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On the range closure of an elementary operator

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To Professor Carl Pearcy on his Seventieth

Abstract

Let $B(\mathscr{H})$ denote the algebra of operators on a Hilbert $\mathscr{H}$. Let $\Delta_{AB} \in B(B(\mathscr{H}))$ and $E \in B(B(\mathscr{H}))$ denote the elementary operators $\Delta_{AB}(X) = AXB - X$ and $E(X) = AXB - CXD$. We answer two questions posed by Turnšek [Mh. Math. 132 (2001) 349–354] to prove that: (i) if A, B are contractions, then $B(\mathscr{H}) = \Delta_{AB}^{-1}(0) \oplus \Delta_{AB}(B(\mathscr{H}))$ if and only if $\Delta_{AB}^{n}(B(\mathscr{H}))$ is closed for some integer $n \geqslant 1$; (ii) if A, B, C and D are normal operators such that A commutes with C and B commutes with D, then $B(\mathscr{H}) = E^{-1}(0) \oplus E(B(\mathscr{H}))$ if and only if $0 \in \operatorname{iso} \sigma(E)$.

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1. Introduction and notation

If T is a Banach space operator, $T \in B(\mathscr{V})$, then a necessary condition for T to have closed range complemented by its kernel $T^{-1}(0)$ is that $0 \in \operatorname{iso} \sigma(T)$ (i.e., 0 is an isolated point of the spectrum $\sigma(T)$ of T). If $\mathscr{V} = B(\mathscr{H})$, the algebra of operators on a complex infinite dimensional Hilbert space $\mathscr{H}$, and $T = \delta_{AB} \in B(B(\mathscr{H}))$ is

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the generalized derivation $\delta_{AB}(X) = AX - XB$, then this condition translates to $0 \in \operatorname{iso}\{\sigma(A) - \sigma(B)\} \iff 0 \in \{\sigma(A) - \sigma(B)\}$ and $\sigma(A) \cap \sigma(B)$ is finite. For normal $A, B \in B(\mathscr{H})$, or (more generally) scalar operators (in the sense of Dunford) $A, B \in B(\mathscr{H})$, the condition $0 \in \operatorname{iso}\sigma(\delta_{AB})$ is also sufficient for $B(\mathscr{H}) = \delta_{AB}^{-1}(0) \oplus \delta_{AB}(B(\mathscr{H}))$ [3].

If M and N are subspaces of $\mathscr{V}$, then M is said to be orthogonal to N, denoted $M \perp N$, if $\|m\| \leqslant \|m + n\|$ for all $m \in M$ and $n \in N$ [12, p. 93]. Recall from Anderson [2] that if $A, B \in B(\mathscr{H})$ are normal, then $\delta_{AB}^{-1}(0) \perp \delta_{AB}(B(\mathscr{H}))$. Various extensions of this orthogonality to the *elementary operators* $\Delta_{AB}(X) = AXB - X$ and $\mathscr{E}_{\mathbf{AB}}(X) = \sum_{i=1}^{m} A_i X B_i$ are to be found in the literature (see, for example, [9,15,19]). Observe that

$$T^{-1}(0) \perp T(\mathscr{V}) \implies T^{-1}(0) \cap T(\mathscr{V}) = \{0\} \iff \operatorname{asc}(T) \leqslant 1$$

