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Applied Mathematics Letters 19 (2006) 22–31

Applied Mathematics Letters

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On the applicability of the Adomian method to initial value problems with discontinuities

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Received 17 September 2004; accepted 1 March 2005

Abstract

In this paper we extend our results of L. Casasús, W. Al-Hayani [The decomposition method for ordinary differential equations with discontinuities, Appl. Math. Comput. 131 (2002) 245–251] to initial value problems with several types of discontinuities, giving relevant examples of linear and nonlinear cases.

Keywords: Decomposition method; Adomian's polynomials; Initial value problems (IVP); Heaviside function; Dirac delta function

1. Introduction

The Adomian decomposition method [2–5] is an approximate analytical procedure to determine the solution series to many functional equations. Although many different kinds of equations have been studied [1,6–8] there is left a great deal of work to do regarding problems of convergence and applicability of the method. We have already explored in [1] the possibilities of this method in the field of ordinary differential equations with Heaviside functions as driving terms.

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doi:10.1016/j.aml.2005.03.004

The main objective of this paper is to analyse first order initial value problems (IVP) with Heaviside functions and other cases of discontinuities. We also make use of the so called Modified Technique [9] to improve the accuracy of the method.

Let us consider the general functional equation

$$y - N(y) = f, \tag{1.1}$$

where N is a nonlinear operator, f is a known function, and we are seeking the solution y satisfying (1.1). We assume that for every f, Eq. (1.1) has one and only one solution.

The Adomian technique consists of approximating the solution of (1.1) as an infinite series

$$y = \sum_{n=0}^{\infty} y_n, \tag{1.2}$$

and decomposing the nonlinear operator N as

$$N(y) = \sum_{n=0}^{\infty} A_n, \tag{1.3}$$

where A_n are polynomials (called Adomian polynomials) of $y_0, \ldots, y_n$ [2–5] given by

$$A_n = \frac{1}{n!} \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} \left[N\left(\sum_{i=0}^{\infty} \lambda^i y_i \right) \right]_{\lambda=0}, \qquad n = 0, 1, 2, \ldots.$$

The proofs of the convergence of the series $\sum_{n=0}^{\infty} y_n$ and $\sum_{n=0}^{\infty} A_n$ are given in [4,10–14]. Substituting (1.2) and (1.3) into (1.1) yields

$$\sum_{n=0}^{\infty} y_n - \sum_{n=0}^{\infty} A_n = f.$$

Thus, we can identify

$$y_0 = f,$$
$$y_{n+1} = A_n(y_0, \ldots, y_n), \qquad n = 0, 1, 2, \ldots.$$

Thus all components of y can be calculated once the A_n are given. We then define the n-term approximant to the solution y by $\phi_n[y] = \sum_{i=0}^{n-1} y_i$ with $\lim_{n\to\infty} \phi_n[y] = y$.

2. Decomposition method applied to an IVP

Consider the general IVP:

$$y' + k^2 y - g(y) = \lambda f(t, y), \qquad y(0) = \alpha, \qquad 0 \le t \le T, \tag{2.1}$$

where k, λ and α are real constants, g is a (possibly) nonlinear function of y and f is a function with some discontinuity.

Applying the decomposition method as in [2–5], Eq. (2.1) can be written as

$$Ly = \lambda f(t, y) - k^2 y + N(y), \tag{2.2}$$

where $L = \frac{d}{dt}$ is the linear operator and $N(y) = g(y)$ is the nonlinear operator. Operating on both sides of Eq. (2.2) with the inverse operator of L (namely $L^{-1}[\cdot] = \int_0^t [\cdot] dt$) yields

$$y(t) = y(0) + \lambda L^{-1} f(t, y) - k^2 L^{-1} y + L^{-1} N(y).$$

Upon using (1.2) and (1.3) it follows that

$$\sum_{n=0}^{\infty} y_n = y(0) + \lambda L^{-1} f(t, y) - k^2 L^{-1} \sum_{n=0}^{\infty} y_n + L^{-1} \sum_{n=0}^{\infty} A_n. \tag{2.3}$$

