma_d} \int_0^{+\infty} \left|\frac{\partial^k}{\partial t^k} \mu_t^{(1/2)}(\mathrm{d}s)\right| \\
&\leq \frac{C_k}{t^k} \|f\|_{p,\gamma_d}. \quad \square
\end{aligned}$$

Let us study some inclusions among the Gaussian Besov–Lipschitz spaces,

Proposition 2.1. *The inclusion $B^{\alpha_1}_{p,q_1}(\gamma_d) \subset B^{\alpha_2}_{p,q_2}(\gamma_d)$ holds if either:*

(i) $\alpha_1 > \alpha_2 > 0$ *(q_1 and q_2 need not to be related), or*

(ii) *If $\alpha_1 = \alpha_2$ and $q_1 \leq q_2$*

Proof. In order to prove (ii), we set $A = \left(\int_0^{+\infty} \left(t^{k-\alpha} \|\frac{\partial^k P_t f}{\partial t^k}\|_{p,\gamma_d}\right)^{q_1} \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q_1}}$ Now, fixing $t_0 > 0$

$$\int_{\frac{t_0}{2}}^{t_0} \left(t^{k-\alpha} \left\|\frac{\partial^k P_t f}{\partial t^k}\right\|_{p,\gamma_d}\right)^{q_1} \frac{\mathrm{d}t}{t} \leq A^{q_1}.$$

By Lemma 2.4, $\|\frac{\partial^k P_t f}{\partial t^k}\|_{p,\gamma_d}$ takes its minimum value at the upper end point ($t = t_0$) of the above integral. So we get

$$\left\|\frac{\partial^k P_{t_0} f}{\partial t^k}\right\|^{q_1}_{p,\gamma_d} \int_{\frac{t_0}{2}}^{t_0} t^{(k-\alpha)q_1} \frac{\mathrm{d}t}{t} \leq A^{q_1}.$$

That is $\|\frac{\partial^k P_{t_0} f}{\partial t^k}\|_{p,\gamma_d} \leq C A t_0^{-k+\alpha}$ but since t_0 is arbitrary then

$$\left\|\frac{\partial^k P_t f}{\partial t^k}\right\|_{p,\gamma_d} \leq C A t^{-k+\alpha},$$

for all $t > 0$. In other words $f \in B^{\alpha}_{p,q_1}$ implies also that $f \in B^{\alpha}_{p,\infty}$. Thus, as $q_2 \geq q_1$

$$\begin{aligned}
&\int_0^{+\infty} \left(t^{k-\alpha} \left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{p,\gamma_d} \right)^{q_2} \frac{\mathrm{d}t}{t} \\
&= \int_0^{+\infty} \left(t^{k-\alpha} \left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{p,\gamma_d} \right)^{q_2-q_1} \left(t^{k-\alpha} \left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{p,\gamma_d} \right)^{q_1} \frac{\mathrm{d}t}{t} \\
&\leq (CA)^{q_2-q_1} \int_0^{+\infty} \left(t^{k-\alpha} \left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{p,\gamma_d} \right)^{q_1} \frac{\mathrm{d}t}{t} \\
&= (CA)^{q_2-q_1} A^{q_1} = CA^{q_2} < +\infty,
\end{aligned}$$

and therefore $f \in B^{\alpha}_{p,q_2}$.

Now in order to prove part (i), by Lemma 2.4 we have

$$\left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{p,\gamma_d} \leq C t^{-k}, \quad t > 0.$$

Now given $f \in B^{\alpha_1}_{p,q_1}$, taking again

$$A = \left(\int_0^{+\infty} \left(t^{k-\alpha_1} \left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{p,\gamma_d} \right)^{q_1} \frac{\mathrm{d}t}{t} \right)^{\frac{1}{q_1}},$$

we get as in part (ii)

$$\left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{p,\gamma_d} \leq CAt^{-k+\alpha_1},$$

for all $t > 0$. Now,

$$\begin{aligned}
&\int_0^{+\infty} \left(t^{k-\alpha_2} \left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{p,\gamma_d} \right)^{q_2} \frac{\mathrm{d}t}{t} \\
&= \int_0^{1} \left(t^{k-\alpha_2} \left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{p,\gamma_d} \right)^{q_2} \frac{\mathrm{d}t}{t} + \int_1^{+\infty} \left(t^{k-\alpha_2} \left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{p,\gamma_d} \right)^{q_2} \frac{\mathrm{d}t}{t} \\
&= \mathrm{I} + \mathrm{II}.
\end{aligned}$$

Now,

$$\begin{aligned}
\mathrm{I} &= \int_0^1 t^{(k-\alpha_2)q_2} \left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{p,\gamma_d}^{q_2} \frac{\mathrm{d}t}{t} \leq \int_0^1 t^{(k-\alpha_2)q_2} (CA)^{q_2} t^{(\alpha_1-k)q_2} \frac{\mathrm{d}t}{t} \\
&= (CA)^{q_2} \int_0^1 t^{(\alpha_1-\alpha_2)q_2} \frac{\mathrm{d}t}{t} = CA^{q_2},
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{II} &= \int_1^{+\infty} t^{(k-\alpha_2)q_2} \left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{p,\gamma_d}^{q_2} \frac{\mathrm{d}t}{t} \leq \int_1^{+\infty} t^{(k-\alpha_2)q_2} C^{q_2} t^{-kq_2} \frac{\mathrm{d}t}{t} \\
&= C^{q_2} \int_1^{+\infty} t^{-\alpha_2 q_2} \frac{\mathrm{d}t}{t} = C.
\end{aligned}$$

Hence,

$$\int_0^{+\infty} \left(t^{k-\alpha_2} \left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{p,\gamma_d} \right)^{q_2} \frac{dt}{t} < +\infty,$$

and so $f \in B^{\alpha_2}_{p,q_2}$. □

The following technical result will be the key to defining Gaussian Triebel–Lizorkin spaces,

Lemma 2.5. *Let $\alpha \geq 0$ and k, l integers such that $k \geq l > \alpha$. Then*

$$\left\| \left(\int_0^{+\infty} \left(t^{k-\alpha} \left| \frac{\partial^k}{\partial t^k} P_t f \right| \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \right\|_{p,\gamma} < \infty$$

if and only if

$$\left\| \left(\int_0^{+\infty} \left(t^{l-\alpha} \left| \frac{\partial^l}{\partial t^l} P_t f \right| \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \right\|_{p,\gamma} < \infty.$$

Moreover, there exist constants $A_{k,l,\alpha}$, $D_{k,l,\alpha}$ such that

$$\begin{aligned} D_{k,l,\alpha} & \left\| \left(\int_0^{+\infty} \left(t^{l-\alpha} \left| \frac{\partial^l}{\partial t^l} P_t f \right| \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \right\|_{p,\gamma} \\ & \leq \left\| \left(\int_0^{+\infty} \left(t^{k-\alpha} \left| \frac{\partial^k}{\partial t^k} P_t f \right| \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \right\|_{p,\gamma} \\ & \leq A_{k,l,\alpha} \left\| \left(\int_0^{+\infty} \left(t^{l-\alpha} \left| \frac{\partial^l}{\partial t^l} P_t f \right| \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \right\|_{p,\gamma}. \end{aligned}$$

Proof. Let $n \in \mathbb{N}$ such that $n > \alpha$. Then it can be proved that

$$\left| \frac{\partial^n}{\partial t^n} P_t f(x) \right| \leq \int_t^{+\infty} \left| \frac{\partial^{n+1}}{\partial s^{n+1}} P_s f(x) \right| ds.$$

Then by Hardy's inequality,

$$\begin{aligned} & \left(\int_0^{+\infty} \left(t^{n-\alpha} \left| \frac{\partial^n}{\partial t^n} P_t f(x) \right| \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \\ & \leq \left(\int_0^{+\infty} \left(t^{n-\alpha} \int_t^{+\infty} \left| \frac{\partial^{n+1}}{\partial s^{n+1}} P_s f(x) \right| ds \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \\ & \leq \frac{1}{n-\alpha} \left(\int_0^{+\infty} \left(s \left| \frac{\partial^{n+1}}{\partial s^{n+1}} P_s f(x) \right| \right)^q s^{(n-\alpha)q-1} ds \right)^{\frac{1}{q}} \end{aligned}$$

$$= \frac{1}{n-\alpha}\left(\int_0^{+\infty}\left(s^{n+1-\alpha}\left|\frac{\partial^{n+1}}{\partial s^{n+1}}P_s f(x)\right|\right)^q \frac{\mathrm{d}s}{s}\right)^{\frac{1}{q}}.$$

Now as $n > \alpha$ is arbitrary, iterating the previous argument $k - l$ times, we have

$$\left(\int_0^{+\infty}\left(t^{l-\alpha}\left|\frac{\partial^{l}}{\partial t^{l}}P_t f(x)\right|\right)^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}}$$
$$\leq \frac{1}{l-\alpha}\left(\int_0^{+\infty}\left(t^{l+1-\alpha}\left|\frac{\partial^{l+1}}{\partial t^{l+1}}P_t f(x)\right|\right)^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}}$$
$$\leq \frac{1}{(l-\alpha).(l+1-\alpha)}\left(\int_0^{+\infty}\left(t^{l+2-\alpha}\left|\frac{\partial^{l+2}}{\partial t^{l+2}}P_t f(x)\right|\right)^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}}$$
$$\ldots$$
$$\leq C_{k,l,\alpha}\left(\int_0^{+\infty}\left(t^{k-\alpha}\left|\frac{\partial^{k}}{\partial t^{k}}P_t f(x)\right|\right)^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}}$$

where $C_{k,l,\alpha} = \frac{1}{(l-\alpha)(l+1-\alpha)\cdots(k-1-\alpha)}$. Thus

$$D_{k,l,\alpha}\left\|\left(\int_0^{+\infty}\left(t^{l-\alpha}\left|\frac{\partial^{l}}{\partial t^{l}}P_t f\right|\right)^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}}\right\|_{p,\gamma} \leq \left\|\left(\int_0^{+\infty}\left(t^{k-\alpha}\left|\frac{\partial^{k}}{\partial t^{k}}P_t f\right|\right)^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}}\right\|_{p,\gamma},$$

where $D_{k,l,\alpha} = 1/C_{k,l,\alpha}$.

The converse inequality is also obtained by an inductive argument from the case $k = l + 1$. Let us remember (26),

$$\frac{\partial^k u(x,t)}{\partial t^k} = \int_0^{+\infty} T_s\left(\frac{\partial^l P_{t_2} f(x)}{\partial t_2^l}\right)\frac{\partial^{k-l}}{\partial t_1^{k-l}}\mu_{t_1}^{(1/2)}(\mathrm{d}s),$$

and since, from (23), $\frac{\partial}{\partial t_1}\mu_{t_1}^{(1/2)}(\mathrm{d}s) = \left(t_1^{-1} - \frac{t_1}{2s}\right)\mu_{t_1}^{(1/2)}(\mathrm{d}s)$ we get

$$\left|\frac{\partial^k u(x,t)}{\partial t^k}\right| \leq \int_0^{+\infty} T_s\left(\left|\frac{\partial^l P_{t_2} f(x)}{\partial t_2^l}\right|\right)\left|\left(t_1^{-1} - \frac{t_1}{2s}\right)\right|\mu_{t_1}^{(1/2)}(\mathrm{d}s)$$
$$\leq t_1^{-1}\int_0^{+\infty} T_s\left(\left|\frac{\partial^l P_{t_2} f(x)}{\partial t_2^l}\right|\right)\mu_{t_1}^{(1/2)}(\mathrm{d}s)$$
$$+ \frac{t_1}{2}\int_0^{+\infty} T_s\left(\left|\frac{\partial^l P_{t_2} f(x)}{\partial t_2^l}\right|\right)\frac{1}{s}\mu_{t_1}^{(1/2)}(\mathrm{d}s).$$

Therefore

$$\left(\int_0^{+\infty}\left(t_2^{k-\alpha}\left|\frac{\partial^k u(x,t)}{\partial t^k}\right|\right)^q \frac{\mathrm{d}t_2}{t_2}\right)^{1/q}$$

$$\leq C_q \left[\left(\int_0^{+\infty} \left(t_2^{k-\alpha} t_1^{-1} \int_0^{+\infty} T_s \left(\left| \frac{\partial^l P_{t_2} f(x)}{\partial t_2^l} \right| \right) \mu_{t_1}^{(1/2)}(\mathrm{d}s) \right)^q \frac{\mathrm{d}t_2}{t_2} \right)^{1/q} \right.$$
$$\left. + \left(\int_0^{+\infty} \left(t_2^{k-\alpha} \frac{t_1}{2} \int_0^{+\infty} T_s \left(\left| \frac{\partial^l P_{t_2} f(x)}{\partial t_2^l} \right| \right) \frac{1}{s} \mu_{t_1}^{(1/2)}(\mathrm{d}s) \right)^q \frac{\mathrm{d}t_2}{t_2} \right)^{1/q} \right]$$
$$= \mathrm{I} + \mathrm{II}.$$