for an operator $T \in B(\mathscr{V})$. (Here, and in the sequel, $\operatorname{asc}(T)$ denotes the ascent, and $T(\mathscr{V})$ denotes the range, of T.) $T^{-1}(0) \perp T(\mathscr{V})$ does not however imply that $T(\mathscr{V})$ is closed, or even when $T(\mathscr{V})$ is closed that $\mathscr{V} = T^{-1}(0) + T(V)$. In his study of the range-kernel orthogonality of the elementary operators Δ_{AB} and $\mathscr{E}_{\mathbf{AB}}$ in von Neumann–Schatten p-classes $\mathscr{C}_p(\mathscr{H})$ [20], Turnšek has posed the following problems: Find conditions (i) for $B(\mathscr{H}) = \Delta_{AB}^{-1}(0) \oplus \Delta_{AB}(B(\mathscr{H})$ to hold, given that $A, B \in B(\mathscr{H})$ are contractions; (ii) for $B(\mathscr{H}) = \mathscr{E}_{\mathbf{AB}}^{-1}(0) \oplus \mathscr{E}_{\mathbf{AB}}(B(\mathscr{H})$ to hold, given that $m = 2$, and (A_1, A_2) and (B_1, B_2) are tuples of mutually commuting normal operators in $B(\mathscr{H})$. These problems were partially answered in [8]. In this note we use techniques from *local spectral theory* to provide a complete answer to these problems by proving that the equality in (i) holds if and only if either Δ_{AB} is Kato type or $\Delta_{AB}^{n}(B(\mathscr{H}))$ is closed for some integer $n \geqslant 1$, and that the equality in (ii) holds if and only if $0 \in \operatorname{iso}\sigma(T)$.

In addition to the notation and terminology already introduced, we shall use the following further notation and terminology.

The (*algebra*) *numerical range* $W(B(\mathscr{V}), T)$ of an operator $T \in B(\mathscr{V})$ is the set

$$\left\{f(T) : f \in B(\mathscr{V})^*, \|f\| = f(I) = 1\right\},$$

where $B(\mathscr{V})^*$ denotes the (Banach space) dual of $B(\mathscr{V})$; $W(B(\mathscr{V}), T) = \overline{\operatorname{co} V(T)}$, where $\overline{\operatorname{co} V(T)}$ denotes the closed convex hull of the *spatial numerical range*

$$V(T) = \left\{F(Ty) : F \in \mathscr{V}^*, y \in \mathscr{V}, \|F\| = \|y\| = F(y) = 1\right\}$$

of T [6, Theorem 9.4]. If we denote the (Banach space) conjugate operator of T by T^*, then $\overline{\operatorname{co} V(T)} = \overline{\operatorname{co} V(T^*)}$ [6, Corollary 9.6(ii)]. Hence:

Proposition 1.1. $W(B(\mathscr{V}), T) = W(B(\mathscr{V}^*), T^*)$.

If M is a linear subspace of $\mathscr{V}$, let

$$M^{\perp} = \left\{\phi \in \mathscr{V}^* : \phi(m) = 0 \text{ for all } m \in M\right\}$$

denote the *annihilator* of M (in the dual space $\mathscr{V}^*$), and if N is a linear subspace of $\mathscr{V}^*$, let

$$^{\perp}N = \{v \in \mathscr{V} : \phi(v) = 0 \text{ for all } \phi \in N\}$$

denote the *pre-annihilator* of N (in $\mathscr{V}$). By the bi-polar theorem, $^{\perp}(M^{\perp})$ is the norm closure of M and $(^{\perp}N)^{\perp}$ is the weak-$*$-closure of N. For every $T \in B(\mathscr{V})$, $T^{*-1}(0) = T(\mathscr{V})^{\perp}$ and $T^{-1}(0) =^{\perp} T^*(\mathscr{V}^*)$.

The ascent (descent) of $T \in B(\mathscr{V})$, denoted asc(T) (resp., dsc(T)), is the least non-negative integer n such that $T^{-n}(0) = T^{-(n+1)}(0)$ (resp., $T^n(\mathscr{V}) = T^{n+1}(\mathscr{V})$). The deficiency indices $\alpha(T)$ and $\beta(T)$ of T are the integers $\alpha(T) = \dim(T^{-1}(0))$ and $\beta(T) = \dim(\mathscr{V}/T(\mathscr{V}))$. Let $\mathbb{C}$ denote the set of complex numbers. An operator T has SVEP (short for *the single-valued extension property*) at a point $\lambda_0 \in \mathbb{C}$ if for every open disc $\mathscr{D}_{\lambda_0}$ centered at λ_0 the only analytic function $f : \mathscr{D}_{\lambda_0} \to \mathscr{V}$ which satisfies

