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Hadwiger's conjecture for powers of cycles and their complements

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Abstract

C. Thomassen showed that powers of certain cycles are counterexamples to Hajós' conjecture. We prove that powers of cycles and their complements satisfy Hadwiger's conjecture, that is, every k-chromatic graph has a k-clique as a minor.

1. Introduction

Hadwiger conjectured that every k-chromatic graph G has a complete graph K_k on k vertices as a minor, that is, there are k connected subgraphs $A_1, A_2, \ldots, A_k$ of G, such that $V(A_i \cap A_j) = \emptyset$ and there is at least one edge between $V(A_i)$ and $V(A_j)$, for $1 \le i < j \le k$. This conjecture is maybe one of the most intriguing conjectures in graph theory. For more details about the conjecture, the reader is referred to [4].

Recently, Thomassen in [3] has given some new classes of graphs which are counterexamples to Hajós' conjecture. These include some certain line graphs, powers of cycles, and complements of Kneser graphs. If Hadwiger's conjecture is false, then counterexamples can be found among the counterexamples to Hajós' conjecture. Reed and Seymour in [2] showed that Hadwiger's conjecture holds for line graphs. In this note, we prove that Hadwiger's conjecture holds for powers of cycles and their complements. For the complements of Kneser graphs, we give some examples which satisfy Hadwiger's conjecture too.

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Let xy be an edge of graph G. The *edge contraction* of xy is obtained be deleting x and y and all incident edges from G and adding a new vertex u and an edge uv from u to v for each vertex v that is a neighbor of x or y or both in G. A graph is a *minor* of G if it is either a subgraph of G, or can be obtained from one by a series of edge contractions. The complement of G is the graph with vertex set $V(G)$ and edge set $E(\bar{G}) = E(K_n) - E(G)$, that is, two vertices x_1 and x_2 are adjacent in $\bar{G}$ if and only if they are not adjacent in G. The p-th power of a cycle C_n, denoted by C_n^p, is the graph with vertex set $\{1, 2, \ldots, n\}$ in which two vertices i and j are adjacent if and only if $|i - j| \pmod n \leq p$. Let A_1 and A_2 be two connected subgraphs; we use $e(A_1, A_2)$ to denote the number of edges with one end in A_1 and the other end in A_2. The k-clique is a set of k-vertices which are pairwise adjacent. For other notation we refer the reader to [6].

2. Powers of cycles

Theorem 1. *Let C_n^p be the p-th power of C_n. Then C_n^p satisfies the Hadwiger's conjecture.*

Before we prove the theorem, we need some lemmas.

Lemma 1. *Let C_{p+k+r}^p be the p-th power of C_{p+k+r} for $2 \leq k < r \leq p$. Then C_{p+k+r}^p has a $(p + \lfloor (r + k - 1)/2 \rfloor + 1)$-clique as a minor.*

Proof. Let $A_i = \{i\}$ for $1 \leq i \leq p + 1$, $A_{p+2} = \{p + 2, p + 2 + \lfloor (r + k - 1)/2 \rfloor\}$, $A_{p+3} = \{p + 3, p + 3 + \lfloor (r + k - 1)/2 \rfloor\}, \ldots, A_{p+1+\lfloor (r+k-1)/2 \rfloor} = \{p + 1 + \lfloor (r + k - 1)/2 \rfloor, p + 1 + 2\lfloor (r + k - 1)/2 \rfloor\}$.

Each A_i induces a connected subgraph since $r + k - 1 < 2p - 1$. $A_1, A_2, \ldots$, and A_{p+1} induce a $(p + 1)$-clique and A_{p+i} for $2 \leq i \leq 1 + \lfloor (r + k - 1)/2 \rfloor$ induce a $\lfloor (r + k - 1)/2 \rfloor$-clique. Since $i - (p + i + 1 + \lfloor (r + k - 1)/2 \rfloor) + p + r + k = \lceil (r + k - 1)/2 \rceil < p$ for $2 \leq i \leq 1 + \lfloor (r + k - 1)/2 \rfloor$ and $1 \leq j \leq p + 1$, we have $e(A_i, A_j) > 0$ for $i \neq j$. Then we obtain a $(p + 1 + \lfloor (r + k - 1)/2 \rfloor)$-clique as a minor. □