Now using twice Minkowski's integral inequality (since T_s is an integral transformation with positive kernel) and the fact that $\mu_{t_1}^{(1/2)}(\mathrm{d}s)$ is a probability, we get

$$\mathrm{I} = C_q \left(\int_0^{+\infty} \left(t_2^{k-\alpha} t_1^{-1} \right)^q \left(\int_0^{+\infty} T_s \left(\left| \frac{\partial^l P_{t_2} f(x)}{\partial t_2^l} \right| \right) \mu_{t_1}^{(1/2)}(\mathrm{d}s) \right)^q \frac{\mathrm{d}t_2}{t_2} \right)^{1/q}$$
$$\leq C_q \int_0^{+\infty} \left(\int_0^{+\infty} \left(t_2^{k-\alpha} t_1^{-1} \right)^q \left(T_s \left(\left| \frac{\partial^l P_{t_2} f(x)}{\partial t_2^l} \right| \right) \right)^q \frac{\mathrm{d}t_2}{t_2} \right)^{1/q} \mu_{t_1}^{(1/2)}(\mathrm{d}s)$$
$$\leq C_q \int_0^{+\infty} T_s \left(\left(\int_0^{+\infty} \left(t_2^{k-\alpha} t_1^{-1} \right)^q \left(\left| \frac{\partial^l P_{t_2} f(x)}{\partial t_2^l} \right| \right)^q \frac{\mathrm{d}t_2}{t_2} \right)^{1/q} \right) \mu_{t_1}^{(1/2)}(\mathrm{d}s)$$
$$\leq C_q T^* \left(\left(\int_0^{+\infty} \left(t_2^{k-\alpha} t_1^{-1} \right)^q \cdot \left(\left| \frac{\partial^l P_{t_2} f(x)}{\partial t_2^l} \right| \right)^q \frac{\mathrm{d}t_2}{t_2} \right)^{1/q} \right)$$

and using the same argument for (II) and (24), we have

$$\mathrm{II} \leq C_q T^* \left(\left(\int_0^{+\infty} \left(t_2^{k-\alpha} t_1 \right)^q \left(\left| \frac{\partial^l P_{t_2} f(x)}{\partial t_2^l} \right| \right)^q \frac{\mathrm{d}t_2}{t_2} \right)^{1/q} \right) \frac{1}{t_1^2}$$
$$= C_q T^* \left(\left(\int_0^{+\infty} \left(t_2^{k-\alpha} t_1^{-1} \right)^q \left(\left| \frac{\partial^l P_{t_2} f(x)}{\partial t_2^l} \right| \right)^q \frac{\mathrm{d}t_2}{t_2} \right)^{1/q} \right).$$

Taking $t_1 = t_2 = \frac{t}{2}$ and changing the variable, we get

$$\mathrm{I} \leq C_q T^* \left(\left(\int_0^{+\infty} \left(t^{l-\alpha} \right)^q \left(\left| \frac{\partial^l P_t f(x)}{\partial t^l} \right| \right)^q \frac{\mathrm{d}t}{t} \right)^{1/q} \right)$$

and

$$\mathrm{II} \leq C_q T^* \left(\left(\int_0^{+\infty} \left(t^{l-\alpha} \right)^q \left(\left| \frac{\partial^l P_t f(x)}{\partial t^l} \right| \right)^q \frac{\mathrm{d}t}{t} \right)^{1/q} \right).$$

Hence, by the L^p boundedness of T^*

$$\left\| \left(\int_0^{+\infty} \left(t^{k-\alpha} \left| \frac{\partial^k u(x,t)}{\partial t^k} \right| \right)^q \frac{\mathrm{d}t}{t} \right)^{1/q} \right\|_{p,\gamma}$$

$$\leq C_{q,k,\alpha}\left\|T^*\left(\left(\left(\int_0^{+\infty}\left(t^{l-\alpha}\left|\frac{\partial^l P_t f(x)}{\partial u^l}\right|\right)^q \frac{\mathrm{d}t}{t}\right)^{1/q}\right)\right\|_{p,\gamma}$$
$$+C_q\left\|T^*\left(\left(\left(\int_0^{+\infty}\left(t^{l-\alpha}\left|\frac{\partial^l P_t f(x)}{\partial u^l}\right|\right)^q \frac{\mathrm{d}t}{t}\right)^{1/q}\right)\right\|_{p,\gamma}$$
$$\leq C_{k,\alpha,q}\left\|\left(\int_0^{+\infty}\left(t^{l-\alpha}\left|\frac{\partial^l P_t f(x)}{\partial t^l}\right|\right)^q \frac{\mathrm{d}t}{t}\right)^{1/q}\right\|_{p,\gamma}. \quad \square$$

Now, we can introduce the Gaussian Triebel–Lizorkin spaces $F^{\alpha}_{p,q}(\gamma_d)$ following the classical case (see [6,18,19]),

Definition 2.2. Let $\alpha \geq 0$, k be the smallest integer greater than α, and $1 \leq p, q < \infty$. The Gaussian Triebel–Lizorkin space $F^{\alpha}_{p,q}(\gamma_d)$ is the set of functions $f \in L^p(\gamma_d)$ for which

$$\left\|\left(\int_0^{\infty}\left(t^{k-\alpha}\left|\frac{\partial^k P_t f}{\partial t^k}\right|\right)^q \frac{\mathrm{d}t}{t}\right)^{1/q}\right\|_{p,\gamma_d} < \infty. \tag{31}$$

The norm of $f \in F^{\alpha}_{p,q}(\gamma_d)$ is defined as

$$\|f\|_{F^{\alpha}_{p,q}} := \|f\|_{p,\gamma_d} + \left\|\left(\int_0^{\infty}\left(t^{k-\alpha}\left|\frac{\partial^k P_t f}{\partial t^k}\right|\right)^q \frac{\mathrm{d}t}{t}\right)^{1/q}\right\|_{p,\gamma_d}. \tag{32}$$

Observe that by Lemma 2.5 the definition of $F^{\alpha,q}_{p}(\gamma_d)$ does not depend on which $k > \alpha$ is chosen and the resulting norms are equivalent.

In [10] the notion of homogeneous Gaussian Besov–Lipschitz and homogeneous Gaussian Triebel–Lizorkin spaces were considered. Nevertheless the definitions of those spaces given there appear to be wrong in the case that $\alpha > 1$. On the other hand, Epperson [3] has considered Triebel–Lizorkin spaces with respect to the Hermite functions expansions which are different to the spaces that we are considering in this article related to Hermite polynomial expansions.

Let us observe that by the $L^p(\gamma_d)$-continuity of the Gaussian Littlewood–Paley g_1-function, see [9]

$$g_1(f)(x) = \left(\int_0^{\infty} t\left|\frac{\partial P_t f}{\partial t}\right|^2 \mathrm{d}t\right)^{1/2} \tag{33}$$

it is immediate to see that for $1 < p < \infty$

$$L^p(\gamma_d) = F^0_{p,2}(\gamma_d),$$

and by the trivial identification of the L^p spaces with the Hardy spaces, see [5], we have also

$$H^p(\gamma_d) = F^0_{p,2}(\gamma_d),$$

For Gaussian Triebel–Lizorkin spaces we have the following inclusion result, which is analogous to Proposition 2.1 (i),

Proposition 2.2. *The inclusion* $F^{\alpha_1}_{p,q_1}(\gamma_d) \subset F^{\alpha_2}_{p,q_2}(\gamma_d)$ *holds for* $\alpha_1 > \alpha_2 > 0$ *and* $q_1 \geq q_2$.

Proof. Let us consider $f \in F_p^{\alpha_1,q_1}(\gamma_d)$. Then

$$\begin{aligned}
&\left(\int_0^{+\infty}\left(t^{k-\alpha_2}\left|\frac{\partial^k P_t f(x)}{\partial t^k}\right|\right)^{q_2}\frac{dt}{t}\right)^{\frac{1}{q_2}}\\
&= \left(\int_0^{1}\left(t^{k-\alpha_2}\left|\frac{\partial^k P_t f(x)}{\partial t^k}\right|\right)^{q_2}\frac{dt}{t} + \int_1^{+\infty}\left(t^{k-\alpha_2}\left|\frac{\partial^k P_t f(x)}{\partial t^k}\right|\right)^{q_2}\frac{dt}{t}\right)^{\frac{1}{q_2}}\\
&\leq \left(\int_0^{1}\left(t^{k-\alpha_2}\left|\frac{\partial^k P_t f(x)}{\partial t^k}\right|\right)^{q_2}\frac{dt}{t}\right)^{\frac{1}{q_2}} + \left(\int_1^{+\infty}\left(t^{k-\alpha_2}\left|\frac{\partial^k P_t f(x)}{\partial t^k}\right|\right)^{q_2}\frac{dt}{t}\right)^{\frac{1}{q_2}}\\
&= \mathrm{I} + \mathrm{II}.
\end{aligned}$$

Let us observe that for the first term I, the case $q_1 = q_2$ is immediate since as $t < 1$, $t^{k-\alpha_2} < t^{k-\alpha_1}$ and then

$$I^{q_2} \leq \int_0^{+\infty}\left(t^{k-\alpha_1}\left|\frac{\partial^k P_t f(x)}{\partial t^k}\right|\right)^{q_1}\frac{dt}{t}.$$

Now, in the case $q_1 > q_2$ taking $r = \frac{q_1}{q_2}$, $s = \frac{q_1}{q_1-q_2}$ then $r, s > 1$ and $\frac{1}{r} + \frac{1}{s} = 1$, then, by the Hölder inequality

$$\begin{aligned}
\mathrm{I}^{q_2} &= \int_0^1 t^{(\alpha_1-\alpha_2)q_2}\left(t^{k-\alpha_1}\left|\frac{\partial^k P_t f(x)}{\partial t^k}\right|\right)^{q_2}\frac{dt}{t}\\
&\leq \left(\int_0^1 t^{(\alpha_1-\alpha_2)q_2 s}\frac{dt}{t}\right)^{\frac{1}{s}}\left(\int_0^1\left(t^{k-\alpha_1}\left|\frac{\partial^k P_t f(x)}{\partial t^k}\right|\right)^{q_2 r}\frac{dt}{t}\right)^{\frac{1}{r}}\\
&= \frac{1}{(\alpha_1-\alpha_2)q_2 s}\left(\int_0^1\left(t^{k-\alpha_1}\left|\frac{\partial^k P_t f(x)}{\partial t^k}\right|\right)^{q_1}\frac{dt}{t}\right)^{\frac{q_2}{q_1}}\\
&\leq C\left(\int_0^{+\infty}\left(t^{k-\alpha_1}\left|\frac{\partial^k P_t f(x)}{\partial t^k}\right|\right)^{q_1}\frac{dt}{t}\right)^{\frac{q_2}{q_1}}.
\end{aligned}$$

Now for the second term II, using Lemma 2.1, we have

$$\begin{aligned}
\mathrm{II} &= \left(\int_1^{+\infty}\left(t^{k-\alpha_2}\left|\frac{\partial^k P_t f(x)}{\partial t^k}\right|\right)^{q_2}\frac{dt}{t}\right)^{\frac{1}{q_2}}\\
&\leq C\,T^* f(x)\left(\int_1^{+\infty}\left(t^{k-\alpha_2}t^{-k}\right)^{q_2}\frac{dt}{t}\right)^{\frac{1}{q_2}}\\
&= CT^* f(x)\left(\int_1^{+\infty} t^{-\alpha_2 q_2}\frac{dt}{t}\right)^{\frac{1}{q_2}} = C\,T^* f(x).
\end{aligned}$$

Then, using the $L^p(\gamma_d)$-continuity of T^*, we get

$$\begin{aligned}
&\left\| \left(\int_0^{+\infty} \left(t^{k-\alpha_2} \left| \frac{\partial^k P_t f}{\partial t^k} \right| \right)^{q_2} \frac{dt}{t} \right)^{\frac{1}{q_2}} \right\|_{p,\gamma_d} \\
&\quad \leq C \left\| \left(\int_0^{+\infty} \left(t^{k-\alpha_1} \left| \frac{\partial^k P_t f}{\partial t^k} \right| \right)^{q_1} \frac{dt}{t} \right)^{\frac{1}{q_1}} \right\|_{p,\gamma_d} + C \|T^* f\|_{p,\gamma_d} \\
&\quad \leq C \left[\left\| \left(\int_0^{+\infty} \left(t^{k-\alpha_1} \left| \frac{\partial^k P_t f}{\partial t^k} \right| \right)^{q_1} \frac{dt}{t} \right)^{\frac{1}{q_1}} \right\|_{p,\gamma_d} + \|f\|_{p,\gamma_d} \right] < +\infty,
\end{aligned}$$