From Eq. (2.3), the iterates defined using the Standard Adomian method are determined in the following recursive way:

$$\begin{aligned} y_0 &= y(0) + \lambda L^{-1} f(t, y) = \alpha + \lambda L^{-1} f(t, y), \\ y_{n+1} &= -k^2 L^{-1} y_n + L^{-1} A_n, \qquad n = 0, 1, 2, \ldots. \end{aligned}$$

Using the Modified Technique, according to (2.3), the iterates are determined in the following recursive way:

$$\begin{aligned} y_0 &= y(0) = \alpha, \\ y_1 &= \lambda L^{-1} f(t, y) - k^2 L^{-1} y_0 + L^{-1} A_0, \\ y_{n+2} &= -k^2 L^{-1} y_{n+1} + L^{-1} A_{n+1}, \qquad n = 0, 1, 2, \ldots. \end{aligned}$$

2.1. Linear case

Let $g(y) = 0$ and $\alpha = 1$.

1. If we take $\lambda = 10$ and the function $f(t, y)$ is continuous, but not differentiable, for example

$$f(t, y) = \begin{cases} -t + \frac{1}{2} & \text{if } t < \frac{1}{2}, \\ t - \frac{1}{2} & \text{if } t \geq \frac{1}{2}, \end{cases}$$

the maximum errors for $0 \leq t \leq 1$ of the Standard and Modified Adomian methods are given in Table 1, where n represents the number of iterations.

The estimated orders of convergence (EOC) of the Standard and Modified Adomian methods for different values of the constant k are given in Table 2.

Fig. 1 represents both the exact solution $y_E(t)$ and our approximation $\phi_{13}(t)$ within the interval $0 \leq t \leq 1$.

In this case, the most appropriate method is the Modified one. For $k \geq 3$, the application of the method requires approximants of order $n \geq 15$ if we want to arrive beyond the discontinuity (at $t = \frac{1}{2}$).

2. Taking $k = 1$, $\lambda = 1$ and

$$f(t, y) = \begin{cases} 0 & \text{if } t < 1 \\ 1 & \text{if } t \geq 1 \end{cases}$$

(Heaviside function) then the following maximum errors are obtained for $0 \leq t \leq 2$ (Table 3).

Table 1

$k = 2$ n	Standard $\|y_E(t) - \phi_n(t)\|_\infty$	Modified $\|y_E(t) - \phi_n(t)\|_\infty$
8	1.62×10^0	0.40×10^{-1}
9	0.73×10^0	0.41×10^{-2}
10	0.29×10^0	0.38×10^{-2}
11	0.11×10^0	0.29×10^{-2}
12	0.36×10^{-1}	0.14×10^{-2}
13	0.11×10^{-1}	0.57×10^{-3}
14	0.31×10^{-2}	0.20×10^{-3}
15	0.84×10^{-3}	0.60×10^{-4}

Table 2

k	Standard		Modified	
	$t = 0.4$	$t = 0.6$	$t = 0.4$	$t = 0.6$
1	1.0574	1.0769	1.0695	1.1016
2	1.0729	1.1008	1.0806	1.0589
3	1.1463	1.2232	1.1482	1.2249

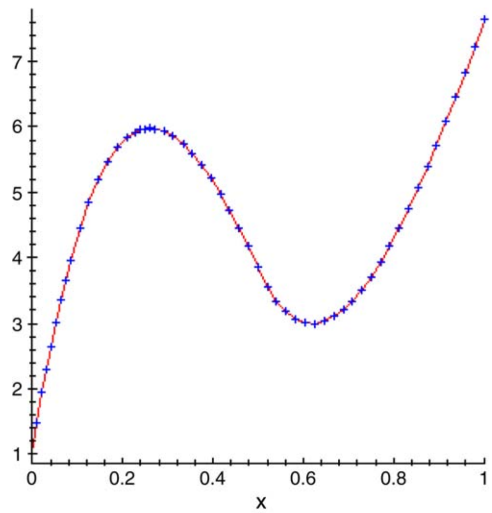

Fig. 1. Continuous line: $y_E(t)$, $+$: $\phi_{13}(t)$, $\lambda = 100$, $k = 2$.