Lemma 2. *For integers p, r, k satisfying $2 \leq k < r \leq p$, we have $\lceil r/k \rceil \leq \lfloor (r + k - 1)/2 \rfloor$.*

Proof. If $k = 2$, $\lceil r/2 \rceil \leq \lfloor (r + 1)/2 \rfloor$ whenever r is odd or even. If $k \geq 3$, $\lceil r/k \rceil \leq \lceil r/3 \rceil \leq (r + 2)/3 < \lfloor (r + 2)/2 \rfloor \leq \lfloor (r + k - 1)/2 \rfloor$. □

Proof of Theorem 1. For the p-th power of the cycle C_n, if $n \leq 2p + 1$, C_n^p is isomorphic to the complete graph K_n. If $p + 1$ divides n, the chromatic number of C_n^p is $p + 1$, and K_{p+1} is an induced subgraph of C_n^p. If $n = (p+1)k+r$, and $0 < r \leq k$, the chromatic number of C_n^p is $p+2$. We partition the vertex set as follows: $V_i = \{(i-1)(p+2)+1, (i-1)(p+2)+2, \ldots, i(p+2)\}$ for $1 \leq i \leq r$, $V_{r+j} = \{r(p+2)+(j-1)(p+1)+1, r(p+2)+(j-1)(p+1)+2, \ldots, r(p+2)+j(p+1)\}$ for $1 \leq j \leq k-r$. Color V_i with $p+2$ colors such that vertex $(i-1)(p+2)+s$ receives color s, and color V_{r+j} with $p+1$ colors such that vertex $r(p+2)+(j-1)(p+1)+t$ receives color t, for $1 \leq s \leq p + 2$, $1 \leq t \leq p + 1$. Now let $A_i = \{i\}$ for $1 \leq i \leq p + 1$, and $A_{p+2} = \{p + 2, p + 3, \ldots, n\}$. Obviously, C_n^p has a $p + 2$-clique as a minor.

For the last case $n = (p+1)k+r$ and $k < r \leq p$, we write $n = (p+\lceil r/k \rceil)k+(r-k(\lceil r/k \rceil - 1))$. As in the above cases, let $s = r - k(\lceil r/k \rceil - 1)$, $|V_1| = \cdots = |V_s| = p + \lceil r/k \rceil + 1$, $|V_{s+1}| = \cdots = |V_k| = p + \lceil r/k \rceil$. So $\chi(C_n^p) \leq p + \lceil r/k \rceil + 1$.

Now we show that C_n^p admits a $(p + \lceil r/k \rceil + 1)$-clique as a minor. By Lemma 2, we show that it admits a $(p+1+\lfloor (r+k-1)/2 \rfloor)$-clique as a minor. Let $A_i = \{i, p+i, 2p+i, \ldots, (k-1)p+i\}$ for $1 \leq i \leq p$. Let $A_{p+1} = \{kp + 1\}$, and let $A_{p+2} = \{kp + 2, kp + 2 + \lfloor (r + k - 1)/2 \rfloor\}$,

$A_{p+3} = \{kp+3, kp+3+\lfloor (r+k-1)/2 \rfloor\}, \ldots, A_{p+1+\lfloor (r+k-1)/2 \rfloor} = \{kp+1+\lfloor (r+k-1)/2 \rfloor, kp+1+2\lfloor (r+k-1)/2 \rfloor\}$. Proceeding as in Lemma 1, we can check that C_n^p has a $(p+1+\lfloor (r+k-1)/2 \rfloor)$-clique as a minor. □

3. Complements of power of cycles

We define a graph G_k^d with vertex set $\{i : 1 \le i \le k\}$, and edge set $\{ij : d \le |j-i| \le k-d\}$ for positive integers k, d and $k \ge 2d$. Obviously, the complement of C_n^p is isomorphic to G_n^{p+1}. Now we consider G_k^d. If $d = 1$, G_k^d is the complete graph K_k. So we may assume $d \ge 2$. Define a coloring of G_k^d as the following: $c(i) = \lfloor i/d \rfloor$ for $0 \le i \le k-1$. It is easy to see that c is a proper coloring, and the chromatic number of G_k^d is at most $\lceil k/d \rceil$. Let $\gcd(k,d) = s$, $k' = k/s$, and $d' = d/s$. Note that $G_{k'}^{d'}$ is an induced subgraph of G_k^d and $\chi(G_{k'}^{d'}) \le \chi(G_k^d) \le \lceil k/d \rceil = \lceil k'/d' \rceil$. In what follows, we assume $\gcd(k,d) = 1$, and $k = pd + r$ $(0 < r < d)$, we need to find a $(p+1)$-clique as a minor of G_k^d. Since $\lceil k/d \rceil = 2$ or 3, it is trivial; we assume that $p \ge 3$.