Thus, $f \in F_p^{\alpha_2,q_2}(\gamma_d)$. □

Let us observe that the Gaussian Besov–Lipschitz spaces and the Gaussian Triebel–Lizorkin spaces are by construction subspaces of $L^p(\gamma_d)$. Moreover since trivially $\|f\|_{p,\gamma_d} \leq \|f\|_{B^\alpha_{p,q}}$ and $\|f\|_{p,\gamma_d} \leq \|f\|_{F^\alpha_{p,q}}$ the inclusions are continuous. On the other hand, from (11) it is clear that for all $t > 0$ and $k \in \mathbb{N}$,

$$\frac{\partial^k}{\partial t^k} P_t h_\beta(x) = (-1)^k |\beta|^{k/2} e^{-t\sqrt{|\beta|}} h_\beta(x),$$

and therefore

$$\begin{aligned}
&\left(\int_0^{+\infty} \left(t^{k-\alpha} \left\| \frac{\partial^k}{\partial t^k} P_t h_\beta \right\|_{p,\gamma} \right)^q \frac{dt}{t} \right)^{1/q} \\
&\quad = \left(\int_0^{+\infty} \left(t^{k-\alpha} \|(-|\beta|^{1/2})^k e^{-t\sqrt{|\beta|}} h_\beta\|_{p,\gamma} \right)^q \frac{dt}{t} \right)^{1/q} \\
&\quad = |\beta|^{k/2} \left(\int_0^{+\infty} t^{(k-\alpha)q} e^{-t\sqrt{|\beta|}q} \frac{dt}{t} \right)^{1/q} \|h_\beta\|_{p,\gamma} \\
&\quad = \frac{|\beta|^{\alpha/2}}{q^{k-\alpha}} (\Gamma((k-\alpha)q))^{1/q} \|h_\beta\|_{p,\gamma} < \infty.
\end{aligned}$$

Thus $h_\beta \in B^\alpha_{p,q}(\gamma_d)$ and

$$\|h_\beta\|_{B^\alpha_{p,q}} = \left(1 + \frac{|\beta|^{\alpha/2}}{q^{k-\alpha}} (\Gamma((k-\alpha)q))^{1/q} \right) \|h_\beta\|_{p,\gamma}.$$

Similarly, $h_\beta \in F^\alpha_{p,q}(\gamma_d)$ and

$$\begin{aligned}
\|h_\beta\|_{F^\alpha_{p,q}} &= \|h_\beta\|_{p,\gamma} + \left\| \left(\int_0^{+\infty} \left(t^{k-\alpha} \left| \frac{\partial^k}{\partial t^k} P_t h_\beta(x) \right| \right)^q \frac{dt}{t} \right)^{1/q} \right\|_{p,\gamma} \\
&= \left(1 + \frac{|\beta|^{\alpha/2}}{q^{k-\alpha}} (\Gamma((k-\alpha)q))^{1/q} \right) \|h_\beta\|_{p,\gamma} = \|h_\beta\|_{B^\alpha_{p,q}}.
\end{aligned}$$

Therefore, the set of polynomials $\mathcal{P}$ is included in $B^\alpha_{p,q}(\gamma_d)$ and in $F^\alpha_{p,q}(\gamma_d)$.

Also we have the following inclusion relations between Gaussian Triebel–Lizorkin spaces and Gaussian Besov–Lipschitz spaces,

Proposition 2.3. *Let $\alpha \geq 0$ and $p, q > 1$*

(i) *If $p = q$ then*

$$F^{\alpha}_{p,p}(\gamma_d) = B^{\alpha}_{p,p}(\gamma_d).$$

(ii) *If $q > p$ then*

$$F^{\alpha}_{p,q}(\gamma_d) \subset B^{\alpha}_{p,q}(\gamma_d).$$

(iii) *If $p > q$ then*

$$B^{\alpha}_{p,q}(\gamma_d) \subset F^{\alpha}_{p,q}(\gamma_d).$$

Proof. (i) Using Tonelli's theorem, we trivially have

$$\begin{aligned}
&\left\| \left(\int_0^{+\infty} \left(t^{k-\alpha} \left| \frac{\partial^k P_t f}{\partial t^k} \right| \right)^p \frac{\mathrm{d}t}{t} \right)^{\frac{1}{p}} \right\|_{p,\gamma_d} \\
&= \left(\int_0^{+\infty} t^{(k-\alpha)p} \int_{\mathbb{R}^d} \left| \frac{\partial^k P_t f(x)}{\partial t^k} \right|^p \gamma_d(\mathrm{d}x) \frac{\mathrm{d}t}{t} \right)^{\frac{1}{p}} \\
&= \left(\int_0^{+\infty} \left(t^{k-\alpha} \left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_p \right)^p \frac{\mathrm{d}t}{t} \right)^{\frac{1}{p}}.
\end{aligned}$$

(ii) Suppose $q > p$, by Minkowski's integral inequality we have,

$$\begin{aligned}
&\left(\int_0^{\infty} \left(t^{k-\alpha} \left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{p,\gamma_d} \right)^q \frac{\mathrm{d}t}{t} \right)^{p/q} \\
&= \left(\int_0^{\infty} t^{(k-\alpha)q} \left(\int_{\mathbb{R}^d} \left| \frac{\partial^k P_t f(x)}{\partial t^k} \right|^p \gamma_d(\mathrm{d}x) \right)^{q/p} \frac{\mathrm{d}t}{t} \right)^{p/q} \\
&\leq \int_{\mathbb{R}^d} \left(\int_0^{\infty} \left(t^{k-\alpha} \left| \frac{\partial^k P_t f(x)}{\partial t^k} \right| \right)^q \frac{\mathrm{d}t}{t} \right)^{p/q} \gamma_d(\mathrm{d}x).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|f\|_{B^{\alpha}_{p,q}} &= \|f\|_{p,\gamma_d} + \left(\int_0^{\infty} \left(t^{k-\alpha} \left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{p,\gamma_d} \right)^q \frac{\mathrm{d}t}{t} \right)^{1/q} \\
&\leq \|f\|_{p,\gamma_d} + \left\| \left(\int_0^{\infty} \left(t^{k-\alpha} \left| \frac{\partial^k P_t f}{\partial t^k} \right| \right)^q \frac{\mathrm{d}t}{t} \right)^{1/q} \right\|_{p,\gamma_d} = \|f\|_{F^{\alpha}_{p,q}}.
\end{aligned}$$

(iii) Finally, if $p > q$, using again Minkowski's integral inequality, we

$$\|f\|_{F^{\alpha}_{p,q}} = \|f\|_{p,\gamma_d} + \left\| \left(\int_0^{\infty} \left(t^{k-\alpha} \left| \frac{\partial^k P_t f}{\partial t^k} \right| \right)^q \frac{\mathrm{d}t}{t} \right)^{1/q} \right\|_{p,\gamma_d}$$

$$\leq \|f\|_{p,\gamma_d} + \left(\int_0^\infty \left(t^{k-\alpha}\left\|\frac{\partial^k P_t f}{\partial t^k}\right\|_{p,\gamma_d}\right)^q \frac{dt}{t}\right)^{1/q} = \|f\|_{B^\alpha_{p,q}}. \quad \square$$

Let us prove now that the Gaussian Sobolev spaces $L^p_\alpha(\gamma_d)$ are contained in some Besov–Lipschitz and Triebel–Lizorkin spaces, and therefore they are "finer scales" to measure the regularity of functions.

Theorem 2.1. *Let us suppose that* $1 < p < +\infty$ *and* $\alpha > 0$. *Then*

(i) $L^p_\alpha(\gamma_d) \subset F^\alpha_{p,2}(\gamma_d)$ *if* $p > 1$.
(ii) $L^p_\alpha(\gamma_d) \subset B^\alpha_{p,p}(\gamma_d) = F^\alpha_{p,p}(\gamma_d)$ *if* $p \geq 2$.
(iii) $L^p_\alpha(\gamma_d) \subset B^\alpha_{p,2}(\gamma_d)$ *if* $p \leq 2$.

Proof. (i). We have to consider two cases:

(1) If $\alpha \geq 1$. Suppose $h \in L^p_\alpha(\gamma_d)$ then $h = \mathcal{J}_\alpha f$, $f \in L^p(\gamma_d)$, by the change of variable $u = t+s$ using the fact the representation of the Bessel potentials (17) and Hardy's inequality to get,

$$\begin{aligned}
&\left(\int_0^{+\infty}\left(t^{k-\alpha}\left|\frac{\partial^k P_t h(x)}{\partial t^k}\right|\right)^2 \frac{dt}{t}\right)^{\frac12} \\
&= \left(\int_0^{+\infty} t^{2(k-\alpha)}\left|\frac{\partial^k P_t \mathcal{J}_\alpha f(x)}{\partial t^k}\right|^2 \frac{dt}{t}\right)^{\frac12} \\
&\leq \frac{1}{\Gamma(\alpha)}\left(\int_0^{+\infty} t^{2(k-\alpha)}\left(\int_0^{+\infty} s^\alpha e^{-s}\left|\frac{\partial^k P_{t+s} f(x)}{\partial (t+s)^k}\right|\frac{ds}{s}\right)^2\frac{dt}{t}\right)^{\frac12} \\
&= \frac{1}{\Gamma(\alpha)}\left(\int_0^{+\infty} t^{2(k-\alpha)}\left(\int_t^{+\infty} (u-t)^{\alpha-1} e^{t-u}\left|\frac{\partial^k P_u f(x)}{\partial u^k}\right| du\right)^2\frac{dt}{t}\right)^{\frac12} \\
&\leq \frac{1}{\Gamma(\alpha)}\left(\int_0^{+\infty}\left(\int_t^{+\infty} u^{\alpha-1}\left|\frac{\partial^k P_u f(x)}{\partial u^k}\right| du\right)^2 t^{2(k-\alpha)-1} dt\right)^{\frac12} \\
&\leq \frac{1}{\Gamma(\alpha)}\frac{1}{k-\alpha}\left(\int_0^{+\infty}\left(u^k\left|\frac{\partial^k P_u f(x)}{\partial u^k}\right|\right)^2\frac{du}{u}\right)^{\frac12}.
\end{aligned}$$

Hence, by the $L^p(\gamma_d)$-continuity of the Gaussian Littlewood–Paley g_k-function, see [4]

$$\begin{aligned}
&\left\|\left(\int_0^{+\infty}\left(t^{k-\alpha}\left|\frac{\partial^k P_t h}{\partial t^k}\right|\right)^2\frac{dt}{t}\right)^{\frac12}\right\|_{p,\gamma} \\
&\leq \frac{1}{\Gamma(\alpha)}\frac{1}{k-\alpha}\left\|\left(\int_0^{+\infty}\left(u^k\left|\frac{\partial^k P_u f}{\partial u^k}\right|\right)^2\frac{du}{u}\right)^{\frac12}\right\|_{p,\gamma} \\
&= C_{k,\alpha}\|g_k f\|_{p,\gamma} \leq C_{k,\alpha}\|f\|_{p,\gamma} = C_{k,\alpha}\|h\|_{p,\alpha},
\end{aligned}$$

thus $h \in F^\alpha_{p,2}(\gamma_d)$.