$$(T - \lambda)f(\lambda) = 0 \quad \text{for all } \lambda \in \mathscr{D}_{\lambda_0}$$

is the function $f \equiv 0$. Trivially, every operator T has SVEP at points of the resolvent $\rho(T) = \mathbb{C} \setminus \sigma(T)$; also T has SVEP at $\lambda \in \operatorname{iso}\sigma(T)$. The *quasi-nilpotent part* $H_0(T)$ and the *analytic core* $K(T)$ of T are defined by

$$H_0(T) = \left\{v \in \mathscr{V} : \lim_{n\to\infty} \|T^n v\|^{\frac{1}{n}} = 0\right\}$$

and

$$K(T) = \big\{v \in \mathscr{V} : \text{there exists a sequence } \{v_n\} \subset \mathscr{V} \text{ and } \delta > 0 \text{ for which } v = v_0, Tv_{n+1} = v_n \text{ and } \|v_n\| \leqslant \delta^n \|v\| \text{ for all } n = 1, 2, \ldots\big\}.$$

We note that $H_0(T)$ and $K(T)$ are (generally) non-closed hyperinvariant subspaces of T such that $T^{-q}(0) \subseteq H_0(T)$ for all $q = 0, 1, 2, \ldots$ and $TK(T) = K(T)$ [17]. An operator T is said to be *semi-regular* if $T(\mathscr{V})$ is closed and $T^{-1}(0) \subset T^{\infty}(\mathscr{V}) = \cap_{n\in\mathbf{N}} T^n(\mathscr{V})$; T admits a *generalized Kato decomposition*, GKD for short, if there exists a pair of T-invariant closed subspaces (M, N) such that $\mathscr{V} = M \oplus N$, the restriction $T|_M$ is quasinilpotent and $T|_N$ is semi-regular. An operator $T \in B(\mathscr{V})$ has a GKD at every $\lambda \in \operatorname{iso}\sigma(T)$, namely $\mathscr{V} = H_0(T - \lambda) \oplus K(T - \lambda)$. We say that T is *Kato type* at a point λ if $(T - \lambda)|_M$ is nilpotent in the GKD for $(T - \lambda)$. Recall that every Fredholm operator is Kato type (with the additional property that $\dim M < \infty$) [14, Theorem 4].

2. Results

Let $\mathscr{V}_p$ denote either of the Banach spaces $B(\mathscr{H})$ and $\mathscr{C}_p$, where $\mathscr{C}_p$, $1 \leqslant p < \infty$, is the von Neumann–Schatten p-class $\mathscr{C}_p(\mathscr{H})$. (Here it is assumed that $\mathscr{H}$ is separable in the case in which $\mathscr{V}_p = \mathscr{C}_p(\mathscr{H})$.) The following theorem answers [20, Question 2]. (We use the convention that if $0 \notin \sigma(T)$, then $0 \in \operatorname{iso}\sigma(T)$.)

Theorem 2.1. *If $A, B \in B(\mathscr{H})$ are contractions, then either of the following conditions is necessary and sufficient for $\mathscr{V}_p = \Delta_{AB}^{-1}(0) \oplus \Delta_{AB}(\mathscr{V}_p)$:*

(i) *Δ_{AB} is Kato type.*
(ii) *$\Delta_{AB}^n(\mathscr{V}_p)$ is closed for some integer $n \geqslant 1$.*

Proof. If $A, B \in B(H)$ are contractions, then

$$W(B(\mathscr{V}_p), \Delta_{AB}) \subseteq \{\lambda \in \mathbb{C} : |\lambda + 1| \leqslant 1|\}.$$

