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Irreflexive and reflexive dimension

Larry J. Langley[1]

Dartmouth College, Hanover, NH 03755-3551, USA

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Abstract

There are three equivalent definitions of dimension for partially ordered sets. When generating these three definitions to $\mathscr{C}$-dimension over an arbitrary class of orders $\mathscr{C}$, the three definitions diverge. We compare these three definitions and determine certain requirements under which they are equivalent.

1. Introduction

The dimension of a partially ordered set, introduced by Dushnik and Miller [1], is the minimum number of linear extensions of a given partial order required to realize that order. An *ordered set* is a pair consisting of a ground set X and a transitive, antisymmetric, reflexive relation $\leqslant$ (a reflexive order). An *extension* of a given order $<$ is an order $<'$ on the same ground set so that $x < y \Rightarrow x <' y$. A collection of ordered sets $\{(X, <_i)\}$ *realize* a given ordered set $P = (X, <)$ if the intersection of their order relations (as sets of ordered pairs) equals $<$. The *dimension* of a partially ordered set $P = (X, <)$ is the minimum κ such that P is realized by $\{L_i\}$, where $\{L_i\}$ is an arbitrary collection of linear extensions of P and the cardinality of $\{L_i\}$ is κ. This concept of dimension was generalized to interval orders [8] and then generalized further to an arbitrary class of orders $\mathscr{C}$ [4].

Let $\mathscr{C}$ be a collection of ordered sets that is hereditary and contains all linear orders. We require $\mathscr{C}$ to be hereditary so that suborders do not have greater dimension, and isomorphic ordered sets have the same dimension. If $\mathscr{C}$ contains all linear orders then $\mathscr{C}$-dimension is well defined and bounded by linear dimension. (Linear dimension is well defined for all orders [1].)

[1] *Correspondence address*: Department of Mathematics, University of Colorado at Denver, Campus Box 170, P.O. Box 173364, Denver, CO 80217-3364, USA. Research supported by ONR Grant N00014-91-J-1019.

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The *$\mathscr{C}$-dimension* of an ordered set $P = (X, \leqslant)$ is the minimum κ such that

$$(X, \leqslant) = \bigcap_{i \in I, |I| = \kappa} (X, \leqslant_i) = (X, \bigcap_{0 \leqslant i < \kappa} \leqslant_i),$$

where $\leqslant_i \in \mathscr{C}$. This is abbreviated $\mathscr{C}$-Dim(P).

A second definition of dimension is also commonly used: the dimension of a partially ordered set is the minimum number of linear orders so that the given partially ordered set can be imbedded in the direct product of these linear orders. This definition gives some justification to the choice of the word dimension. It is a classical result that the two definitions are equivalent [7]. The intersection dimension of a partial order is always equal to the direct product dimension of a partial order. In addition there is a third definition of dimension obtained by considering the product of irreflexive linear orders. Milner and Pouzet [5] prove that these three definitions of dimension are equivalent. In the same manner as intersection dimension, we can generalize these two definitions of product dimension to arbitrary classes of ordered sets:

The *$\mathscr{C}$-direct product dimension* of an ordered set (abbreviated $\mathscr{C}$-DPD(P)) is the minimum κ such that

$$(X, \leqslant) \text{ is a restriction of } \bigotimes_{i \in I, |I| = \kappa} (X_i, \leqslant_i),$$

where $\leqslant_i \in \mathscr{C}$ and $\bigotimes$ is the usual product of ordered sets.

The *$\mathscr{C}$-strict product dimension* of an ordered set (abbreviated $\mathscr{C}$-SPD(P)) is the minimum κ such that

$$(X, \leqslant) \text{ is a restriction of } \bigodot_{i \in I, |I| = \kappa} (X_i, \leqslant_i),$$

where $\bigodot_{i \in I, |I| = \kappa} (X', \leqslant_i)$ is given by $(a_0, a_1, \ldots, a_i, \ldots) \leqslant (b_0, b_1, \ldots, b_i, \ldots)$ if and only if $\forall i \in I$, $|I| = \kappa$, $a_i = b_i$ or $a_i < b_i$. $\bigodot$ is called the *irreflexive product*.