(2) If $0 < \alpha < 1$. Suppose $h \in L^p_\alpha(\gamma_d)$, then $h = \mathcal{J}_\alpha f$, $f \in L^p(\gamma_d)$, again using (17),

$$
\begin{aligned}
&\left(\int_0^{+\infty}\left(t^{k-\alpha}\left|\frac{\partial^k P_t h(x)}{\partial t^k}\right|\right)^2\frac{dt}{t}\right)^{\frac12}\\
&\le \frac{1}{\Gamma(\alpha)}\left(\int_0^{+\infty} t^{2(k-\alpha)}\left(\int_0^{+\infty} s^\alpha e^{-s}\left|\frac{\partial^k P_{t+s} f(x)}{\partial(t+s)^k}\right|\frac{ds}{s}\right)^2\frac{dt}{t}\right)^{\frac12}\\
&\le \frac{C}{\Gamma(\alpha)}\left(\int_0^{+\infty} t^{2(k-\alpha)-1}\left[\left(\int_0^t s^\alpha e^{-s}\left|\frac{\partial^k P_{t+s} f(x)}{\partial(t+s)^k}\right|\frac{ds}{s}\right)^2\right.\right.\\
&\quad+\left.\left.\left(\int_t^{+\infty} s^\alpha e^{-s}\left|\frac{\partial^k P_{t+s} f(x)}{\partial(t+s)^k}\right|\frac{ds}{s}\right)^2\right]dt\right)^{\frac12}\\
&\le \frac{C}{\Gamma(\alpha)}\left(\int_0^{+\infty} t^{2(k-\alpha)-1}\left(\int_0^t s^{\alpha-1} e^{-s}\left|\frac{\partial^k P_{t+s} f(x)}{\partial(t+s)^k}\right|ds\right)^2 dt\right)^{\frac12}\\
&\quad+\frac{C}{\Gamma(\alpha)}\left(\int_0^{+\infty} t^{2(k-\alpha)-1}\left(\int_t^{+\infty} s^{\alpha-1} e^{-s}\left|\frac{\partial^k P_{t+s} f(x)}{\partial(t+s)^k}\right|ds\right)^2 dt\right)^{\frac12}\\
&= \mathrm{I}+\mathrm{II}.
\end{aligned}
$$

Now, since $e^{-s} < 1$, $s^{\alpha-1} < t^{\alpha-1}$ as $\alpha < 1$, and using the change of variables $u = t + s$ and Hardy inequality we get,

$$
\begin{aligned}
\mathrm{II} &\le \left(\int_0^{+\infty} t^{2(k-1)-1}\left(\int_t^{+\infty}\left|\frac{\partial^k P_{t+s} f(x)}{\partial(t+s)^k}\right|ds\right)^2 dt\right)^{\frac12}\\
&= \left(\int_0^{+\infty} t^{2(k-1)-1}\left(\int_{2t}^{+\infty}\left|\frac{\partial^k P_u f(x)}{\partial u^k}\right|du\right)^2 dt\right)^{\frac12}\\
&\le \left(\int_0^{+\infty} t^{2(k-1)-1}\left(\int_t^{+\infty}\left|\frac{\partial^k P_u f(x)}{\partial u^k}\right|du\right)^2 dt\right)^{\frac12}\\
&\le \left(\int_0^{+\infty}\left(u\left|\frac{\partial^k P_u f(x)}{\partial u^k}\right|\right)^2 u^{2(k-1)-1}du\right)^{\frac12}\\
&= \left(\int_0^{+\infty}\left(\left|u^k\frac{\partial^k P_u f(x)}{\partial u^k}\right|\right)^2\frac{du}{u}\right)^{\frac12} = g_k f(x).
\end{aligned}
$$

On the other hand, again since $e^{-s} < 1$,

$$
\begin{aligned}
I^2 &\le \int_0^{+\infty} t^{2(k-\alpha)-1}\left(\int_0^t s^{\alpha-1}\left|\frac{\partial^k P_{t+s} f(x)}{\partial(t+s)^k}\right|ds\right)^2 dt\\
&= \frac{1}{\alpha^2}\int_0^{+\infty} t^{2k-1}\left(\frac{\alpha}{t^\alpha}\int_0^t s^{\alpha-1}\left|\frac{\partial^k P_{t+s} f(x)}{\partial(t+s)^k}\right|ds\right)^2 dt.
\end{aligned}
$$

Now, as $\alpha > 0$ using Jensen's inequality for the measure $\frac{\alpha}{t^\alpha} s^{\alpha-1} \mathrm{d}s$ and Tonelli's Theorem,

$$I^2 \leq \frac{1}{\alpha^2} \int_0^{+\infty} t^{2k-1} \left(\frac{\alpha}{t^\alpha} \int_0^t s^{\alpha-1} \left| \frac{\partial^k P_{t+s} f(x)}{\partial (t+s)^k} \right|^2 \mathrm{d}s \right) \mathrm{d}t$$
$$\leq \frac{1}{\alpha} \int_0^{+\infty} s^{\alpha-1} \left(\int_s^{+\infty} (t+s)^{2k-\alpha-1} \left| \frac{\partial^k P_{t+s} f(x)}{\partial (t+s)^k} \right|^2 \mathrm{d}t \right) \mathrm{d}s,$$

since $2k - \alpha - 1 > 0$. Finally, again using the change of variables $u = t + s$ and the Hardy inequality

$$I^2 \leq \frac{1}{\alpha} \int_0^{+\infty} s^{\alpha-1} \left(\int_{2s}^{+\infty} u^{2k-\alpha-1} \left| \frac{\partial^k P_u f(x)}{\partial u^k} \right|^2 \mathrm{d}u \right) \mathrm{d}s$$
$$\leq \frac{1}{\alpha} \int_0^{+\infty} s^{\alpha-1} \left(\int_{s}^{+\infty} u^{2k-\alpha-1} \left| \frac{\partial^k P_u f(x)}{\partial u^k} \right|^2 \mathrm{d}u \right) \mathrm{d}s$$
$$\leq \frac{1}{\alpha} \int_0^{+\infty} \left(u^k \left| \frac{\partial^k P_u f(x)}{\partial u^k} \right| \right)^2 \frac{\mathrm{d}u}{u} = \frac{1}{\alpha} g_k^2 f(x).$$

Hence, again by the $L^p(\gamma_d)$-continuity of the Gaussian Littlewood–Paley g_k-function,

$$\left\| \left(\int_0^{+\infty} \left(t^{k-\alpha} \left| \frac{\partial^k P_t h}{\partial t^k} \right| \right)^2 \frac{\mathrm{d}t}{t} \right)^{\frac{1}{2}} \right\|_{p,\gamma} \leq C_{k,\alpha} \| g_k f \|_{p,\gamma}$$
$$\leq C_{k,\alpha} \| f \|_{p,\gamma} = C_{k,\alpha} \| h \|_{p,\alpha}.$$

Thus $h \in F^\alpha_{p,2}(\gamma_d)$, for $0 < \alpha < 1$.

Let us prove now (ii). Suppose $h \in L^p_\alpha(\gamma_d)$ with $p \geq 2$ then $h = \mathcal{J}_\alpha f$, $f \in L^p(\gamma_d)$. Using the inequality $(a+b)^p \leq C_p(a^p + b^p)$ if $a, b \geq 0$, $p \geq 1$ we get

$$\left(\int_0^{+\infty} \left(t^{k-\alpha} \left\| \frac{\partial^k P_t \mathcal{J}_\alpha f}{\partial t^k} \right\|_{p,\gamma_d} \right)^p \frac{\mathrm{d}t}{t} \right)^{\frac{1}{p}}$$
$$\leq \frac{1}{\Gamma(\alpha)} \left(\int_0^{+\infty} \left(t^{k-\alpha} \int_0^{+\infty} s^\alpha \mathrm{e}^{-s} \left\| \frac{\partial^k P_{t+s} f}{\partial (t+s)^k} \right\|_{p,\gamma} \frac{\mathrm{d}s}{s} \right)^p \frac{\mathrm{d}t}{t} \right)^{\frac{1}{p}}$$
$$\leq \frac{C}{\Gamma(\alpha)} \left(\int_0^{+\infty} t^{p(k-\alpha)} \left(\int_0^t s^\alpha \left\| \frac{\partial^k P_{s+t} f}{\partial (s+t)^k} \right\|_{p,\gamma} \frac{\mathrm{d}s}{s} \right)^p \right.$$
$$\left. + \left(\int_t^{+\infty} s^\alpha \left\| \frac{\partial^k P_{s+t} f}{\partial (s+t)^k} \right\|_{p,\gamma} \frac{\mathrm{d}s}{s} \right)^p \frac{\mathrm{d}t}{t} \right)^{\frac{1}{p}}.$$

Using the inequality $(a+b)^{1/p} \leq a^{1/p} + b^{1/p}$ if $a, b \geq 0$, $p \geq 1$

$$\frac{C}{\Gamma(\alpha)} \left(\int_0^{+\infty} t^{p(k-\alpha)} \left(\int_0^t s^\alpha \left\| \frac{\partial^k P_{s+t} f}{\partial (s+t)^k} \right\|_{p,\gamma} \frac{\mathrm{d}s}{s} \right)^p \right.$$

$$
\begin{aligned}
&+ \left(\int_t^{+\infty} s^{\alpha} \left\| \frac{\partial^k P_{s+t} f}{\partial (s+t)^k} \right\|_{p,\gamma} \frac{\mathrm{d}s}{s} \right)^p \frac{\mathrm{d}t}{t} \Bigg)^{\frac{1}{p}} \\
&\leq \frac{C}{\Gamma(\alpha)} \left(\int_0^{+\infty} t^{(k-\alpha)p} \left(\int_0^t s^{\alpha} \left\| \frac{\partial^k P_{s+t} f}{\partial (s+t)^k} \right\|_{p,\gamma} \frac{\mathrm{d}s}{s} \right)^p \frac{\mathrm{d}t}{t} \right)^{\frac{1}{p}} \\
&+ \frac{C}{\Gamma(\alpha)} \left(\int_0^{+\infty} t^{(k-\alpha)p} \left(\int_t^{+\infty} s^{\alpha} \left\| \frac{\partial^k P_{s+t} f}{\partial (s+t)^k} \right\|_{p,\gamma} \frac{\mathrm{d}s}{s} \right)^p \frac{\mathrm{d}t}{t} \right)^{\frac{1}{p}} \\
&= \mathrm{I} + \mathrm{II}.
\end{aligned}
$$

Now, using again the Hardy's inequality, since $k > \alpha$ and Lemma 2.4

$$
\begin{aligned}
\mathrm{II} &= \frac{C}{\Gamma(\alpha)} \left(\int_0^{+\infty} t^{p(k-\alpha)} \left(\int_t^{+\infty} s^{\alpha} \left\| \frac{\partial^k P_{s+t} f}{\partial (s+t)^k} \right\|_{p,\gamma} \frac{\mathrm{d}s}{s} \right)^p \frac{\mathrm{d}t}{t} \right)^{\frac{1}{p}} \\
&\leq \frac{C}{\Gamma(\alpha)} \left(\int_0^{+\infty} t^{p(k-\alpha)} \left(\int_t^{+\infty} s^{\alpha} \left\| \frac{\partial^k P_s f}{\partial s^k} \right\|_{p,\gamma} \frac{\mathrm{d}s}{s} \right)^p \frac{\mathrm{d}t}{t} \right)^{\frac{1}{p}} \\
&\leq \frac{C}{\Gamma(\alpha)} \frac{1}{k-\alpha} \left(\int_0^{+\infty} \left(s^{\alpha} \left\| \frac{\partial^k}{\partial s^k} P_s f \right\|_{p,\gamma} \right)^p s^{(k-\alpha)p-1} \mathrm{d}s \right)^{\frac{1}{p}} \\
&= C_{k,\alpha} \left(\int_0^{+\infty} \left(s^k \left\| \frac{\partial^k}{\partial s^k} P_s f \right\|_{p,\gamma} \right)^p \frac{\mathrm{d}s}{s} \right)^{\frac{1}{p}} \\
&= C_{k,\alpha} \left\| \left(\int_0^{+\infty} |s^k \frac{\partial^k P_s f}{\partial s^k}|^p \frac{\mathrm{d}s}{s} \right)^{\frac{1}{p}} \right\|_p,
\end{aligned}
$$

by Tonelli's Theorem.

Now since $p \geq 2$ using Lemma 2.1, we have

$$
\begin{aligned}
\int_0^{+\infty} \left| u^k \frac{\partial^k P_u f(x)}{\partial u^k} \right|^p \frac{\mathrm{d}u}{u} &= \int_0^{+\infty} \left(u^k \left| \frac{\partial^k}{\partial u^k} P_u f(x) \right| \right)^{p-2} \left(u^k \left| \frac{\partial^k}{\partial u^k} P_u f(x) \right| \right)^2 \frac{\mathrm{d}u}{u} \\
&\leq C \left(T^* f(x) \right)^{p-2} \int_0^{+\infty} \left(u^k \left| \frac{\partial^k}{\partial u^k} P_u f(x) \right| \right)^2 \frac{\mathrm{d}u}{u}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\left\| \left(\int_0^{+\infty} \left| u^k \frac{\partial^k P_u f}{\partial u^k} \right|^p \frac{\mathrm{d}u}{u} \right)^{\frac{1}{p}} \right\|_p^p \\
&= \int_{\mathbb{R}^d} \left(\int_0^{+\infty} \left| u^k \frac{\partial^k P_u f(x)}{\partial u^k} \right|^p \frac{\mathrm{d}u}{u} \right) \gamma_d(\mathrm{d}x) \\
&\leq C \int_{\mathbb{R}^d} \left(\left(T^* f(x) \right)^{p-2} \int_0^{+\infty} \left(u^k \left| \frac{\partial^k P_u f(x)}{\partial u^k} \right| \right)^2 \frac{\mathrm{d}u}{u} \right) \gamma_d(\mathrm{d}x).
\end{aligned}
$$