For both methods (Standard and Modified), the EOC are 1.0827 at $t = 0.9$ and 1.0933 at $t = 1.1$. So, they have essentially the same value on both sides of the discontinuity. In Fig. 2 we represent both the exact solution $y_E(t)$ and our approximation $\phi_8(t)$ within the interval $0 \leq t \leq 2$.

In this case, the method is applicable until the value $k \simeq 2$.

Table 3

n	Standard $\|y_E(t) - \phi_n(t)\|_\infty$	Modified $\|y_E(t) - \phi_n(t)\|_\infty$
5	0.20×10^{0}	0.19×10^{0}
6	0.69×10^{-1}	0.67×10^{-1}
7	0.20×10^{-1}	0.20×10^{-1}
8	0.52×10^{-2}	0.52×10^{-2}
9	0.12×10^{-2}	0.12×10^{-2}
10	0.24×10^{-3}	0.24×10^{-3}

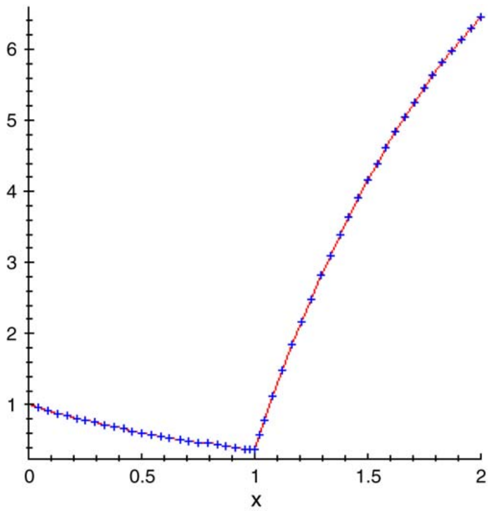

Fig. 2. Continuous line: $y_E(t)$, +: $\phi_8(t)$, $\lambda = 10$, $k = 1$.

Table 4

Standard		Modified	
$t = 0.9$	$t = 1.1$	$t = 0.9$	$t = 1.1$
1.0929	1.1044	1.0929	1.1161

3. Letting $k = 1$, $\lambda = 1$ and $f(t, y) = \delta(t - 1)$, the Dirac delta function at $t = 1$. The EOC are given in Table 4.

   So, they have essentially the same value on both sides of the discontinuity. Fig. 3 gives both the exact solution $y_E(t)$ and our approximation $\phi_9(t)$ for the interval $0 \le t \le 2$.

   Again, in this case, the method is applicable for the values $k \precsim 2.2$.

4. Now we take $k = 1$, $\lambda = 1$ and $f(t, y) = \delta\left(t - \frac{1}{2}\right) + \delta(t - 1) + \delta\left(t - \frac{3}{2}\right)$, Dirac delta function at $t = \frac{1}{2}, 1, \frac{3}{2}$. The EOC of the Standard method are given in Table 5.

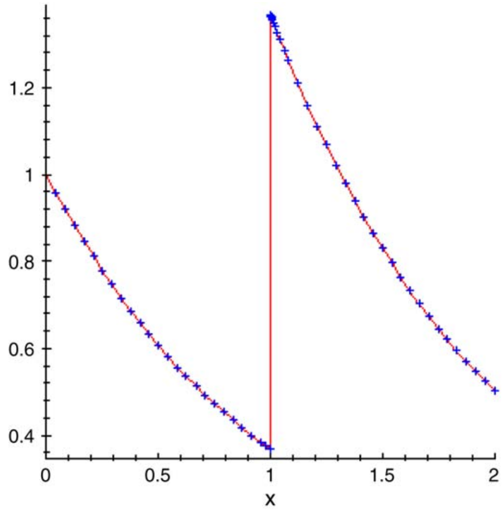

Fig. 3. Continuous line: $y_E(t)$, $+$: $\phi_9(t)$.