(This is proved in [8, Theorem 1] for the case in which $\mathscr{V}_p = B(\mathscr{H})$; the proof for the case in which $\mathscr{V}_p = \mathscr{C}_p$ follows from this since $W(B(\mathscr{C}_p), \Delta_{AB}) = W(B(B(\mathscr{H})), \Delta_{AB})$.) In particular, the point 0 (whenever it is in $\sigma(T)$) is a boundary point of both $\sigma(\Delta_{AB})$ and $\sigma(\Delta_{AB}^*)$ (see Proposition 1.1). Applying the Nirschl-Schneider theorem [4, Theorem 10.10] it follows that $\operatorname{asc}(\Delta_{AB}) \leqslant 1$ and $\operatorname{asc}(\Delta_{AB}^*) \leqslant 1$, both Δ_{AB} and ${\Delta_{AB}}^*$ have SVEP at 0, and

$$\Delta_{AB}^{-1}(0) \cap \Delta_{AB}(\mathscr{V}_p) = \{0\} = {\Delta_{AB}^*}^{-1}(0) \cap \Delta_{AB}^*(\mathscr{V}_p^*)$$

[5, p. 25].

(i) The *only if part* being obvious, we prove the *if part*. If T is Kato type, then there exists a GKD (M, N) such that $\mathscr{V}_p = M \oplus N$, $\Delta_{AB}|_M$ is nilpotent and $\Delta_{AB}|_N$ is semi-regular. Since $\operatorname{asc}(\Delta_{AB}) \leqslant 1$, $\Delta_{AB}|_M$ is 1-nilpotent. Again, since Δ_{AB}^* has SVEP at 0 (and Δ_{AB} is Kato type), $\operatorname{dsc}(\Delta_{AB}) < \infty$ [1, Theorem 2.9]. Thus $\operatorname{asc}(\Delta_{AB}) = \operatorname{dsc}(\Delta_{AB}) \leqslant 1$ and $\mathscr{V}_p = \Delta_{AB}^{-1}(0) \oplus \Delta_{AB}(\mathscr{V}_p)$ [16, Proposition 4.10.4].

(ii) Once again, the *only if part* being obvious, we prove the *if part*. Recall from [16, Lemma 4.10.2] that if $\operatorname{asc}(T) \leqslant 1$ and $T^{-1}(0) + T(\mathscr{V})$ is closed for a Banach space operator $T \in B(\mathscr{V})$, then $T(\mathscr{V})$ is closed; again, if $\operatorname{asc}(T) \leqslant 1$ and $T^n(\mathscr{V})$ is closed for some integer $n > 1$, then $T^{-1}(0) + T(\mathscr{V})$ is closed [16, Proposition 4.10.4]. Hence the hypothesis $\Delta_{AB}^n(\mathscr{V}_p)$ is closed for some integer $n \geqslant 1$ implies that $\Delta_{AB}^{-1}(0) + \Delta_{AB}(\mathscr{V}_p)$ is closed, which in turn implies that $\Delta_{AB}(\mathscr{V}_p)$, and so also $\Delta_{AB}^*(\mathscr{V}_p^*)$, is closed. Since

$$\begin{aligned}\left\{\Delta_{AB}^{-1}(0) + \Delta_{AB}(\mathscr{V}_p)\right\}^{\perp} &= \Delta_{AB}(\mathscr{V}_p)^{\perp} \cap \Delta_{AB}^{-1}(0)^{\perp} \\ &= {\Delta_{AB}^*}^{-1}(0) \cap \left\{{}^{\perp}\Delta_{AB}^*(\mathscr{V}_p^*)\right\}^{\perp} \\ &= {\Delta_{AB}^*}^{-1}(0) \cap \Delta_{AB}^*(\mathscr{V}_p^*) = \{0\},\end{aligned}$$

it follows that

$$\Delta_{AB}^{-1}(0) + \Delta_{AB}(\mathscr{V}_p) = \mathscr{V}_p.$$

Hence $\operatorname{dsc}(\Delta_{AB}) \leqslant 1$, which (see above) implies that $\mathscr{V}_p = \Delta_{AB}^{-1}(0) \oplus \Delta_{AB}(\mathscr{V}_p)$. □

Theorem 2.1 subsumes [8, Theorem 2], as the following corollary shows.