Using the Hölder inequality, with $\theta = \frac{2}{p}$, and the $L^p(\gamma_d)$-continuity of T^* and g_k, we have

$$\begin{aligned}
&\left\| \left(\int_0^{+\infty} \left| u^k \frac{\partial^k P_u f}{\partial u^k} \right|^p \frac{\mathrm{d}u}{u} \right)^{\frac{1}{p}} \right\|_p^p \\
&\leq C \int_{\mathbb{R}^d} \left(\left(T^* f(x)\right)^{p-2} \int_0^{+\infty} \left(u^k \left| \frac{\partial^k P_u f(x)}{\partial u^k} \right| \right)^2 \frac{\mathrm{d}u}{u} \right) \gamma_d(\mathrm{d}x) \\
&\leq C \int_{\mathbb{R}^d} \left(\left(T^* f(x)\right)^{(p-2)\cdot \frac{1}{1-\theta}} \gamma_d(\mathrm{d}x) \right)^{1-\theta} \\
&\quad \times \left(\int_{\mathbb{R}^d} \left(\int_0^{+\infty} \left(u^k \left| \frac{\partial^k}{\partial u^k} P_u f(x) \right| \right)^2 \frac{\mathrm{d}u}{u} \right)^{\frac{1}{\theta}} \gamma_d(\mathrm{d}x) \right)^{\theta} \\
&= C \int_{\mathbb{R}^d} \left(\left(T^* f(x)\right)^p \gamma_d(\mathrm{d}x) \right)^{\frac{p-2}{p}} \\
&\quad \times \left(\int_{\mathbb{R}^d} \left(\int_0^{+\infty} \left(u^k \left| \frac{\partial^k}{\partial u^k} P_u f(x) \right| \right)^2 \frac{\mathrm{d}u}{u} \right)^{\frac{p}{2}} \gamma_d(\mathrm{d}x) \right)^{\frac{2}{p}} \\
&= C \|T^* f\|_{p,\gamma}^{p-2} \|g_k f\|_{p,\gamma}^2 \leq C \|f\|_{p,\gamma}^p.
\end{aligned}$$

Thus,

$$\mathrm{II} \leq C_{k,\alpha} \|h\|_{p,\alpha}.$$

Now, using again Lemma 2.4 and since $\alpha > 0$

$$\begin{aligned}
\mathrm{I} &= \frac{C}{\Gamma(\alpha)} \left(\int_0^{+\infty} t^{p(k-\alpha)} \left(\int_0^t s^\alpha \left\| \frac{\partial^k}{\partial (s+t)^k} P_{s+t} f \right\|_{p,\gamma} \frac{\mathrm{d}s}{s} \right)^p \frac{\mathrm{d}t}{t} \right)^{\frac{1}{p}} \\
&\leq \frac{C}{\Gamma(\alpha)} \left(\int_0^{+\infty} t^{p(k-\alpha)} \left(\int_0^t s^\alpha \left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{p,\gamma} \frac{\mathrm{d}s}{s} \right)^p \frac{\mathrm{d}t}{t} \right)^{\frac{1}{p}} \\
&= \frac{1}{\alpha} \frac{C}{\Gamma(\alpha)} \left(\int_0^{+\infty} t^k \left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{p,\gamma}^p \frac{\mathrm{d}t}{t} \right)^{\frac{1}{p}} \leq C_{k,\alpha} \|h\|_{p,\alpha}.
\end{aligned}$$

So $h \in B^\alpha_{p,p}(\gamma_d)$, if $p \geq 2$.

(iii) can be proved using similar arguments as in (i) and (ii) but it is immediate consequences of (i) and of Proposition 2.3(ii). □

In [10], using Theorem 3.2, it is claimed that the Gaussian Sobolev spaces $L^p_\alpha(\gamma_d)$ coincide with the homogeneous Gaussian Triebel–Lizorkin $\dot{F}^\alpha_{p,2}$ but the proof of that theorem is wrong since it is assumed that the operator involved is linear, but it is actually only sublinear. We suspect that, as in the case of homogeneous spaces see [8], the equality in (i) is actually true, but we have not been able to prove it so far.

Now, let us prove some interpolation results for the Gaussian Besov–Lipschitz spaces and for the Gaussian Triebel–Lizorkin spaces,

Theorem 2.2. *We have the following interpolation results:*

(i) *For* $1 < p_j, q_j < +\infty$ *and* $\alpha_j \geq 0$, *if* $f \in B^{\alpha_j}_{p_j,q_j}(\gamma_d)$, $j = 0, 1$, *then* $f \in B^{\alpha}_{p,q}(\gamma_d)$, *where* $\alpha = \alpha_0(1-\theta) + \alpha_1\theta$, *and*

$$\frac{1}{p} = \frac{1}{p_0}(1-\theta) + \frac{\theta}{p_1}, \qquad \frac{1}{q} = \frac{1}{q_0}(1-\theta) + \frac{\theta}{q_1}, \quad 0 < \theta < 1.$$

(ii) *For* $1 < p_j, q_j < +\infty$ *and* $\alpha_j \geq 0$, *if* $f \in F^{\alpha_j}_{p_j,q_j}(\gamma_d)$, $j = 0, 1$, *then* $f \in F^{\alpha}_{p,q}(\gamma_d)$, *where* $\alpha = \alpha_0(1-\theta) + \alpha_1\theta$, *and*

$$\frac{1}{p} = \frac{1}{p_0}(1-\theta) + \frac{\theta}{p_1}, \qquad \frac{1}{q} = \frac{1}{q_0}(1-\theta) + \frac{\theta}{q_1}, \quad 0 < \theta < 1.$$

Proof. The proofs of both results are based on the following interpolation result for $L^p(\gamma_d)$ spaces (actually true for any measure μ) that is obtained using the Hölder inequality:

For $1 < r_0, r_1 < \infty$ and $\frac{1}{r} = \frac{1}{r_0}(1-\eta) + \frac{\eta}{r_1}, 0 < \eta < 1$. If $f \in L^{r_j}(\gamma_d)$, $j = 0, 1$ then $f \in L^r(\gamma_d)$ and

$$\|f\|_{r,\gamma_d} \leq \|f\|^{1-\eta}_{r_0,\gamma_d} \|f\|^{\eta}_{r_1,\gamma_d}. \tag{34}$$

Let us prove (i). Let k be any integer greater than α_0 and α_1, by using the above result we get for $\alpha = \alpha_0(1-\theta) + \alpha_1\theta$,

$$\begin{aligned}
&\int_0^{+\infty} \left(t^{k-\alpha} \left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{p,\gamma_d} \right)^q \frac{\mathrm{d}t}{t} \\
&\leq \int_0^{+\infty} \left(t^{k-(\alpha_0(1-\theta)+\alpha_1\theta)} \left\| \frac{\partial^k P_t f}{\partial t^k} \right\|^{1-\theta}_{p_0,\gamma_d} \left\| \frac{\partial^k P_t f}{\partial t^k} \right\|^{\theta}_{p_1,\gamma_d} \right)^q \frac{\mathrm{d}t}{t} \\
&= \int_0^{+\infty} \left(t^{(1-\theta)(k-\alpha_0)+\theta(k-\alpha_1)} \left\| \frac{\partial^k P_t f}{\partial t^k} \right\|^{1-\theta}_{p_0,\gamma_d} \left\| \frac{\partial^k P_t f}{\partial t^k} \right\|^{\theta}_{p_1,\gamma_d} \right)^q \frac{\mathrm{d}t}{t} \\
&= \int_0^{+\infty} \left(t^{k-\alpha_0} \left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{p_0,\gamma_d} \right)^{(1-\theta)q} \left(t^{k-\alpha_1} \left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{p_1,\gamma_d} \right)^{\theta q} \frac{\mathrm{d}t}{t}.
\end{aligned}$$

Now, if $\lambda = \frac{\theta q}{q_1}$ then $0 < \lambda < 1$ and $q = (1-\lambda)q_0 + \lambda q_1$. Therefore by using again the Hölder inequality,

$$\begin{aligned}
&\int_0^{+\infty} \left(t^{k-\alpha} \left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{p,\gamma_d} \right)^q \frac{\mathrm{d}t}{t} \\
&\leq \left(\int_0^{+\infty} \left(t^{k-\alpha_0} \left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{p_0,\gamma_d} \right)^{q_0} \frac{\mathrm{d}t}{t} \right)^{1-\lambda} \left(\int_0^{+\infty} \left(t^{k-\alpha_1} \left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{p_1,\gamma_d} \right)^{q_1} \frac{\mathrm{d}t}{t} \right)^{\lambda} \\
&< \infty,
\end{aligned}$$

and so $f \in B^{\alpha}_{p,q}(\gamma_d)$.

(ii) Analogously, by taking $\beta = \frac{p\theta}{p_1}$, $\lambda = \frac{q\theta}{q_1}$, we have $0 < \beta, \lambda < 1$ and $p = (1-\beta)p_0 + \beta p_1$, $q = (1-\lambda)q_0 + \lambda q_1$. Let k be any integer greater than α_0 and α_1, by using the Hölder inequality we get for $\alpha = \alpha_0(1-\theta) + \alpha_1\theta$,

$$\begin{aligned}
&\int_0^{+\infty} \left(t^{k-\alpha} \left| \frac{\partial^k P_t f}{\partial t^k} \right| \right)^q \frac{\mathrm{d}t}{t} \\
&\quad = \int_0^{+\infty} \left(t^{k-\alpha_0} \left| \frac{\partial^k P_t f}{\partial t^k} \right| \right)^{(1-\theta)q} \left(t^{k-\alpha_1} \left| \frac{\partial^k P_t f}{\partial t^k} \right| \right)^{\theta q} \frac{\mathrm{d}t}{t} \\
&\quad = \int_0^{+\infty} \left(t^{k-\alpha_0} \left| \frac{\partial^k P_t f}{\partial t^k} \right| \right)^{(1-\lambda)q_0} \left(t^{k-\alpha_1} \left| \frac{\partial^k P_t f}{\partial t^k} \right| \right)^{\lambda q_1} \frac{\mathrm{d}t}{t} \\
&\quad \leq \left(\int_0^{+\infty} \left(t^{k-\alpha_0} \left| \frac{\partial^k P_t f}{\partial t^k} \right| \right)^{q_0} \frac{\mathrm{d}t}{t} \right)^{1-\lambda} \left(\int_0^{+\infty} \left(t^{k-\alpha_1} \left| \frac{\partial^k P_t f}{\partial t^k} \right| \right)^{q_1} \frac{\mathrm{d}t}{t} \right)^{\lambda}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\left\| \left(\int_0^{+\infty} \left(t^{k-\alpha} \left| \frac{\partial^k P_t f}{\partial t^k} \right| \right)^q \frac{\mathrm{d}t}{t} \right)^{\frac{1}{q}} \right\|_{p,\gamma_d}^p \\
&\quad = \int_{\mathbb{R}^d} \left(\int_0^{+\infty} \left(t^{k-\alpha} \left| \frac{\partial^k P_t f}{\partial t^k} \right| \right)^q \frac{\mathrm{d}t}{t} \right)^{\frac{p}{q}} \gamma_d(\mathrm{d}x) \\
&\quad \leq \int_{\mathbb{R}^d} \left(\int_0^{+\infty} \left(t^{k-\alpha_0} \left| \frac{\partial^k P_t f}{\partial t^k} \right| \right)^{q_0} \frac{\mathrm{d}t}{t} \right)^{\frac{(1-\lambda)p}{q}} \\
&\qquad \times \left(\int_0^{+\infty} \left(t^{k-\alpha_1} \left| \frac{\partial^k P_t f}{\partial t^k} \right| \right)^{q_1} \frac{\mathrm{d}t}{t} \right)^{\frac{\lambda p}{q}} \gamma_d(\mathrm{d}x) \\
&\quad = \int_{\mathbb{R}^d} \left(\int_0^{+\infty} \left(t^{k-\alpha_0} \left| \frac{\partial^k P_t f}{\partial t^k} \right| \right)^{q_0} \frac{\mathrm{d}t}{t} \right)^{\frac{(1-\theta)p}{q_0}} \\
&\qquad \times \left(\int_0^{+\infty} \left(t^{k-\alpha_1} \left| \frac{\partial^k P_t f}{\partial t^k} \right| \right)^{q_1} \frac{\mathrm{d}t}{t} \right)^{\frac{\theta p}{q_1}} \gamma_d(\mathrm{d}x) \\
&\quad = \int_{\mathbb{R}^d} \left(\int_0^{+\infty} \left(t^{k-\alpha_0} \left| \frac{\partial^k P_t f}{\partial t^k} \right| \right)^{q_0} \frac{\mathrm{d}t}{t} \right)^{\frac{(1-\beta)p_0}{q_0}} \\
&\qquad \times \left(\int_0^{+\infty} \left(t^{k-\alpha_1} \left| \frac{\partial^k P_t f}{\partial t^k} \right| \right)^{q_1} \frac{\mathrm{d}t}{t} \right)^{\frac{\beta p_1}{q_1}} \gamma_d(\mathrm{d}x),
\end{aligned}$$

and then again using the Hölder inequality,

$$\left\| \left(\int_0^{+\infty} \left(t^{k-\alpha} \left| \frac{\partial^k P_t f}{\partial t^k} \right| \right)^q \frac{\mathrm{d}t}{t} \right)^{\frac{1}{q}} \right\|_{p,\gamma_d}^p$$

$$\leq \left(\int_{\mathbb{R}^d}\left(\int_0^{+\infty}\left(t^{k-\alpha_0}\left|\frac{\partial^k P_t f}{\partial t^k}\right|\right)^{q_0}\frac{\mathrm{d}t}{t}\right)^{\frac{p_0}{q_0}}\gamma_d(\mathrm{d}x)\right)^{1-\beta}$$

$$\times\left(\int_{\mathbb{R}^d}\left(\int_0^{+\infty}\left(t^{k-\alpha_1}\left|\frac{\partial^k P_t f}{\partial t^k}\right|\right)^{q_1}\frac{\mathrm{d}t}{t}\right)^{\frac{p_1}{q_1}}\gamma_d(\mathrm{d}x)\right)^{\beta}$$

$$= \left\|\left(\int_0^{+\infty}\left(t^{k-\alpha_0}\left|\frac{\partial^k P_t f}{\partial t^k}\right|\right)^{q_0}\frac{\mathrm{d}t}{t}\right)^{\frac{1}{q_0}}\right\|_{p_0,\gamma_d}^{p_0(1-\beta)}$$

$$\times\left\|\left(\int_0^{+\infty}\left(t^{k-\alpha_1}\left|\frac{\partial^k P_t f}{\partial t^k}\right|\right)^{q_1}\frac{\mathrm{d}t}{t}\right)^{\frac{1}{q_1}}\right\|_{p_1,\gamma_d}^{p_1\beta} < +\infty.$$

Hence $f \in F^{\alpha}_{p,q}(\gamma_d)$. □

Now, we are going to study the continuity properties of the Ornstein–Uhlenbeck semigroup, the Poisson–Hermite semigroup and the Bessel potentials on the Besov–Lipschitz and Triebel–Lizorkin spaces.

