An optimal control problem for a prey–predator system with a general functional response

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ABSTRACT

An optimal control problem is studied for a prey–predator system with a general functional response. The control functions represent the rate of mixture of the populations and the cost functional is of Mayer type. The number of switching points of the optimal control is discussed in terms of the sign of a specific constant.

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1. Introduction

In this work, we find necessary optimality conditions for a problem related to the prey–predator system

\begin{align}
    y_1' &= ry_1 \left( 1 - \frac{y_1}{k} \right) - y_2 F(y_1, y_2) \\
    y_2' &= y_2 \left[ -d + cF(y_1, y_2) \right], \quad t \in [0, T].
\end{align}

Here $y_1(t)$ and $y_2(t)$ represent the densities of prey and predators at the moment $t \in [0, T]$ and $r, k, c, d > 0$ are given parameters. Function $f(y_1) = ry_1 \left( 1 - \frac{y_1}{k} \right)$ is the logistic growth rate of the prey species. The parameter $r$ controls the prey population growth, while the coefficient $k$ is the prey-carrying capacity.

The predator functional response $F(y_1, y_2)$ signifies the number of prey consumed per predator in unit time. It is supposed to satisfy the following conditions:

(i) $F$ is well defined and $F(y_1, y_2) > 0$, for all $y_1, y_2 > 0$.
(ii) $F \in C^1([0, \infty)^2)$; $F$ is bounded with respect to $y_2$.
(iii) $\frac{\partial F}{\partial y_1}(y_1, y_2) > 0$ and $F(y_1, y_2) + y_2 \frac{\partial F}{\partial y_2}(y_1, y_2) > 0$, for all $y_1, y_2 > 0$.

Function $F$ above includes as particular cases various classical functional responses (see [1,2]). For example:

\begin{align*}
    F(y_1, y_2) &= by_1 \quad (\text{Holling type I}). \\
    F(y_1, y_2) &= \frac{by_1}{1 + my_1} \quad (\text{Holling type II}). \\
    F(y_1, y_2) &= \frac{by_1^2}{1 + my_1^2} \quad (\text{Holling type III}). \\
    F(y_1, y_2) &= k \left( 1 - e^{-by_1} \right) \quad (\text{Ivlev functional response}).
\end{align*}
The parameters $b$ and $m$ are positive. All of these functional responses are only prey dependent and they satisfy conditions (i)–(iii).

Two functional responses that depend both on prey and predators are:

$$F (y_1, y_2) = \frac{by_1}{y_2 + b y_1}, \quad m, n, b, l > 0 \ (\text{Hassell} \& \text{Valery}).$$

$$F (y_1, y_2) = \frac{by_1}{y_1 + l y_2 + y_0}, \quad b, l > 0 \ (\text{De Angelis et al.; Beddington}).$$

Both functions verify hypotheses (i)–(iii), the first one for $0 < m \leq 1$.

The predator’s numerical response $G(y_1, y_2) = -d + cF(y_1, y_2)$ shows the per capita growth rate of the predator population. Parameters $d$ and $c$ are the per capita predator death rate and the maximal per capita predator birth rate, respectively.

One separates the prey from the predators with the aid of a control function $u : [0, T] \to \mathbb{R}, 0 \leq u(t) \leq 1$ a.e. on $[0, T]$. Then the functional response $F(y_1, y_2)$ is multiplied by $u$. The separation rate at the moment $t$ is $1 - u(t)$. If $u(t) = 0$, then prey and predators are completely separated from each other at the moment $t$; if $u(t) = 1$, then they are not separated at all, that is the ecosystem coincides with the original one. Similarly, we separate the prey individuals from each other by a control function $v : [0, T] \to \mathbb{R}$. Then the second term of the intrinsic growth rate of the prey population will be multiplied by $v$. Suppose that the prey individuals cannot be completely isolated from each other, i.e. $v(t) > 0$. More exactly, assume that $0 < v_0 \leq v(t) \leq 1$ a.e. on $[0, T]$, where $v_0$ is a fixed value in $(0, 1)$. The control functions represent the rate of mixture of the populations: $u$ is the rate of mixture between prey and predators, while $v$ is the rate of mixture between prey individuals.

The dynamics of the controlled ecosystem is given by

$$\begin{align*}
\dot{y}_1 &= r y_1 \left(1 - \frac{y_1 v}{k}\right) - u y_2 F(y_1, y_2), \quad t \in [0, T]. \\
\dot{y}_2 &= y_2 \left[-d + c u F(y_1, y_2)\right], \quad t \in [0, T].
\end{align*}$$

(1.2)

Initial value conditions of the form

$$y_1(0) = y_1^0 > 0, \quad y_2(0) = y_2^0 > 0$$

(1.3)

are associated with system (1.2).

Assumptions (i)–(iii) assure the existence and uniqueness of a local solution $y = (y_1, y_2)$ of problem (1.2) and (1.3), defined on a maximal interval $[0, \delta), \delta > 0$. Since system (1.2) admits the zero solution and $y_1^0 > 0, y_2^0 > 0$, it follows by a comparison theorem that $y_1 > 0, y_2 > 0$ on $[0, \delta)$. Condition (i) implies the boundedness of $y_1$ and $y_2$ and, consequently, the solution of (1.2) and (1.3) is defined on the whole of $[0, T]$.