Table 5

$t = 0.4$	$t = 0.6$	$t = 0.9$	$t = 1.1$	$t = 1.4$	$t = 1.6$
1.0227	1.0715	1.0847	1.0936	1.1062	1.1157

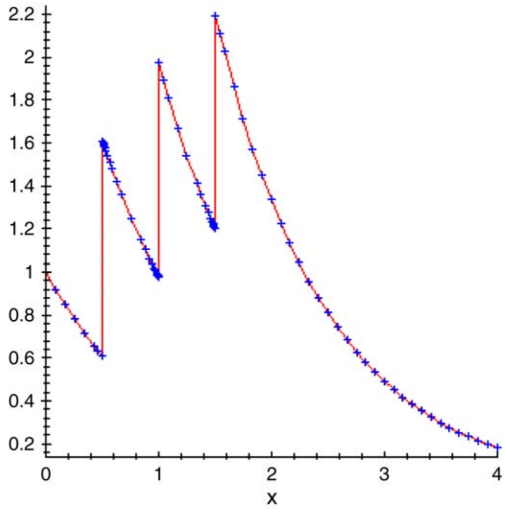

Fig. 4. $\lambda = 1$, $k = 1$, $0 \le t \le 4$.

For the Modified method, the EOC reduces by 10%. In Figs. 4 and 5, we depict the exact solution $y_E(t)$ by a continuous line and our approximation $\phi_{15}(t)$ by $+$.

The validity of the approximation $\phi_{15}(t)$ only till the second discontinuity can be easily noticed in Fig. 5.

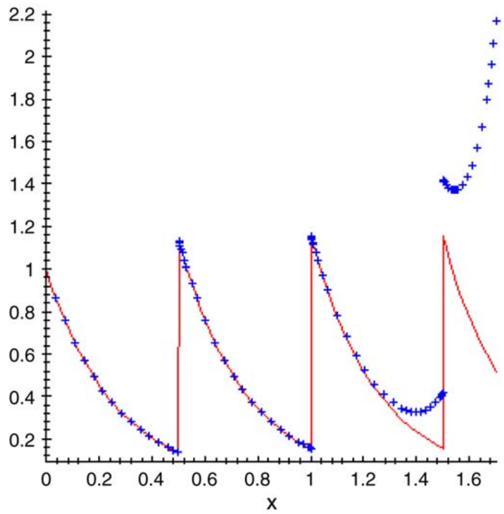

Fig. 5. $\lambda = 1$, $k = 2$, $0 \le t \le 1.7$.

2.2. Nonlinear case

Let $g(y) = y^2$ and $\alpha = 1$. In this case the nonlinear term is

$$Ny = g(y) = y^2 = \sum_{n=0}^{\infty} A_n,$$

and the Adomian polynomials can be derived as follows [15]:

$$\begin{aligned} y^2 &= (y_0 + y_1 + y_2 + y_3 + \cdots)^2 \\ &= \underbrace{y_0^2}_{A_0} + \underbrace{2y_0y_1}_{A_1} + \underbrace{2y_0y_2 + y_1^2}_{A_2} + \underbrace{2y_0y_3 + 2y_1y_2}_{A_3} \\ &\quad + \underbrace{2y_0y_4 + 2y_1y_3 + y_2^2}_{A_4} + \underbrace{2y_0y_5 + 2y_1y_4 + 2y_2y_3}_{A_5} + \cdots. \end{aligned} \tag{2.4}$$

After collecting and rearranging terms in Eq. (2.4) we get the following Adomian polynomials:

$$A_n = \sum_{i=0}^{n} y_i y_{n-i}, \qquad n \ge i, n = 0, 1, \ldots.$$

1. If we take $k = 1$, $\lambda = 1$ and

$$f(t, y) = \begin{cases} 0 & \text{if } t < 1 \\ 1 & \text{if } t \ge 1 \end{cases}$$

the EOC for both Standard and Modified methods is 1.0098 at $t = 1.1$. Fig. 6 represents both the numeric solution $y_N(t)$ with a very small error and our approximations $\phi_{15}(t)$ for $0 \le t \le 2$.
For all values of λ the method is applicable in this case, when $k \lesssim 1.5$.