Theorem 2.3. *For the Besov–Lipschitz spaces $B^{\alpha}_{p,q}(\gamma_d)$ and Triebel–Lizorkin spaces $F^{\alpha}_{p,q}(\gamma_d)$, we have*

(i) *The Ornstein–Uhlenbeck semigroup $\{T_t\}$ and the Poisson–Hermite semigroup $\{P_t\}$ are bounded on $B^{\alpha}_{p,q}(\gamma_d)$.*
(ii) *For every $\beta > 0$, the Bessel potentials $\mathcal{J}^{\gamma}_{\beta}$ are bounded on $B^{\alpha}_{p,q}(\gamma_d)$.*
(iii) *The Ornstein–Uhlenbeck semigroup $\{T_t\}$, the Poisson–Hermite semigroup $\{P_t\}$ are bounded on $F^{\alpha}_{p,q}$.*
(iv) *The Bessel potentials $\mathcal{J}^{\gamma}_{\beta}$ are bounded on $F^{\alpha}_{p,q}(\gamma_d)$.*

Proof. (i) Let us prove the $B^{\alpha}_{p,q}(\gamma_d)$-continuity of P_t for any $t > 0$, the proof for T_t is totally analogous. By the L^p-continuity of the Poisson–Hermite semigroup, Lebesgue's dominated convergence theorem and Jensen's inequality we get

$$\begin{aligned}\int_{\mathbb{R}^d}\left|\frac{\partial^k P_t(P_s f)}{\partial t^k}(x)\right|^p\gamma_d(\mathrm{d}x) &= \int_{\mathbb{R}^d}\left|P_s\left(\frac{\partial^k P_t f}{\partial t^k}\right)(x)\right|^p\gamma_d(\mathrm{d}x)\\ &\leq \int_{\mathbb{R}^d}P_s\left(\left|\frac{\partial^k P_t f(x)}{\partial t^k}\right|^p\right)\gamma_d(\mathrm{d}x)\\ &= \int_{\mathbb{R}^d}\left|\frac{\partial^k P_t f(x)}{\partial t^k}\right|^p\gamma_d(\mathrm{d}x).\end{aligned}$$

Thus,

$$\left\|\frac{\partial^k P_t(P_s f)}{\partial t^k}\right\|_{p,\gamma_d} \leq \left\|\frac{\partial^k P_t f}{\partial t^k}\right\|_{p,\gamma_d},$$

and therefore

$$\|P_s f\|_{B^{\alpha}_{p,q}} = \|P_s f\|_{p,\gamma_d} + \left(\int_0^{+\infty}\left(t^{k-\alpha}\left\|\frac{\partial^k P_t(P_s f)}{\partial t^k}\right\|_{p,\gamma_d}\right)^q\frac{\mathrm{d}t}{t}\right)^{1/q}$$

$$\leq \|f\|_{p,\gamma_d} + \left(\int_0^{+\infty} \left(t^{k-\alpha} \left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{p,\gamma_d}\right)^q \frac{\mathrm{d}t}{t}\right)^{1/q}$$
$$= \|f\|_{B^\alpha_{p,q}}.$$

(ii) Now let us see that $\mathcal{J}_\beta$ is bounded on $B^\alpha_{p,q}(\gamma_d)$. Using Lebesgue's dominated convergence theorem and Minkowski's integral inequality and Jensen's inequality, we have

$$\left\| \frac{\partial^k P_t}{\partial t^k} \left(\mathcal{J}_\beta f\right) \right\|^q_{p,\gamma_d} = \left(\int_{\mathbb{R}^d} \left| \frac{\partial^k P_t}{\partial t^k} \left(\frac{1}{\Gamma(\beta)} \int_0^{+\infty} s^\beta \mathrm{e}^{-s} P_s f(x) \frac{\mathrm{d}s}{s}\right)\right|^p \gamma_d(\mathrm{d}x)\right)^{\frac{q}{p}}$$
$$\leq \left(\frac{1}{\Gamma(\beta)} \int_0^{+\infty} s^\beta \mathrm{e}^{-s} \left(\int_{\mathbb{R}^d} \left|\frac{\partial^k P_t P_s f(x)}{\partial t^k}\right|^p \gamma_d(\mathrm{d}x)\right)^{\frac{1}{p}} \frac{\mathrm{d}s}{s}\right)^q$$
$$\leq \frac{1}{\Gamma(\beta)} \int_0^{+\infty} s^\beta \mathrm{e}^{-s} \left\| \frac{\partial^k P_t P_s f}{\partial t^k}\right\|^q_{p,\gamma_d} \frac{\mathrm{d}s}{s},$$

and then using Tonelli's Theorem,

$$\int_0^{+\infty} \left(t^{k-\alpha} \left\| \frac{\partial^k P_t}{\partial t^k} \left(\mathcal{J}_\beta f\right)\right\|_{p,\gamma_d}\right)^q \frac{\mathrm{d}t}{t}$$
$$\leq \frac{1}{\Gamma(\beta)} \int_0^{+\infty} s^\beta \mathrm{e}^{-s} \left(\int_0^{+\infty} \left(t^{k-\alpha} \left\| \frac{\partial^k P_t \left(P_s f\right)}{\partial t^k}\right\|_{p,\gamma_d}\right)^q \frac{\mathrm{d}t}{t}\right) \frac{\mathrm{d}s}{s}$$
$$\leq \frac{1}{\Gamma(\beta)} \int_0^{+\infty} s^\beta \mathrm{e}^{-s} \left(\int_0^{+\infty} \left(t^{k-\alpha} \left\| \frac{\partial^k P_t f}{\partial t^k}\right\|_{p,\gamma_d}\right)^q \frac{\mathrm{d}t}{t}\right) \frac{\mathrm{d}s}{s}$$
$$= \int_0^{+\infty} \left(t^{k-\alpha} \left\| \frac{\partial^k P_t f}{\partial t^k}\right\|_{p,\gamma_d}\right)^q \frac{\mathrm{d}t}{t}.$$

Therefore

$$\|\mathcal{J}_\beta f\|_{B^\alpha_{p,q}} = \|\mathcal{J}_\beta f\|_{p,\gamma_d} + \int_0^{+\infty} \left(t^{k-\alpha} \left\| \frac{\partial^k P_t}{\partial t^k} \left(\mathcal{J}_\beta f\right)\right\|_{p,\gamma_d}\right)^q \frac{\mathrm{d}t}{t}$$
$$\leq \|f\|_{p,\gamma_d} + \int_0^{+\infty} \left(t^{k-\alpha} \left\| \frac{\partial^k P_t f}{\partial t^k}\right\|_{p,\gamma_d}\right)^q \frac{\mathrm{d}t}{t} = \|f\|_{B^\alpha_{p,q}}.$$

(iii) Let us prove the $F^\alpha_{p,q}$-continuity of P_t for any $t > 0$, the proof for T_t is totally analogous. By Lebesgue's dominated convergence theorem and Minkowski's integral inequality, we have

$$\left(\int_0^\infty \left(s^{k-\alpha} \left| \frac{\partial^k P_t(P_s g)}{\partial s^k}(x)\right|\right)^q \frac{\mathrm{d}s}{s}\right)^{1/q}$$
$$= \left(\int_0^\infty \left(s^{k-\alpha} \left| \int_{\mathbb{R}^d} p(t,x,y) \frac{\partial^k P_s g(y)}{\partial s^k} \mathrm{d}y\right|\right)^q \frac{\mathrm{d}s}{s}\right)^{1/q}$$
$$\leq \int_{\mathbb{R}^d} p(t,x,y) \left(\int_0^\infty \left(s^{k-\alpha} \left| \frac{\partial^k P_s g(y)}{\partial s^k}\right|\right)^q \frac{\mathrm{d}s}{s}\right)^{1/q} \mathrm{d}y$$

$$= P_t\left(\left(\int_0^\infty \left(s^{k-\alpha}\left|\frac{\partial^k P_s g}{\partial s^k}\right|\right)^q \frac{ds}{s}\right)^{1/q}\right)(x).$$

Therefore, by the L^p-continuity of P_t we get

$$\left\|\left(\int_0^\infty \left(s^{k-\alpha}\left|\frac{\partial^k P_s(P_t g)}{\partial s^k}\right|\right)^q \frac{ds}{s}\right)^{1/q}\right\|_{p,\gamma_d}$$
$$\leq \left\|P_t\left(\left(\int_0^\infty \left(s^{k-\alpha}\left|\frac{\partial^k P_s g}{\partial s^k}\right|\right)^q \frac{ds}{s}\right)^{1/q}\right)\right\|_{p,\gamma_d}$$
$$\leq \left\|\left(\int_0^\infty \left(s^{k-\alpha}\left|\frac{\partial^k P_s g}{\partial s^k}\right|\right)^q \frac{ds}{s}\right)^{1/q}\right\|_{p,\gamma_d}.$$

Thus,

$$\|P_t g\|_{F^\alpha_{p,q}} = \|P_t g\|_{p,\gamma_d} + \left\|\left(\int_0^\infty \left(s^{k-\alpha}\left|\frac{\partial^k P_s(P_t g)}{\partial s^k}\right|\right)^q \frac{ds}{s}\right)^{1/q}\right\|_{p,\gamma_d}$$
$$\leq \|g\|_{p,\gamma_d} + \left\|\left(\int_0^\infty \left(s^{k-\alpha}\left|\frac{\partial^k P_s g}{\partial s^k}\right|\right)^q \frac{ds}{s}\right)^{1/q}\right\|_{p,\gamma_d}$$
$$= \|g\|_{F^\alpha_{p,q}}.$$

(iv) Now let us see that $\mathcal{J}_\beta$ is bounded on $F^\alpha_{p,q}(\gamma_d)$. By Lebesgue's dominated convergence theorem, Minkowski's integral inequality and (iii), we have

$$\left(\int_0^\infty \left(s^{k-\alpha}\left|\frac{\partial^k P_s}{\partial s^k}\left(\mathcal{J}^\gamma_\beta g\right)(x)\right|\right)^q \frac{ds}{s}\right)^{1/q}$$
$$= \left(\int_0^\infty \left(s^{k-\alpha}\left|\frac{\partial^k P_s}{\partial s^k}\left(\frac{1}{\Gamma(\beta)}\int_0^{+\infty} t^\beta e^{-t} P_t g(x)\frac{dt}{t}\right)\right|\right)^q \frac{ds}{s}\right)^{1/q}$$
$$\leq \frac{1}{\Gamma(\beta)}\int_0^{+\infty} t^\beta e^{-t}\left(\int_0^\infty \left(s^{k-\alpha}\left|\frac{\partial^k P_s(P_t g)}{\partial s^k}(x)\right|\right)^q \frac{ds}{s}\right)^{1/q}\frac{dt}{t},$$

then, again by Minkowski's integral inequality and (iii)

$$\left\|\left(\int_0^\infty \left(s^{k-\alpha}\left|\frac{\partial^k P_s}{\partial s^k}\left(\mathcal{J}^\gamma_\beta g\right)\right|\right)^q \frac{ds}{s}\right)^{1/q}\right\|_{p,\gamma_d}$$
$$\leq \left\|\frac{1}{\Gamma(\beta)}\int_0^{+\infty} t^\beta e^{-t}\left(\int_0^\infty \left(s^{k-\alpha}\left|\frac{\partial^k P_s(P_t g)}{\partial s^k}\right|\right)^q \frac{ds}{s}\right)^{1/q}\frac{dt}{t}\right\|_{p,\gamma_d}$$
$$\leq \frac{1}{\Gamma(\beta)}\int_0^{+\infty} t^\beta e^{-t}\left\|\left(\int_0^\infty \left(s^{k-\alpha}\left|\frac{\partial^k P_s(P_t g)}{\partial s^k}\right|\right)^q \frac{ds}{s}\right)^{1/q}\right\|_{p,\gamma_d}\frac{dt}{t}$$

$$\leq \frac{1}{\Gamma(\beta)} \int_0^{+\infty} t^\beta \mathrm{e}^{-t} \left\| \left(\int_0^\infty \left(s^{k-\alpha} \left| \frac{\partial^k P_s g}{\partial s^k} \right| \right)^q \frac{\mathrm{d}s}{s} \right)^{1/q} \right\|_{p,\gamma_d} \frac{\mathrm{d}t}{t}$$

$$= \left\| \left(\int_0^\infty \left(s^{k-\alpha} \left| \frac{\partial^k P_s g}{\partial s^k} \right| \right)^q \frac{\mathrm{d}s}{s} \right)^{1/q} \right\|_{p,\gamma_d} .$$