Corollary 2.2. *If $A, B \in B(\mathscr{H})$ are contractions such that the isolated points λ of $\sigma(A)$ and $\sigma(B)$ with $|\lambda| = 1$ are eigen-values, then we have the implications*

$$B(\mathscr{H}) = \Delta_{AB}^{-1}(0) \oplus \Delta_{AB}(B(\mathscr{H})) \iff 0 \in \operatorname{iso}\sigma(\Delta_{AB}).$$

Proof. The forward implication being obvious, we prove the reverse implication. For this it will suffice to prove that $\Delta_{AB}(B(\mathscr{H}))$ is closed: the proof will then follow from Theorem 2.1(ii). Since a large part of the argument is the same as that of the proof of [8, Theorem 2], we shall be economical with the detail. Thus assume that $0 \in \operatorname{iso}\sigma(\Delta_{AB})$. Arguing as in [8] it is seen that Δ_{AB} has a matrix representation $\Delta_{AB}(X) = [\Delta_{A_iB_j}(X_{ij})]_{i,j=1,2}^2$, where $0 \notin \sigma(\Delta_{A_iB_j})$ for all $i, j \neq 1$, and where A_1, B_1 are unitary operators with finite spectrum. Consequently, $\Delta_{A_iB_j}$ has closed range for all $0 \leqslant i, j \leqslant 2$; hence $\Delta_{AB}(B(\mathscr{H}))$ is closed. □

Examples of operators in $B(\mathscr{H})$ for which isolated points of the spectrum are eigen-values of the spectrum occur in abundance. Call an operator $T \in B(\mathscr{H})$ *totally hereditarily normaloid*, $T \in THN$, if every part of T (i.e., its restriction to an invariant subspace, including $\mathscr{H}$), and T_p^{-1} for every invertible part T_p of T, is normaloid: if $T \in THN$, then isolated points of the spectrum of T are eigen-values of T [11, Lemma 2.1]. Hyponormal operators T ($|T^*|^2 \leqslant |T|^2$) and (more generally) paranormal operators T ($\|Tx\|^2 \leqslant \|T^2x\|$ for every unit vector $x \in \mathscr{H}$) are examples of THN operators.

Theorem 2.1 extends to the operator $\Phi(X) = \sum_{i=1}^m A_iXB_i - X$, where $A_i, B_i \in B(\mathscr{H})$ are such that $\{\|\sum_{i=1}^m A_iA_i^*\|\|\sum_{i=1}^m B_i^*B_i\|\}^{\frac{1}{2}} \leqslant c$, $c = 1$ if $\Phi \in B(B(\mathscr{H}))$ and $c = m^{\frac{-1}{p}}$ if $\Phi \in B(\mathscr{C}_p)$.

Corollary 2.3. *If the operators $A_i, B_i \in B(\mathscr{H})$, $1 \leqslant i \leqslant m$, and the operator $\Phi \in B(\mathscr{V}_p)$ are defined as above, then either of the conditions* (i) *and* (ii) *of Theorem* 2.1 *is both necessary and sufficient for $\mathscr{V}_p = \Phi^{-1}(0) \oplus \Phi(\mathscr{V}_p)$.*

Proof. Define the row vector **A** and the column vector **B** by $\mathbf{A} = [A_1, A_2, \ldots, A_m]$ and $\mathbf{B} = [B_1, B_2, \ldots, B_m]^{\mathrm{T}}$. Then $\phi(X) = \sum_{i=1}^m A_iXB_i = \mathbf{A}(X \otimes I_m)\mathbf{B}$, where I_m is the identity of $\mathbf{M}_m(\mathbb{C})$. Clearly, ϕ is a contraction, and the argument of the proof of Theorem 2.1 applies. □

Remark 2.4. The conclusion $\mathscr{V}_p = \Delta_{AB}^{-1}(0) \oplus \Delta_{AB}(\mathscr{V}_p)$ implies that $0 \in \operatorname{iso}\sigma(\Delta_{AB})$. Thus, the hypothesis $0 \in \operatorname{iso}\sigma(\Delta_{AB})$ is necessary for $\mathscr{V}_p = \Delta_{AB}^{-1}(0) \oplus \Delta_{AB}(\mathscr{V}_p)$: the following example shows that this condition is not sufficient. Let V denote the Voltera (integral) operator on $\mathscr{H} = L^2(0, 1)$. Define $A, B \in B(\mathscr{H})$ by $A = (I + V)^{-1}$ and $B = I$. Then A, B are contractions and $\sigma(\Delta_{AB}) = \{0\}$.