It is not true, however, that these three definitions of dimension are equivalent for all possible classes of orders. For example, the interval direct product dimension of a partial order is in general less than the intersection dimension of the order (Fig. 1), but the interval strict product dimension is equivalent to the interval intersection dimension. An elegant method of understanding this equivalence and a geometric representation of the product can be seen in the box orders introduced by Felsner et al. [2]. This equivalence has been attributed to Cogis as well.

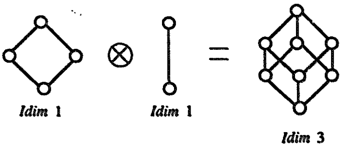

Fig. 1.

In fact the gap between product and intersection dimension can become arbitrarily large. The partial order consisting of subsets of a $2n$-element set ordered by set inclusion is the product of n interval orders on four elements of the type given in Fig. 1. It contains the order on single-element and $(2n-1)$-element subsets which have interval dimension $2n$ [8].

We would like to find the relationships among the three definitions of dimension for general classes $\mathscr{C}$. We will show that both definitions of product dimension are dominated by intersection dimension, however neither product dimension dominates the other.

2. Conditions for equality

To begin with, it is easy to show that the product dimensions can be no greater than the intersection dimension [5, 6].

Lemma 1. *For any $\mathscr{C}$ and any ordered set P,*
(1) $\mathscr{C}$-DPD$(P) \leqslant \mathscr{C}$-Dim$(P)$,
(2) $\mathscr{C}$-SPD$(P) \leqslant \mathscr{C}$-Dim$(P)$.

Proof. Suppose $P = (X, \leqslant)$ is isomorphic to the intersection of the ordered sets $P_i = (X, \leqslant_i)$, $i \in I$, $|I| < \kappa$ where $P_i \in \mathscr{C}$. Assign to each vertex x of X the vertex $(x, x, \ldots)$ of the appropriate product of the P_i's. $(x, x, \ldots) < (y, y, \ldots)$ if and only if $x < y$. $(x, x, \ldots) = (y, y, \ldots)$ if and only if $x = y$. Therefore P is a restriction of the product of the P_i's and hence the $\mathscr{C}$-product dimension of P, direct or strict, is no greater than the $\mathscr{C}$-dimension of P. □

We will find cases of equality by constructing the required intersection of extensions of the product. To "substitute" one ordered set into another is to replace each vertex of the first ordered set by the second ordered set. More precisely we order the collection of pairs (x_i, y_j) lexicographically so that $(x_1, y_1) < (x_2, y_2)$ if $x_1 < x_2$ or if $x_1 = x_2$ and $y_1 < y_2$.

One condition of equivalence is known.

Theorem 1 (Mitas [6]). *If $\mathscr{C}$ is closed under "substitution by chains" then* $\mathscr{C}$-dim$(P) = \mathscr{C}$-DPD(P).

Proof. See [6]. □

Theorem 2. *If $\mathscr{C}$ is closed under "substitution by chains" then* $\mathscr{C}$-Dim$(P) = \mathscr{C}$-SPD(P).

Proof. Suppose $P = (X, <)$ is contained in $\bigodot_{i \in I, |I| = \kappa} P_i$. Define $P_i' = (\times_{i \in I, |I| = \kappa} X_i, <_i')$ by taking the dual of a linear extension of $\bigotimes_{j \in I, |I| = \kappa, j \neq i} P_j$