The goal of the work is to find the optimal control $(u, v)$ such that, at the end of the time interval $[0, T]$, the total density of the two populations is maximal. The optimal control problem associated with system (1.2) and (1.3) is

$$\text{Min } \{-y_1(T) - y_2(T)\},$$

(1.4)

where $u : [0, T] \to [0, 1], \quad v : [0, T] \to [v_0, 1]$, and $(y_1, y_2)$ verifies (1.2) and (1.3).

The cost functional is of Mayer type. The case of linear growth rate $f(y_1) = r y_1$ and linear functional response $F(y_1, y_2) = b y_1$ (Holling type I) was studied in [3]. Paper [4] is devoted to the optimality conditions for a three-population ecosystem. Basic results on the optimal control theory can be found in [5]. Other applications of the control theory in biology are presented in [6–9].

Section 2 of the present work is devoted to the maximum principle for our problem. One finds that the control variable $u$ is bang–bang, while $v$ is $v_0$ on the entire interval $[0, T]$. In Section 3 we establish the number of switching points of $u$ in terms of the sign of the constant $c - 1$.

2. The maximum principle

The boundedness of the solution $y$ of the control system (1.2) and (1.3) permits us to take a compact target set at $t = T$. Then, according to [5, Theorem 1.2, pp. 43], it follows that our optimal control problem admits at least one solution $(y, (u, v))$, where $y = (y_1, y_2)$.

We apply Pontryagin’s maximum principle to find the form of the optimal control $(u, v)$ for problem (1.2)–(1.4). To this end, we associate the Hamiltonian function

$$H(y, p, u, v) = r y_1 p_1 - d y_2 p_2 - v \frac{r y_2^2 p_1}{k} + u y_2 F(y_1, y_2)(c p_2 - p_1).$$

(2.1)

where $p_1, p_2$ are the adjoint variables. If $(u, v)$ is the optimal control, $y = (y_1, y_2)$ is the optimal state, then $p_1$ and $p_2$ verify the adjoint system

$$\begin{align*}
p'_1 &= -r p_1 + 2 r y_1 p_1 \frac{F'(y_1, y_2}(y_1, y_2)(p_1 - c p_2) \\
p'_2 &= dp_2 + u(p_1 - c p_2)
\end{align*}$$

(2.2)
and the transversality condition
\[ p_1(T) = p_2(T) = 1. \]  
\( \text{(2.3)} \)

The optimal control \((u, v)\) should maximize the Hamiltonian function \(H\) for fixed \(y_1, y_2, p_1, p_2\). Then, hypothesis (i) leads to the following form of the optimal variables \(u\) and \(v\):
\[ u(t) = \begin{cases} 0, & (p_1 - cp_2)(t) > 0 \\ 1, & (p_1 - cp_2)(t) < 0 \end{cases} \]
\[ v(t) = \begin{cases} v_0, & p_1(t) > 0 \\ 1, & p_1(t) < 0 \end{cases} \]
\( a.e. \) on \([0, T]\). In the sequel, we show that \(v = v_0\) on \([0, T]\), while \(u\) can be either 0, or 1, or it has a unique switching point in \((0, T)\).

**Remark.** From (2.4) we can easily see that
\[ u(t)(p_1 - cp_2)(t) \leq 0 \quad a.e. \text{ on } [0, T]. \]  
\( \text{(2.5)} \)

Regarding the first equation from (2.2) as a linear equation in \(p_1\) of the form \(p_1' = -\gamma (t) p_1 + \alpha (t)\), with \(\gamma (t) = r - v \frac{2\kappa_1}{k}\), \(\alpha (t) = u y_2 \frac{\partial F}{\partial y_1} (y_1, y_2) (p_1 - cp_2)\), and the end-point value \(p_1(T) = 1\), we can write
\[ p_1(t) = e^{\int_t^T \gamma(s) ds} \left\{ 1 - \int_t^T u y_2 \frac{\partial F}{\partial y_1} (y_1, y_2) (p_1 - cp_2) \right\} (s) e^{-\int_s^T \gamma(t) dt} ds \].

Analogously we have
\[ p_2(t) = e^{-d(T-t)} \left\{ 1 - \int_t^T u (p_1 - cp_2) (F (y_1, y_2) + y_2 \frac{\partial F}{\partial y_2} (y_1, y_2)) \right\} (s) e^{d(T-s)} ds. \]

Using hypothesis (iii), together with the estimates (2.5), one observes that \(p_1(t) > 0\) and \(p_2(t) > 0\), \(\forall t \in [0, T]\). Hence \(v(t) = v_0\), \(\forall t \in [0, T]\). The control function \(u\) is bang–bang. We shall discuss the form of \(u\) as a function of the sign of \(c - 1\).