Lemma 3. *G_{2p+1}^2 has a $(p+1)$-clique as a minor.*

Proof. Let $A_0 = \{0\}$. We split the proof into two cases according to the parity of p.

Case 1. p is even. Assume $p = 2s$. Let $A_{2i-1} = \{4i-3, 4i-1\}$, $A_{2i} = \{4i-2, 4i\}$ for $1 \le i \le s$. Each A_i induces a connected subgraph, $V(A_i \cap A_j) = \emptyset$ and $e(A_i, A_j) > 0$ for $i \ne j$.

Case 2. p is odd. Assume $p = 2s+1$. If $s = 1$, $G_{2p+1}^2 = G_7^2$, and let $A_1 = \{1,3\}$, $A_2 = \{2,5\}$, $A_3 = \{4,6\}$. If $s \ge 2$, let $A_1 = \{1,3\}$, $A_2 = \{2,5\}$, $A_3 = \{4,6\}$, and for other A_i's, let $A_{2i} = \{4i-1, 4i+1\}$, $A_{2i+1} = \{4i, 4i+2\}$ for $2 \le i \le s$. Like in Case 1, each A_i induces a connected graph and $e(A_i, A_j) > 0$ for $i \ne j$. □

Lemma 4. *G_k^d has a $(p+1)$-clique as a minor, where $k = pd + r$, $1 \le r < d$, $d \ge 3$ and $p \ge 3$.*

Proof. Let $A_i = \{id^2, (id+1)d, \ldots, ((i+1)d-1)d\}$ for $0 \le i \le p-1$, $A_p = \{pd^2, (pd+1)d, \ldots, (p-1)d+r\}$. Obviously, $|A_i| = d$ for $0 \le i \le p-1$, and $|A_p| = r$, and each A_i induces a connected subgraph. Now we show that $e(A_i, A_j) > 0$ for $0 \le i < j \le p$.

Case 1. $i \ne p$ and $j \ne p$. If $d \le id^2 - jd^2 \le pd + r - d$, then id^2 is adjacent to jd^2. If $0 \le (i-j)d^2 < d$, since $(id+1)d \in A_i$, $d \le (i-j)d^2 + d < 2d$, then we have $(id+1)d$ is adjacent to jd^2. If $pd + r - d < (i-j)d^2 < pd + r$, $-d < (i-j)d^2 - (pd+r) < 0$, and $d < (i-j)d^2 + 2d \pmod{pd+r} < 2d$, then $(id+2)d$ is adjacent to jd^2. So we have $e(A_i, A_j) > 0$.

Case 2. One of i and j is p, say $j = p$. $pd + r - d = -d \in A_p$. If $d \le id^2 + d \le pd + r - d$, i.e. $0 \le id^2 \le pd + r - 2d$, we have $e(A_i, A_j) > 0$; otherwise we have either $0 < id^2 + d < d$ or $pd + r - d < id^2 + d < pd + r$. For $0 < id^2 + d < d$, $(id+1)d$ is adjacent to $-d$, and for the case $pd + r - d < id^2 + d < pd + r$, $(id+2)d$ is adjacent to $-d$.