and substituting this linearly ordered set for each vertex of P_i. *Claim*: $\bigodot_{i\in I,|I|=\kappa} P_i = \bigcap_{i\in I,|I|=\kappa} P_i'$. Suppose $(a_1,a_2,\ldots) <_{\odot} (b_1,b_2,\ldots)$ then $a_i < b_i$ so $(a_1,a_2,\ldots) <_i' (b_1,b_2,\ldots)$. On the other hand, suppose $(a_1,a_2,\ldots) \nless_{\odot} (b_1,b_2,\ldots)$. There must be some i so that $a_i \nless b_i$. If $a_i \nleq b_i$ then $(a_1,a_2,\ldots) \nless_i' (b_1,b_2,\ldots)$. If $a_i = b_i$ for all i, $(a_1,a_2,\ldots) = (b_1,b_2,\ldots)$ and hence $(a_1,a_2,\ldots) \nless_i' (b_1,b_2,\ldots)$ for any i. If neither of the former cases hold there must be some $a_i = b_i$ and some $a_j < b_j$. So, in $\bigotimes_{j\in I,|I|=\kappa,j\neq i} P_j$, $(a_1,a_2,\ldots) \nless' (b_1\, b_2,\ldots)$, and hence $(a_1,a_2,\ldots) \nless (b_1\, b_2,\ldots)$, in the dual of any linear extension of $\bigotimes_{j\in I,|I|=\kappa,j\neq i} P_j$ and therefore $(a_1,a_2,\ldots) \nless_i' (b_1\, b_2,\ldots)$. So this gives us $\bigodot_{i\in I,|I|=\kappa} P_i = \bigcap_{i\in I,|I|=\kappa} P_i'$. To find the representation for the original order P take each P_i' and restrict to the vertices of $\times_{i\in I,|I|=\kappa} X_i$ which form an order isomorphic to P under $<_{\odot}$. This proves that $\mathscr{C}$-Dim$(P) \leqslant \mathscr{C}$-SPD$(P)$ and together with Lemma 1 the theorem is complete. □

Theorem 3. *If $\mathscr{C}$ is closed under "substitution by antichains" then* $\mathscr{C}$-Dim$(P) = \mathscr{C}$-SPD(P).

Proof. The proof follows as above; however, instead of substituting a linear extension of $\bigotimes_{j\in I,|I|=\kappa,j\neq i} P_j$, we take an antichain on the vertices of $\bigotimes_{j\in I,|I|=\kappa,j\neq i} P_j$ and substitute into P_i'. □

Many classes of orders are defined by restricted suborders, for example an order is an interval order if and only if it contains no suborder isomorphic to a $\underline{2} + \underline{2}$ (a pair of two-element chains). An autonomous suborder is a collection of vertices $V \subseteq X$ so that $\forall x \in X - V$, either $\forall v \in V, x < v$, $\forall v \in V, x > v$ or, $\forall v \in V$, x is not related to v. That is, for each x that is not in V, x is above, below or not related to every element of V.

Mitas [6] noted that any class of orders that is defined by a collection of restricted suborders, such that no suborder contains an autonomous 2-chain (a chain of two elements), is closed under substitution by chains and hence satisfies the conditions of the above theorems. Similarly, if the restricted orders contain no autonomous antichain of two elements the class is closed under substitution by antichains and $\mathscr{C}$-SPD equals intersection dimension. From this and from the example in Fig. 1 we obtain Table 1.

For example, semi-orders are defined as having no restriction isomorphic to either a $\underline{2} + \underline{2}$ or a $\underline{1} + \underline{3}$ (a single element together with a three-element chain). Neither restriction contains an autonomous two-element antichain, so semi-orders are closed under substitution by antichains. Hence semi-order strict product dimension equals semi-order intersection dimension.

The last example in the list is a collection of orders that are closed under neither substitution by chains nor antichains. This class has the property that there is an order in $\mathscr{C}$ that has direct product dimension three and strict product dimension two, and a different order in $\mathscr{C}$ that has strict product dimension three but direct product dimension two. Any order in this class consists of a chain, together with a possible

Table 1

	Restricted suborders	Dim = DPDim?	Dim = SPDim?
Linear orders	o o	Yes	Yes
Weak orders	o o o	No	Yes
Interval orders	o o o o	No	Yes
Semi-orders	o o o, o o o, o o	No	Yes
Series–parallel orders	o o o o	Yes	Yes
Forests	o o o	Yes	Yes
(See Fig. 2)	o, o o, o o o, o o o o, o o, o o o, o o	No	No