Obviously, Δ_{AB} has the GKD $(B(\mathscr{H}), 0)$, Δ_{AB} is injective, $\Delta_{AB}(B(\mathscr{H}))$ is not closed and $B(\mathscr{H}) \neq 0 \oplus \Delta_{AB}(B(\mathscr{H}))$.

Remark 2.5. The hypothesis $0 \in \operatorname{iso}\sigma(T)$, $T \in B(\mathscr{V})$, implies that $\mathscr{V} = H_0(T) \oplus K(T)$, where both $H_0(T)$ and $K(T)$ are closed. The following argument shows that if also $\dim(H_0(T)) < \infty$, then there exists an integer $n \geqslant 1$ such that $\mathscr{V} = T^{-n}(0) \oplus T^n(\mathscr{V})$. Recall that $T^{-1}(0) \subseteq H_0(T)$ and $K(T) \subseteq T(\mathscr{V})$; if $\dim(H_0(T)) < \infty$, then the deficiency indices $\alpha(T)$ and $\beta(T)$ are (both) finite, and T is Fredholm. Obviously, both T and T^* have SVEP at 0; hence $\operatorname{asc}(T)$ and $\operatorname{dsc}(T)$ are finite [1, Theorems 2.6 and 2.9]. There exists an integer $n \geqslant 1$ such that $\operatorname{asc}(T) = \operatorname{dsc}(T) \leqslant n < \infty$ and $\mathscr{V} = T^{-n}(0) \oplus T^n(\mathscr{V})$ [16, Proposition 4.10.6]. The condition $\dim(H_0(T)) < \infty$ is fairly restrictive: a more general, but in many ways equally restrictive, condition is that $H_0(T) = T^{-n}(0)$. In the following we consider just such an operator.

Elementray operator $\mathscr{E}_{\mathbf{AB}}$. Let $\mathbf{A} = (A_1, A_2, \ldots, A_m)$ and $\mathbf{B} = (B_1, B_2, \ldots, B_m)$ be m-tuples of mutually commuting normal operators $A_i, B_i \in B(\mathscr{H})$. The *elementary operator* $\mathscr{E}_{\mathbf{AB}} \in B(B(\mathscr{H}))$ is defined by

$$\mathscr{E}_{\mathbf{AB}}(X) = \sum_{i=1}^{m} A_i X B_i.$$

Recall that an operator $T \in B(\mathscr{V})$ is a *generalized scalar operator* if there exists a continuous algebra homomorphism $\Phi : C^\infty \to B(\mathscr{V})$ for which $\Phi(1) = I$ and $\Phi(Z) = T$, where $C^\infty(\mathbb{C})$ is the Fréchet algebra of all infinitely differentiable functions on $\mathbb{C}$ (endowed with its usual topology of uniform convergence on compact sets for the functions and their partial derivatives) and Z is the identity function on $\mathbb{C}$ (see [7] or [16, p. 4]).