Then G_k^d has a $(p+1)$-clique as a minor. □

Theorem 2. *Hadwiger's conjecture holds for G_k^d.*

Proof. If $\gcd(k,d) = s \ge 2$, let $k' = k/s$, and $d' = d/s$. Since $G_{k'}^{d'}$ is a minor of G_k^d, $G_{k'}^{d'}$ has a $\chi(G_{k'}^{d'})$-minor and so does G_k^d. If $\gcd(k,d) = 1$, by Lemmas 3 and 4, the theorem is straightforward. □

4. Remarks

C. Thomassen pointed out that all Kneser graphs satisfy Hajós' conjecture while some complements of certain Kneser graphs are counterexamples to this conjecture. We do not know whether Hadwiger's conjecture holds for the complements of all Kneser graphs. For some classes, the conjecture holds. In what follows, we give some examples. Suppose that $n \geq k \geq 1$ are integers and let $[n] = \{1, 2, \ldots, n\}$. Then Kneser graph $K(n, k)$ has as vertices the k-subset of $[n]$. Two vertices are adjacent if the corresponding k-subsets are disjoint. Suppose that k divides n, and let $n = pk$. Observe that $\binom{n}{k} = p\binom{n-1}{k-1}$. Let $M = \binom{n-1}{k-1}$. A Baranyai partition (see [5]) of the complete hypergraph $\binom{[n]}{k}$ is a family of M partitions of $[n]$ such that for any given i, $1 \leq i \leq M$, $\mathcal{A}_i = \{A_i^1, A_i^2, \ldots, A_i^p\}$ and $A_i^1 \cup A_i^2 \cup \cdots \cup A_i^p = [n]$ and $|A_i^1| = \cdots = |A_i^p| = k$ and that each k subset of $[n]$ occurs among the A_i^j's exactly once. Obviously, a Baranyai partition is a clique partition of $K(n, k)$ since each $\mathcal{A}_i$ for $1 \leq i \leq M$ induces a p-clique of the graph. Then we have that the chromatic number of the complement of $K(n, k)$ is at most M. By the Erdös–Ko–Rado theorem in [1], the independence number of $K(n, k)$ is M. Then the chromatic number of the complement of $K(n, k)$ is M and the complement of $K(n, k)$ has an M-clique as an induced subgraph. For such a class of Kneser graphs, Hadwiger's conjecture holds.

The Petersen graph is the graph with vertex set $\{x_i, y_i : 0 \leq i \leq 4\}$ and edge set $\{x_i x_{i+1} : 0 \leq i \leq 4\} \cup \{y_0 y_2, y_2 y_4, y_4 y_1, y_1 y_3, y_3 y_0\}$. It is well known that the Petersen graph has clique number two. The chromatic number of the complement of the Petersen graph is five. Let $A_i = \{x_i y_{i+1}\}$ for $0 \leq i \leq 4$. Then the complement of the Petersen graph has a 5-clique as a minor. $K(7, 2)$ has a clique partition as follows: $B_1 = \{\{1, 2\}, \{3, 4\}, \{5, 6\}\}$, $B_2 = \{\{1, 3\}, \{2, 4\}, \{7, 6\}\}$, $B_3 = \{\{3, 2\}, \{6, 4\}, \{5, 7\}\}$, $B_4 = \{\{1, 5\}, \{7, 4\}, \{2, 6\}\}$, $B_5 = \{\{1, 7\}, \{2, 4\}, \{3, 6\}\}$, $B_6 = \{\{1, 6\}, \{7, 3\}, \{2, 5\}\}$, and $B_7 = \{\{1, 4\}, \{7, 2\}, \{3, 5\}\}$. By the Erdös–Ko–Rado theorem, the independence number of $K(7, 2)$ is 7. Then the chromatic number of the complement of $K(7, 2)$ is 7. Let $A_i = \{1, i + 1\}$ for $1 \leq i \leq 6$, and $A_7 = V(K(7, 2)) - \cup_{i=1}^{6} A_i$. It is easy to see that $V(A_i \cap A_j) = \emptyset$ and $e(A_i, A_j) > 0$. Then Hadwiger's conjecture holds for the complement of $K(7, 2)$. Furthermore, for the complement of $K(n, 2)$, let $p = \lfloor n/2 \rfloor$. The chromatic number of the complement of $K(n, 2)$ is $\binom{n}{2}/p$, and it has a minor of size n as follows: $A_i = \{1, i + 1\}$ for $1 \leq i \leq n - 1$, and $A_n = V(K(n, 2)) - \cup_{i=1}^{n-1} A_i$.

Let $\lfloor n/k \rfloor = p$. p divides $\binom{n}{k}$, and let $M = \binom{n}{k}/p$. We wonder whether the Kneser graph $K(n, k)$ admits a p-clique partition $\mathcal{A}_i$ for $1 \leq i \leq M$.

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