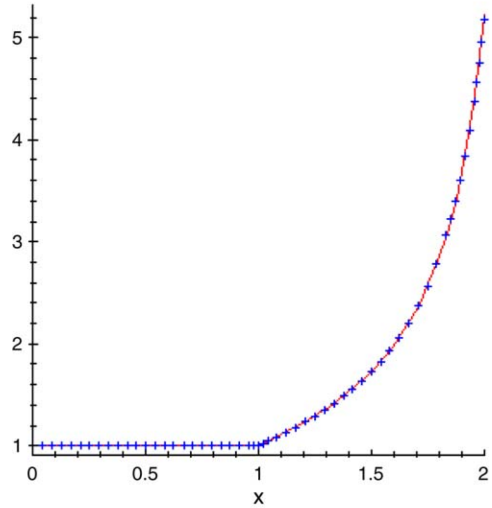

Fig. 6. Continuous line: $y_N(t)$, $+$: $\phi_{15}(t)$.

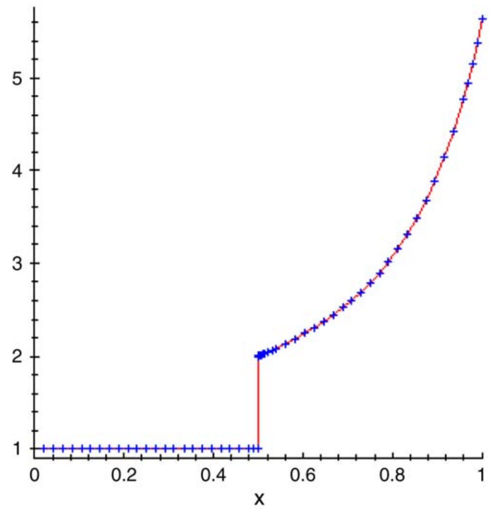

Fig. 7. Continuous line: $\phi_{15}(t)$, $+$: $\phi_{14}(t)$.

2. Taking $k = 1$, $\lambda = 1$ and $f(t, y) = \delta\left(t - \frac{1}{2}\right)$, Dirac delta function at $t = \frac{1}{2}$. For both methods, Standard and Modified, the EOC is 1.0000 at $t = 0.6$. In Fig. 7 we show our approximations $\phi_{15}(t)$ and $\phi_{14}(t)$ for $0 \leq t \leq 1$.

   For any value of λ the method is applicable in this case, when $k \lesssim 2.3$.
3. Finally, we take $k = 1$, $\lambda = 1$ and $f(t, y) = \delta\left(t - \frac{1}{4}\right) + \delta\left(t - \frac{1}{2}\right)$, the Dirac delta function at $t = \frac{1}{4}, \frac{1}{2}$. Fig. 8 represents our approximations $\phi_{15}(t)$, $\phi_{14}(t)$ for $0 \leq t \leq 0.6$.

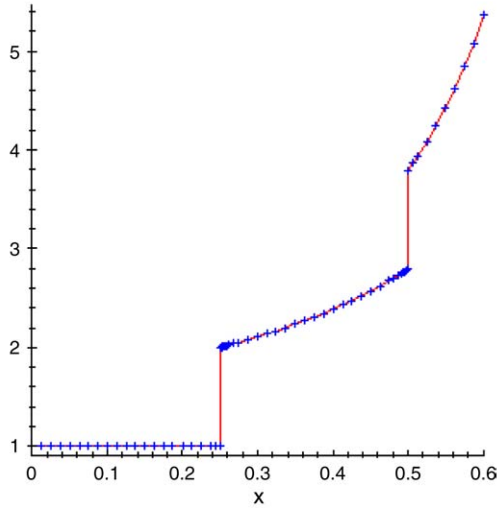

Fig. 8. Continuous line: $\phi_{15}(t)$, $+$: $\phi_{14}(t)$.

3. Conclusions and future work

1. As in [1], the size of the jump (given by λ) does not affect the convergence of the method, which behaves equally well on both sides of the discontinuity. In the nonlinear problems with large values of λ, sometimes a computation with more digits is required in order to avoid unstable oscillations.
2. The error in the Modified Technique is always smaller than the error of the Standard Adomian method for all the values of k.
3. We are already improving the method for the most difficult cases, as the nonlinear problem with $g(y) = y^2$. In this IVP, a direct application of the decomposition method with $k = 3$, does not converge even for small values of the parameter like $\lambda = \frac{1}{1000}$.
4. In the nonlinear cases, it may be of interest to consider the approximation of multiple solutions.

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