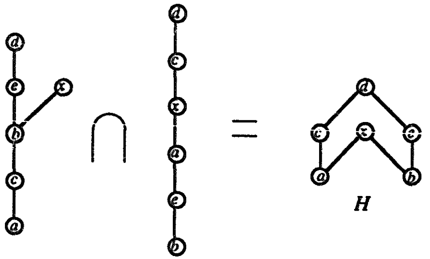

Fig. 2.

extra vertex which is greater than some subset (perhaps empty) of the chain. We will call this element a *protrusion*. The six-element order H given in Fig. 2 has linear dimension three. However, it does have $\mathscr{C}$-intersection dimension two, with the only requirement that at least one of the two orders has a protrusion (corresponding to the element labeled x). An order that includes three separate copies of H cannot have $\mathscr{C}$-intersection dimension two, since if H is the intersection of two orders from $\mathscr{C}$ there must be a distinct protrusion for each H and in two orders we can have at most two protrusions. The products given in Figs. 3 and 4 demonstrate that it is possible to give three copies of H in an order and retain either strict or direct product dimension two. On the other hand the strict product dimension two order cannot have direct product dimension two because it is impossible to have an isolated six-element order if the three vertices labeled x are mutually incomparable. The direct product dimension two order cannot have strict product dimension two because the elements labeled x cannot be related in any suborder of a $\mathscr{C}$-strict product dimension two order.

Although the theorems presented give sufficient conditions for equality between intersection dimension and one or both product dimensions, the question of which conditions are necessary for equality to hold remains open. Also, there is the question of how closely tied together are the different definitions. The example of interval direct product dimension given in the introduction has half the interval strict product dimension. Are there examples with different orders of magnitude?

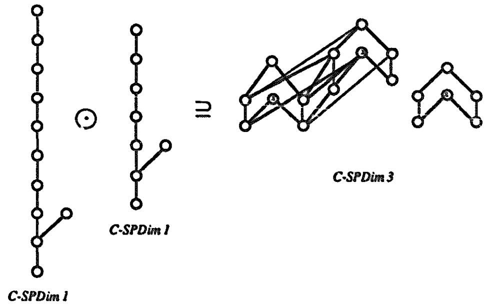

Fig. 3.

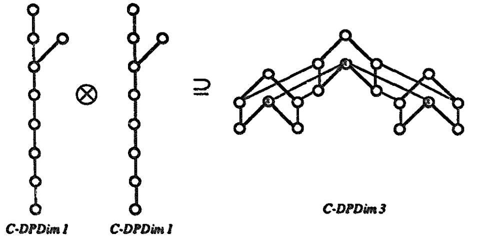

Fig. 4.

References

[1] B. Dushnik and E.W. Miller, Partially ordered sets, Amer. J. Math. 63 (1941) 600-610.
[2] S. Felsner, M. Habib and R.H. Möhring, On the interplay of interval dimension and dimension, Preprint 285, Technische Universität, Berlin.
[3] P.C. Fishburn, Interval Orders and Interval Graphs (Wiley, New York, 1985).
[4] M. Habib, D. Kelly and R.H. Möhring, Interval dimension is a comparability invariant, Discrete Math. 88 (1991) 211-229.
[5] E.C. Milner and M. Pouzet, A note on the dimension of a Poset, Order 7 (1990) 101-102.
[6] J. Mitas, Interval orders based on arbitrary ordered sets, Preprint, TH Darmstadt, Darmstadt, Germany (1992).
[7] O. Ore, Theory of Graphs, American Mathematical Society Colloquium Publications 38 (American Mathematical Society, Providence, RI, 1962).
[8] W.T. Trotter and K. Bogart, Maximal dimensional partially ordered sets III: a characterization of Hiraguchi's inequality for interval dimension, Discrete Math. 15 (1976) 389-400.