Thus

$$\|\mathcal{J}_\beta^\gamma g\|_{F^\alpha_{p,q}} = \|\mathcal{J}_\beta^\gamma g\|_{p,\gamma_d} + \left\| \left(\int_0^\infty \left(s^{k-\alpha} \left| \frac{\partial^k P_s}{\partial s^k} \left(\mathcal{J}_\beta^\gamma g \right) \right| \right)^q \frac{\mathrm{d}s}{s} \right)^{1/q} \right\|_{p,\gamma_d}$$

$$\leq \|g\|_{p,\gamma_d} + \left\| \left(\int_0^\infty \left(s^{k-\alpha} \left| \frac{\partial^k P_s g}{\partial s^k} \right| \right)^q \frac{\mathrm{d}s}{s} \right)^{1/q} \right\|_{p,\gamma_d} = \|g\|_{F^\alpha_{p,q}}. \quad \square$$

Actually we can say more,

Theorem 2.4. *Suppose that $\alpha \geq 0$, $\beta > 0$. Then*

(i) *$\mathcal{J}_\beta$ is bounded from $B^\alpha_{p,q}(\gamma_d)$ to $B^{\alpha+\beta}_{p,q}(\gamma_d)$.*

(ii) *$\mathcal{J}_\beta$ is bounded from $F^\alpha_{p,q}(\gamma_d)$ to $F^{\alpha+\beta}_{p,q}(\gamma_d)$.*

Proof. (i) Let us denote $u(x,t) = P_t f(x)$ and $U(x,t) = P_t \mathcal{J}_\beta f(x)$, using the representation of P_t (6) we have,

$$U(x,t) = \int_0^{+\infty} T_s(\mathcal{J}_\beta f)(x) \mu_t^{(1/2)}(\mathrm{d}s).$$

Therefore,

$$U(x, t_1 + t_2) = P_{t_1}(P_{t_2}(\mathcal{J}_\beta f))(x) = \int_0^{+\infty} T_s(P_{t_2}(\mathcal{J}_\beta f))(x) \mu_{t_1}^{(1/2)}(\mathrm{d}s).$$

Now, let k, l be integer greater than α, β respectively, by differentiating k times respect to t_2 and l times respect to t_1,

$$\frac{\partial^{k+l} U(x, t_1 + t_2)}{\partial (t_1 + t_2)^{k+l}} = \int_0^{+\infty} T_s \left(\frac{\partial^k P_{t_2}}{\partial t_2^k} (\mathcal{J}_\beta f) \right)(x) \frac{\partial^l}{\partial t_1^l} \mu_{t_1}^{(1/2)}(\mathrm{d}s).$$

Thus

$$\frac{\partial^{k+l} U(x,t)}{\partial t^{k+l}} = \int_0^{+\infty} T_s \left(\frac{\partial^k P_{t_2}}{\partial t_2^k} (\mathcal{J}_\beta f) \right)(x) \frac{\partial^l}{\partial t_1^l} \mu_{t_1}^{(1/2)}(\mathrm{d}s),$$

if $t = t_1 + t_2$ and therefore, using the L^p-continuity of T_s and (25)

$$\left\| \frac{\partial^{k+l} U(\cdot,t)}{\partial t^{k+l}} \right\|_{p,\gamma} \leq \int_0^{+\infty} \left\| T_s \left(\frac{\partial^k P_{t_2}}{\partial t_2^k} (\mathcal{J}_\beta f) \right) \right\|_{p,\gamma} \left| \frac{\partial^l}{\partial t_1^l} \mu_{t_1}^{(1/2)}(\mathrm{d}s) \right|$$

$$\leq \int_0^{+\infty} \left\| \frac{\partial^k P_{t_2}}{\partial t_2^k} (\mathcal{J}_\beta f) \right\|_{p,\gamma} \left| \frac{\partial^l}{\partial t_1^l} \mu_{t_1}^{(1/2)}(\mathrm{d}s) \right|$$

$$= \left\| \frac{\partial^k P_{t_2}}{\partial t_2^k}(\mathcal{J}_\beta f) \right\|_{p,\gamma} \int_0^{+\infty} \left| \frac{\partial^l}{\partial t_1^l} \mu_{t_1}^{(1/2)}(\mathrm{d}s) \right|$$

$$\leq C(t_1)^{-l} \left\| \frac{\partial^k}{\partial t_2^k} P_{t_2} \mathcal{J}_\beta f \right\|_{p,\gamma}. \tag{35}$$

On the other hand, using the representation of Bessel potential (17) we have

$$P_t(\mathcal{J}_\beta f)(x) = \frac{1}{\Gamma(\beta)} \int_0^{+\infty} s^\beta \mathrm{e}^{-s} P_{t+s} f(x) \frac{\mathrm{d}s}{s}$$

then

$$\frac{\partial^k P_t}{\partial t^k}(\mathcal{J}_\beta f)(x) = \frac{1}{\Gamma(\beta)} \int_0^{+\infty} s^\beta \mathrm{e}^{-s} \frac{\partial^k P_{t+s} f(x)}{\partial t^k} \frac{\mathrm{d}s}{s}$$
$$= \frac{1}{\Gamma(\beta)} \int_0^{+\infty} s^\beta \mathrm{e}^{-s} \frac{\partial^k P_{t+s} f(x)}{\partial (t+s)^k} \frac{\mathrm{d}s}{s},$$

and this implies that

$$\left\| \frac{\partial^k P_t}{\partial t^k}(\mathcal{J}_\beta f) \right\|_{p,\gamma} \leq \frac{1}{\Gamma(\beta)} \int_0^{+\infty} s^\beta \mathrm{e}^{-s} \left\| \frac{\partial^k P_{t+s} f}{\partial (t+s)^k} \right\|_{p,\gamma} \frac{\mathrm{d}s}{s},$$

since $f \in B^\alpha_{p,q}(\gamma_d)$. Now due to the fact that the definition of $B^\alpha_{p,q}(\gamma_d)$ is independent on the integer $k > \alpha$ that we can choose, let us take $k > \alpha + \beta$ and $l > \beta$, then $k + l > \alpha + 2\beta > \alpha + \beta$, this is $k + l$ is an integer greater than $\alpha + \beta$. Let us see now that

$$\left(\int_0^{+\infty} \left(t^{k+l-(\alpha+\beta)} \left\| \frac{\partial^{k+l} U(\cdot,t)}{\partial t^{k+l}} \right\|_{p,\gamma} \right)^q \frac{\mathrm{d}t}{t} \right)^{\frac{1}{q}} < +\infty.$$

In fact, taking $t_1 = t_2 = t/2$ in (35), we get

$$\left(\int_0^{+\infty} \left(t^{k+l-(\alpha+\beta)} \left\| \frac{\partial^{k+l} U(\cdot,t)}{\partial t^{k+l}} \right\|_{p,\gamma} \right)^q \frac{\mathrm{d}t}{t} \right)^{\frac{1}{q}}$$
$$\leq C \left(\int_0^{+\infty} \left(t^{k+l-(\alpha+\beta)} \left\| \frac{\partial^k P_{\frac{t}{2}}}{\partial \left(\frac{t}{2}\right)^k}(\mathcal{J}_\beta f) \right\|_{p,\gamma} \left(\frac{t}{2}\right)^{-l} \right)^q \frac{\mathrm{d}t}{t} \right)^{\frac{1}{q}}$$
$$\leq \frac{C}{\Gamma(\beta)} \left(\int_0^{+\infty} \left(t^{k-(\alpha+\beta)} \left(\int_0^{+\infty} s^\beta \mathrm{e}^{-s} \left\| \frac{\partial^k P_{s+\frac{t}{2}} f}{\partial (\frac{t}{2})^k} \right\|_{p,\gamma} \frac{\mathrm{d}s}{s} \right) \right)^q \frac{\mathrm{d}t}{t} \right)^{\frac{1}{q}}$$
$$\leq \frac{C}{\Gamma(\beta)} \left[\int_0^{+\infty} t^{(k-(\alpha+\beta))q} \left(\int_0^t s^\beta \left\| \frac{\partial^k P_{s+\frac{t}{2}} f}{\partial (s+\frac{t}{2})^k} \right\|_{p,\gamma} \frac{\mathrm{d}s}{s} \right)^q \right.$$
$$\left. + \left(\int_t^{+\infty} s^\beta \left\| \frac{\partial^k P_{s+\frac{t}{2}} f}{\partial (s+\frac{t}{2})^k} \right\|_{p,\gamma} \frac{\mathrm{d}s}{s} \right)^q \frac{\mathrm{d}t}{t} \right]^{\frac{1}{q}}.$$

Using again that $(a+b)^q \le C_q(a^q+b^q)$ if $a, b \ge 0$, $q \ge 1$, but since $(a+b)^{1/q} \le a^{1/q}+b^{1/q}$ if $a, b \ge 0, q \ge 1$,

$$\frac{C}{\Gamma(\beta)}\left[\int_0^{+\infty} t^{(k-(\alpha+\beta))q}\left(\int_0^t s^\beta \left\|\frac{\partial^k P_{s+\frac{t}{2}}f}{\partial(s+\frac{t}{2})^k}\right\|_{p,\gamma}\frac{\mathrm{d}s}{s}\right)^q\right.$$
$$\left.+\left(\int_t^{+\infty} s^\beta \left\|\frac{\partial^k P_{s+\frac{t}{2}}f}{\partial(s+\frac{t}{2})^k}\right\|_{p,\gamma}\frac{\mathrm{d}s}{s}\right)^q\frac{\mathrm{d}t}{t}\right]^{\frac{1}{q}}$$
$$\le \frac{C}{\Gamma(\beta)}\left[\int_0^{+\infty} t^{(k-(\alpha+\beta))q}\left(\int_0^t s^\beta \left\|\frac{\partial^k P_{s+\frac{t}{2}}f}{\partial(s+\frac{t}{2})^k}\right\|_{p,\gamma}\frac{\mathrm{d}s}{s}\right)^q\frac{\mathrm{d}t}{t}\right]^{1/q}$$
$$+\frac{C}{\Gamma(\beta)}\left[\int_0^{+\infty} t^{(k-(\alpha+\beta))q}\left(\int_t^{+\infty} s^\beta \left\|\frac{\partial^k P_{s+\frac{t}{2}}f}{\partial\left(s+\frac{t}{2}\right)^k}\right\|_{p,\gamma}\frac{\mathrm{d}s}{s}\right)^q\frac{\mathrm{d}t}{t}\right]^{\frac{1}{q}}$$
$$= \mathrm{I}+\mathrm{II}.$$

Now, using Lemma 2.4 and since $\beta > 0$

$$\mathrm{I} = \frac{C}{\Gamma(\beta)}\left[\int_0^{+\infty} t^{(k-(\alpha+\beta))q}\left(\int_0^t s^\beta \left\|\frac{\partial^k P_{s+\frac{t}{2}}f}{\partial(s+\frac{t}{2})^k}\right\|_{p,\gamma}\frac{\mathrm{d}s}{s}\right)^q\frac{\mathrm{d}t}{t}\right]^{\frac{1}{q}}$$
$$\le \frac{C}{\Gamma(\beta)}\left[\int_0^{+\infty} t^{(k-(\alpha+\beta))q}\left(\int_0^t s^\beta \left\|\frac{\partial^k P_{\frac{t}{2}}f}{\partial(\frac{t}{2})^k}\right\|_{p,\gamma}\frac{\mathrm{d}s}{s}\right)^q\frac{\mathrm{d}t}{t}\right]^{\frac{1}{q}}$$
$$= \frac{C}{\beta\Gamma(\beta)}\left(\int_0^{+\infty}\left(t^{k-\alpha}\left\|\frac{\partial^k P_{\frac{t}{2}}f}{\partial(\frac{t}{2})^k}\right\|_{p,\gamma}\right)^q\frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}}$$
$$= C_{\alpha,\beta}\left(\int_0^{+\infty}\left(u^{k-\alpha}\left\|\frac{\partial^k P_u f}{\partial u^k}\right\|_{p,\gamma}\right)^q\frac{\mathrm{d}u}{u}\right)^{\frac{1}{q}} < +\infty,$$

since $f \in B_p^{\alpha,q}(\gamma_d)$.