Let L_A and R_A, $A \in B(\mathscr{V})$, denote the operators of "left multiplication by A" and "right multiplication by A", respectively. If A, B are generalized scalar operators, then L_A and R_B are commuting generalized scalar operators with two commuting spectral distributions, and $L_A R_B$ and $L_A + R_B$ are generalized scalar operators (see [7, Theorem 3.3, Proposition 4.2 and Theorem 4.3, Chapter 4]). Let $\mathbf{A} = (A_1, A_2, \ldots, A_m)$ and $\mathbf{B} = (B_1, B_2, \ldots, B_m)$ be m-tuples of mutually commuting generalized scalar operators in $B(\mathscr{V})$. Since L_{A_i} commutes with R_{B_j} for all $1 \leqslant i, j \leqslant m$, the mutual commutativity of the m-tuples implies that $L_{A_i} R_{B_i}$ commutes with $L_{A_j} R_{B_j}$ for all $1 \leqslant i, j \leqslant m$, the generalized scalar operators $L_{A_i} R_{B_i}$ and $L_{A_j} R_{B_j}$ have two commuting spectral distributions, and (hence) $L_{A_i} R_{B_i} + L_{A_j} R_{B_j}$ is a generalized scalar operator. Since normal operators are generalized scalar operators, a finitely repeated application of this argument implies that $\mathscr{E}_{\mathbf{AB}}$ is a generalized scalar operator.

Theorem 2.6. *A necessary and sufficient condition for*

$$\mathscr{V}_p = \mathscr{E}_{\mathbf{AB}}^{-n}(0) \oplus \mathscr{E}_{\mathbf{AB}}^{n}(\mathscr{V}_p)$$

for some integer $n \geqslant 1$ *is that* $0 \in \operatorname{iso}\sigma(\mathscr{E}_{\mathbf{AB}})$.

Proof. The operator $\mathscr{E}_{\mathbf{AB}}$ being a generalized scalar operator, there exists an integer $n \geqslant 1$ such that $H_0(\mathscr{E}_{\mathbf{AB}}) = \mathscr{E}_{\mathbf{AB}}^{-n}(0)$ [7, Theorem 4.4.5]. If $0 \in \operatorname{iso}\sigma(\mathscr{E}_{\mathbf{AB}})$, then

$$\begin{aligned}\mathscr{V}_p &= H_0(\mathscr{E}_{\mathbf{AB}}) \oplus K(\mathscr{E}_{\mathbf{AB}}) = \mathscr{E}_{\mathbf{AB}}^{-n}(0) \oplus K(\mathscr{E}_{\mathbf{AB}}) \\ &\Longrightarrow \mathscr{E}_{\mathbf{AB}}^{n}(\mathscr{V}_p) = 0 \oplus \mathscr{E}_{\mathbf{AB}}^{n}(K(\mathscr{E}_{\mathbf{AB}})) = 0 \oplus K(\mathscr{E}_{\mathbf{AB}}) \\ &\Longrightarrow \mathscr{V}_p = \mathscr{E}_{\mathbf{AB}}^{-n}(0) \oplus \mathscr{E}_{\mathbf{AB}}^{n}(\mathscr{V}_p).\end{aligned}$$

Since the necessity of the conditon is obvious, the proof is complete. □

One cannot always choose $n = 1$ in Theorem 2.6, for the reason that there exist elementary operators $\mathscr{E}_{\mathbf{AB}}$ with $\operatorname{asc}(\mathscr{E}_{\mathbf{AB}}) > 1$ [18]. However, if we restrict the length of $\mathscr{E}_{\mathbf{AB}}$ to 2 (i.e., if $m = 2$), then $\operatorname{asc}(\mathscr{E}_{\mathbf{AB}}) \leqslant 1$ is guaranteed [9, Theorem 2.7]. Hence:

Corollary 2.7. *If* $\mathbf{A} = (A_1, A_2)$ *and* $\mathbf{B} = (B_1, B_2)$ *are tuples of commuting normal operators in* $B(\mathscr{H})$, *then a necessary and sufficient condition for*

$$\mathscr{V}_p = \mathscr{E}_{\mathbf{AB}}^{-1}(0) \oplus \mathscr{E}_{\mathbf{AB}}(\mathscr{V}_p)$$

is that $0 \in \operatorname{iso}\sigma(\mathscr{E}_{\mathbf{AB}})$.