### 3. The number of switching points of \(u\)

To establish the number of switching points of \(u\), we analyze the sign of \(p_1 - cp_2\). One specifies three cases according to the sign of \(c - 1\). Recall that \(p_1 > 0, p_2 > 0\) on \([0, T]\).

Using system (2.2) and (2.3) with \(v = v_0\) on \([0, T]\), one finds that
\[ (cp_2 - p_1)' = cp_2 + rp_1 - v_0 \frac{2y_1 p_1}{k} - u (cp_2 - p_1) \left\{ c F (y_1, y_2) + c y_2 \frac{\partial F}{\partial y_2} (y_1, y_2) \right\} \quad \forall t \in [0, T]. \]  
\( \text{(3.1)} \)

Define \(y_1^{\text{max}} = \max \{y_1(t), t \in [0, T]\}\). In the sequel, we chose \(v_0\) such that
\[ 0 < v_0 < \min \left\{ 1, \frac{k}{2y_1^{\text{max}}} \right\}. \]  
\( \text{(3.2)} \)

- **Case 1:** \(c < 1\). Since \((cp_2 - p_1)(T) = c - 1 < 0\), it follows that \(cp_2 - p_1 < 0\) in a neighborhood \((\tau, T)\) of \(T\). Suppose it is maximal with respect to this property. If \(\tau \in (0, T)\), then \((cp_2 - p_1)(\tau) = 0\) and the optimal control \(u\) is 0 on \((\tau, T)\). Equalities (3.1) and (3.2) lead to
\[ (cp_2 - p_1)' = cp_2 + rp_1 \left( 1 - v_0 \frac{2y_1}{k} \right) > 0, \]
i.e. \(cp_2 - p_1\) is monotonically increasing and negative on \((\tau, T)\). This contradicts the condition \((cp_2 - p_1)(\tau) = 0\). Therefore \(\tau = 0\) and \(u(t) = 0\), for every \(t \in [0, T]\).

- **Case 2:** \(c = 1\). Since \((cp_2 - p_1)(T) = 0\) and \((cp_2 - p_1)'(T) = cd + r \left( 1 - v_0 \frac{2y_1(T)}{k} \right) > 0\) (from (3.1)), it follows that \(cp_2 - p_1\) is monotonically increasing in a left neighborhood of \(T\). As in Case 1, we infer again that \(u(t) = 0\) for each \(t \in [0, T]\).

- **Case 3:** \(c > 1\). In this case, \((cp_2 - p_1)(T) = c - 1 > 0\), so \(cp_2 - p_1 > 0\) in a neighborhood of \((\tau, T)\) of \(T\), which can be chosen maximal. Then \(u(t) = 1\) on \((\tau, T)\). There are two subcases:

  - **(a)** \(\tau = 0\). Then \(u = 1\) on the whole interval \([0, T]\).
  - **(b)** \(\tau \in (0, T)\). Then \(cp_2 - p_1 > 0\) on \((\tau, T)\) and \((cp_2 - p_1)(\tau) = 0\). With the aid of (3.1), we obtain that the function \(cp_2 - p_1\) is increasing in \(\tau\), that is \(cp_2 - p_1 < 0\) at least in a left neighborhood of \(\tau\). According to (2.4), here \(u = 0\). As long as \(u = 0\), function \(cp_2 - p_1\) is increasing, so we can repeat the reasoning of Case 2 with \(\tau\) instead of \(T\), to deduce that \(u(t) = 0\), for all \(t \in [0, \tau]\). Therefore in Case 3, the optimal control \(u\) either equals 1 on \([0, T]\), or has a unique switching time \(\tau \in (0, T)\).
Thus we have stated the following result.

**Theorem 3.1.** Assume that the constants $r, k, c, d, y_0^1, y_0^2$ from system (1.2) and (1.3) are positive, condition (3.2) holds, and function $F$ satisfies hypotheses (i)–(iii). If $(u, v)$ is the optimal control for problem (1.4), then $v = v_0$ on $[0, T]$ and $u$ is bang–bang, namely it has at most one switching time. More exactly, we have the following cases:

(I) If $c \leq 1$, then $u(t) = 0$, $(\forall) t \in [0, T]$. The corresponding optimal state is $y_1(t) = y_0^1e^{rt}$, $y_2(t) = y_0^2e^{-dt}$, $t \in [0, T]$.

(II) If $c > 1$, then $u$ admits at most one switching time $\tau$, which is the solution in $(0, T)$ of the equation $cp_2 - p_1 = 0$. Here $p = (p_1, p_2)$ is the solution of the adjoint system (2.2) and (2.3). If equation $cp_2 - p_1 = 0$ has no solution in $(0, T)$, then $u(t) = 1$, $(\forall) t \in [0, T]$. If equation $cp_2 - p_1 = 0$ has a unique solution $\tau$ in $(0, T)$, then $u$ has the form

$$u(t) = \begin{cases} 0, & t \in [0, \tau) \\ 1, & t \in [\tau, T] \end{cases}$$

**Remark.** In the particular case when $F(y_1, y_2) = by_1$ and $v = 0$ on $[0, T]$, we obtain again the result from Yosida [3].

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**References**