On the other hand, using the Hardy inequality, since $k > \alpha+\beta$ and Lemma 2.4 we get

$$\mathrm{II} = \frac{C}{\Gamma(\beta)}\left(\int_0^{+\infty} t^{(k-(\alpha+\beta))q}\left(\int_t^{+\infty} s^\beta \left\|\frac{\partial^k P_{s+\frac{t}{2}}f}{\partial\left(s+\frac{t}{2}\right)^k}\right\|_{p,\gamma}\frac{\mathrm{d}s}{s}\right)^q\frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}}$$
$$\le \frac{C}{\Gamma(\beta)}\left(\int_0^{+\infty} t^{(k-(\alpha+\beta))q}\left(\int_t^{+\infty} s^\beta \left\|\frac{\partial^k P_s f}{\partial s^k}\right\|_{p,\gamma}\frac{\mathrm{d}s}{s}\right)^q\frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}}$$
$$\le \frac{C}{\Gamma(\beta)}\frac{1}{k-(\alpha+\beta)}\left(\int_0^{+\infty}\left(s^{k-\alpha}\left\|\frac{\partial^k}{\partial s^k}P_s f\right\|_{p,\gamma}\right)^q\frac{\mathrm{d}s}{s}\right)^{\frac{1}{q}} < +\infty$$

since $f \in B^{\alpha}_{p,q}(\gamma_d)$. Thus $\mathcal{J}_\beta f \in B^{\alpha+\beta}_{p,q}(\gamma_d)$ and moreover,

$$\|\mathcal{J}_\beta f\|_{B^{\alpha+\beta}_{p,q}} \leq C_{\alpha,\beta}\|f\|_{B^{\alpha}_{p,q}}.$$

(ii) Let $k > \alpha + \beta + 1$ a fixed integer, let $f \in F^{\alpha}_{p,q}(\gamma_d)$ and let us consider $h = \mathcal{J}_\beta f$.

Let us consider two cases:

(a) If $\beta \geq 1$. By the change of variable $u = t + s$ and using Hardy's inequality, we get

$$\left(\int_0^{+\infty}\left(t^{k-(\alpha+\beta)}\left|\frac{\partial^k P_t h(x)}{\partial t^k}\right|\right)^q \frac{\mathrm{d}t}{t}\right)^{1/q}$$
$$\leq \frac{1}{\Gamma(\beta)}\left(\int_0^{+\infty} t^{q(k-(\alpha+\beta))}\left(\int_0^{+\infty} s^{\beta}\mathrm{e}^{-s}\left|\frac{\partial^k P_{t+s}f(x)}{\partial(t+s)^k}\right|\frac{\mathrm{d}s}{s}\right)^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}}$$
$$\leq \frac{1}{\Gamma(\beta)}\left(\int_0^{+\infty} t^{q(k-(\alpha+\beta))}\left(\int_t^{+\infty}(u-t)^{\beta-1}\left|\frac{\partial^k P_u f(x)}{\partial u^k}\right|\mathrm{d}u\right)^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}}$$
$$\leq \frac{1}{\Gamma(\beta)}\left(\int_0^{+\infty}\left(\int_t^{+\infty} u^{\beta-1}\left|\frac{\partial^k P_u f(x)}{\partial u^k}\right|\mathrm{d}u\right)^q t^{q(k-(\alpha+\beta))-1}\mathrm{d}t\right)^{\frac{1}{q}}$$
$$\leq \frac{1}{\Gamma(\beta)}\frac{1}{k-(\alpha+\beta)}\left(\int_0^{+\infty}\left(u^{k-\alpha}\left|\frac{\partial^k P_u f(x)}{\partial u^k}\right|\right)^q \frac{\mathrm{d}u}{u}\right)^{\frac{1}{q}}.$$

Therefore,

$$\left\|\left(\int_0^{+\infty}\left(t^{k-(\alpha+\beta)}\left|\frac{\partial^k P_t h}{\partial t^k}\right|\right)^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}}\right\|_{p,\gamma}$$
$$\leq \frac{1}{\Gamma(\beta)(k-(\alpha+\beta))}\left\|\left(\int_0^{+\infty}\left(u^{k-\alpha}\left|\frac{\partial^k P_u f}{\partial u^k}\right|\right)^q \frac{\mathrm{d}u}{u}\right)^{\frac{1}{q}}\right\|_{p,\gamma} < \infty,$$

since $f \in F^{\alpha}_{p,q}(\gamma_d)$. Thus $\mathcal{J}_\beta f \in F^{\alpha+\beta}_{p,q}(\gamma_d)$.

(b) If $0 < \beta < 1$.

$$\left(\int_0^{+\infty}\left(t^{k-(\alpha+\beta)}\left|\frac{\partial^k P_t h(x)}{\partial t^k}\right|\right)^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}}$$
$$\leq \frac{1}{\Gamma(\beta)}\left(\int_0^{+\infty} t^{q(k-(\alpha+\beta))}\left(\int_0^{+\infty} s^{\beta}\mathrm{e}^{-s}\left|\frac{\partial^k P_{t+s}f(x)}{\partial(t+s)^k}\right|\frac{\mathrm{d}s}{s}\right)^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}}$$
$$\leq \frac{C}{\Gamma(\beta)}\left(\int_0^{+\infty} t^{q(k-(\alpha+\beta))-1}\left(\int_0^{t} s^{\alpha-1}\mathrm{e}^{-s}\left|\frac{\partial^k P_{t+s}f(x)}{\partial(t+s)^k}\right|\mathrm{d}s\right)^q \mathrm{d}t\right)^{\frac{1}{q}}$$
$$+\frac{C}{\Gamma(\beta)}\left(\int_0^{+\infty} t^{q(k-(\alpha+\beta))-1}\left(\int_t^{+\infty} s^{\alpha-1}\mathrm{e}^{-s}\left|\frac{\partial^k P_{t+s}f(x)}{\partial(t+s)^k}\right|\mathrm{d}s\right)^q \mathrm{d}t\right)^{\frac{1}{q}}$$
$$= \mathrm{I} + \mathrm{II}.$$

Now, $\mathrm{e}^{-s} < 1$ and as $\beta < 1$, then $s^{\beta-1} < t^{\beta-1}$ for $t < s$.

Hence again by the change of variable $u = t + s$ and using Hardy's inequality, we get

$$\begin{aligned} \mathrm{II} &\le \frac{C}{\Gamma(\beta)} \left(\int_0^{+\infty} t^{q(k-\alpha-1)-1} \left(\int_t^{+\infty} \left| \frac{\partial^k P_{t+s} f(x)}{\partial (t+s)^k} \right| \mathrm{d}s \right)^q \mathrm{d}t \right)^{\frac{1}{q}} \\ &\le \frac{C}{\Gamma(\beta)} \left(\int_0^{+\infty} t^{q(k-\alpha-1)-1} \left(\int_t^{+\infty} \left| \frac{\partial^k P_u f(x)}{\partial u^k} \right| \mathrm{d}u \right)^q \mathrm{d}t \right)^{\frac{1}{q}} \\ &\le \frac{C}{\Gamma(\beta)} \left(\int_0^{+\infty} \left(u^{k-\alpha} \left| \frac{\partial^k P_u f(x)}{\partial u^k} \right| \right)^q \frac{\mathrm{d}u}{u} \right)^{\frac{1}{q}}. \end{aligned}$$

On the other hand, using again $\mathrm{e}^{-s} < 1$,

$$\begin{aligned} I^q &\le \frac{C}{\Gamma(\beta)} \int_0^{+\infty} t^{q(k-(\alpha+\beta))-1} \left(\int_0^t s^{\beta-1} \left| \frac{\partial^k P_{t+s} f(x)}{\partial (t+s)^k} \right| \mathrm{d}s \right)^q \mathrm{d}t \\ &= \frac{C}{\Gamma(\beta)\beta^q} \int_0^{+\infty} t^{q(k-\alpha)-1} \left(\frac{\beta}{t^\beta} \int_0^t s^{\beta-1} \left| \frac{\partial^k P_{t+s} f(x)}{\partial (t+s)^k} \right| \mathrm{d}s \right)^q \mathrm{d}t. \end{aligned}$$

Now, as $\beta > 0$, $\int_0^t s^{\beta-1}\mathrm{d}s = \frac{t^\beta}{\beta}$, then using Jensen's inequality for the probability measure $\frac{\beta}{t^\beta} s^{\beta-1}\mathrm{d}s$ and Fubini's theorem

$$\begin{aligned} I^q &\le \frac{C}{\Gamma(\beta)\beta^q} \int_0^{+\infty} t^{q(k-\alpha)-1} \left(\frac{\beta}{t^\beta} \int_0^t s^{\beta-1} \left| \frac{\partial^k P_{t+s} f(x)}{\partial (t+s)^k} \right|^q \mathrm{d}s \right) \mathrm{d}t \\ &= \frac{C}{\Gamma(\beta)\beta^{q-1}} \int_0^{+\infty} s^{\beta-1} \left(\int_s^{+\infty} t^{q(k-\alpha)-\beta-1} \left| \frac{\partial^k P_{t+s} f(x)}{\partial (t+s)^k} \right|^q \mathrm{d}t \right) \mathrm{d}s \\ &\le \frac{C}{\Gamma(\beta)\beta^{q-1}} \int_0^{+\infty} s^{\beta-1} \left(\int_s^{+\infty} (t+s)^{q(k-\alpha)-\beta-1} \left| \frac{\partial^k P_{t+s} f(x)}{\partial (t+s)^k} \right|^q \mathrm{d}t \right) \mathrm{d}s \end{aligned}$$

as $q(k-\alpha) - \beta - 1 > 0$, since $0 < \beta < 1$. Finally, again by the change of variable $u = t + s$ and using Hardy's inequality, we get

$$\begin{aligned} I^q &\le \frac{C}{\Gamma(\beta)\beta^{q-1}} \int_0^{+\infty} s^{\beta-1} \left(\int_{2s}^{+\infty} u^{q(k-\alpha)-\beta-1} \left| \frac{\partial^k P_u f(x)}{\partial u^k} \right|^q \mathrm{d}u \right) \mathrm{d}s \\ &\le \frac{C}{\Gamma(\beta)\beta^{q-1}} \int_0^{+\infty} s^{\beta-1} \left(\int_s^{+\infty} u^{q(k-\alpha)-\beta-1} \left| \frac{\partial^k P_u f(x)}{\partial u^k} \right|^q \mathrm{d}u \right) \mathrm{d}s \\ &\le \frac{C}{\Gamma(\beta)\beta^{q-1}} \int_0^{+\infty} \left(u^{k-\alpha} \left| \frac{\partial^k P_u f(x)}{\partial u^k} \right| \right)^q \frac{\mathrm{d}u}{u}. \end{aligned}$$

Therefore

$$\begin{aligned} &\left\| \left(\int_0^{+\infty} \left(t^{k-(\alpha+\beta)} \left| \frac{\partial^k P_t h}{\partial t^k} \right| \right)^q \frac{\mathrm{d}t}{t} \right)^{\frac{1}{q}} \right\|_{p,\gamma} \\ &\quad \le C_{k,\alpha,\beta} \left\| \left(\int_0^{+\infty} \left(u^{k-\alpha} \left| \frac{\partial^k P_u f}{\partial u^k} \right| \right)^q \frac{\mathrm{d}u}{u} \right)^{\frac{1}{q}} \right\|_{p,\gamma} < \infty. \end{aligned}$$

Thus $\mathcal{J}_\beta f \in F_{p,q}^{\alpha+\beta}(\gamma_d)$, for $0 < \beta < 1$.

In both cases we have,

$$\begin{aligned}
\|\mathcal{J}_\beta f\|_{F^{\alpha+\beta}_{p,q}} &= \|\mathcal{J}_\beta f\|_{p,\gamma} + \left\| \left(\int_0^{+\infty} \left(t^{k-(\alpha+\beta)} \left| \frac{\partial^k P_t \mathcal{J}_\beta f}{\partial t^k} \right| \right)^q \frac{\mathrm{d}t}{t} \right)^{\frac{1}{q}} \right\|_{p,\gamma} \\
&\le C_\beta \|f\|_{p,\gamma} + C_{k,\alpha,\beta} \left\| \left(\int_0^{+\infty} \left(u^{k-\alpha} \left| \frac{\partial^k P_u}{\partial u^k} \right| \right)^q \frac{\mathrm{d}u}{u} \right)^{\frac{1}{q}} \right\|_{p,\gamma} \\
&\le C_{k,\alpha,\beta} \|f\|_{F^{\alpha}_{p,q}}. \quad \square
\end{aligned}$$

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