Corollary 2.7 answers [20, Question 1]; it was proved in [8] under the more restrictive hypothesis that $0 \in \sigma(\mathscr{E}_{\mathbf{AB}})$ is isolated in the set $S = \{a_1b_1 + a_2b_2 : a_i \in \sigma(A_i), b_i \in \sigma(B_i), i = 1, 2\}$. (Observe that $\sigma(\mathscr{E}_{\mathbf{AB}}) \subseteq S$ [13].) Variants of Corollary 2.7 for Δ_{AB}, and the *generalized derivations* $\delta_{AB}(X) = AX - XB$, have been considered in [8,10]. Observe that for the generalized derivation δ_{AB}, $0 \in \operatorname{iso}\sigma(\delta_{AB})$ if and only if $\sigma(A) \cap \sigma(B)$ is finite. Thus, if A, B are totally hereditarily normaloid operators in $B(\mathscr{H})$ for which isolated points are normal eigen-values (in particular, if A, B are hyponormal), then $B(\mathscr{H}) = \delta_{AB}^{-1}(0) \oplus \delta_{AB}(B(\mathscr{H}))$ if and only if $\sigma(A) \cap \sigma(B)$ is finite.

References

[1] P. Aiena, O. Monsalve, The single valued extension property and the generalized Kato decomposition property, Acta Sci. Math. (Szeged) 67 (2001) 461–477.

[2] J. Anderson, On normal derivations, Proc. Amer. Math. Soc. 38 (1973) 136–140.

[3] J. Anderson, C. Foiaş, Properties which normal operators share with normal derivations and related operators, Pac. J. Math. 61 (1975) 313–325.

[4] F.F. Bonsall, J. Duncan, Numerical ranges I, Lond. Math. Soc. Lecture Notes Series 2 (1971).

[5] F.F. Bonsall, J. Duncan, Numerical ranges II, Lond. Math. Soc. Lecture Notes Series 10 (1973).

[6] F.F. Bonsall, J. Duncan, Complete Normed Algebras, Ergebnisse der Mathematik und ihrer Grenzgebiete Band, vol. 80, Springer-Verlag, 1973.

[7] I. Colojoařa, C. Foiaş, Theory of Generalized Spectral Operators, Gordon and Breach, New York, 1968.

[8] B.P. Duggal, Robin E. Harte, Range-kernel orthogonality and range closure of an elementary operator, Monatsh. Math. 143 (2004) 179–187.
[9] B.P. Duggal, Skew exactness, subspace gaps and range-kernel orthogonality, Linear Algebra Appl. 383 (2004) 93–106.
[10] B.P. Duggal, Weyl's theorem for a generalized derivation and an elementary operator, Math. Vesnik 54 (2002) 71–81.
[11] B.P. Duggal, S.V. Djordjević, Generalized Weyl's theorem for a class of operators satisfying a norm condition, Math. Proc. Royal Irish Acad. 104A (2004) 75–81.
[12] N. Dunford, J.T. Schwartz, Linear Operators, Part I, Interscience, New York, 1964.
[13] M.R. Embry, M. Rosenblum, Spectra, tensor products, and linear operator equations, Pac. J. Math. 53 (1974) 95–107.
[14] T. Kato, Perturbation theory for nullity, deficiency and other quantities of linear operators, J. Math. Anal. 6 (1958) 261–322.
[15] Dragoljub Kečkić, Orthogonality of the range and the kernel of some elementary operators, Proc. Amer. Math. Soc. 128 (2000) 3369–3377.
[16] K.B. Laursen, M.M. Neumann, An Introduction to Local Spectral Theory, London Mathematical Society Monographs (N.S.), Oxford University Press, 2000.
[17] M. Mbekhta, Généralisation de la décomposition de Kato aux opérateurs paranormaux et spectraux, Glasgow Math. J. 29 (1987) 159–175.
[18] V.S. Shulman, On linear equations with normal coefficients, Dokl. Akad. Nauk USSR 270-5 (1983). 1070–1073, English transl. in Soviet Math. Dokl. 27 (1983) 726–729.
[19] A. Turnšek, Generalized Anderson's inequality, J. Math. Anal. Appl. 263 (2001) 121–134.
[20] A. Turnšek, Orthogonality in $\mathscr{C}_p$ classes, Mh. Math. 132 (2001) 349–354.