Notes

The weighted independent domination problem is NP-complete for chordal graphs

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Abstract

An independent dominating set of a graph $G = (V, E)$ is a pairwise non-adjacent subset $D$ of $V$ such that every vertex not in $D$ is adjacent to at least one vertex in $D$. Suppose each vertex in $V$ is associated with a weight which is a real number. The weighted independent domination problem is to find an independent domination set of minimum total weights. This paper records an unpublished result of 20 years ago that the weighted independent domination problem is NP-complete for chordal graphs.

Keywords: Chordal graph; Independent domination; NP-complete; Weight

The concept of domination in graph theory is a natural model for many location problems in operations research. In a graph $G = (V, E)$, a dominating set is a subset $D$ of $V$ such that every vertex not in $D$ is adjacent to at least one vertex in $D$. The domination problem is to determine, for a graph $G$ and a positive integer $k$, if $G$ has a dominating set of size at most $k$. Domination and its variations have been extensively studied in the literature (see [1,4,5]).

Domination and most of its variations are NP-complete for chordal graphs (even for the subclass of split graphs) with the exception of independence domination (see [3]). On the other hand, an unpublished proof for the NP-completeness of the weighted independent domination in chordal graphs by the author 20 years ago (see [3,2]) has been queried from time to time. Someone even claimed that he has an efficient algorithm for the problem. The purpose of this paper is to make a record of the proof for those who are interested.

Recall that an independent dominating set is a dominating set whose elements are pairwise non-adjacent. For the weighted version, each vertex is associated with a weight which is a real number. A graph is chordal if every cycle of length at least 4 has at least one chord, which is an edge joining two non-consecutive vertices in the cycle.

Theorem 1. The weighted independent domination problem is NP-complete in chordal graphs.

Proof. We shall transform the domination problem in general graphs to the weighted independent domination problem in chordal graphs. Therefore, the NP-completeness of the weighted independent domination problem in chordal graphs follows from that of the domination problem in general graph.

For any graph $G = (V, E)$, consider the chordal graph $G' = (V', E')$ with vertex set

$$V' = \{v_1, v_2, v_3, v_4 \mid v \in V\}$$

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and edge set
\[ E' = \{v_1v_2, v_2v_3, v_3v_4 \mid v \in V\} \cup \{u_1v_4 \mid uv \in E\} \cup \{u_4v_4 \mid u, v \in V, u \neq v\}. \]

Also, each \( v_4 \) is associated with weight \( 2|V| \), and each other vertex with weight 1. It is easy to see that \( G' \) is chordal. Fig. 1 shows an example of the transformation.

We claim that \( G \) has a dominating set of size at most \( k \) if and only if \( G' \) has an independent dominating set of total weight at most \( |V| + k \).

First, suppose \( G \) has a dominating set of size at most \( k \). Then \( D' = \{v_1, v_3 \mid v \in D\} \cup \{v_2 \mid v \in V - D\} \) is clearly an independent dominating set of \( G' \) with total weight at most \( |V| + k \).

On the other hand, suppose \( G' \) has an independent dominating set \( D' \) of total weight at most \( |V| + k \). If \( k \geq |V| \), then \( V \) is a dominating set of \( G \) of size at most \( k \). If \( k < |V| \), then \( D' \) cannot contain any \( v_4 \) and hence is of the form \( \{v_1, v_3 \mid v \in D\} \cup \{v_2 \mid v \in V - D\} \) for some subset \( D \) of \( V \). For any vertex \( u \in V \), since \( u_4 \) is dominated by some \( v_3 \) in \( G' \), we have that \( u \) is dominated by \( v \) in \( D \). Hence, \( D \) is a dominating set of \( G \). Also, the total weight of \( D' \) is \( |V| + |D| \) implying that \( D \) is of size at most \( k \). \( \Box \)

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