A compact metric space that is universal for orbit spectra of homeomorphisms

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Abstract
We say a space $X$ with property $\mathcal{P}$ is a universal space for orbit spectra of homeomorphisms with property $\mathcal{P}$ provided that if $Y$ is any space with property $\mathcal{P}$ and the same cardinality as $X$ and $h : Y \to Y$ is any (auto)homeomorphism then there is a homeomorphism $g : X \to X$ such that the orbit equivalence classes for $h$ and $g$ are isomorphic. We construct a compact metric space $X$ that is universal for homeomorphisms of compact metric spaces of cardinality of the continuum $\mathfrak{c}$ and prove that there is no such space that is countably infinite. In the presence of some set theoretic assumptions we also give a separable metric space of size $\mathfrak{c}$ that is universal for homeomorphisms on separable metric spaces.

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1. Introduction

In [6] Good et al. classified up to orbit equivalence all (auto)homeomorphisms of compact metric spaces. They showed that if the collection of orbits of a bijection $T$ on a set $X$ satisfies a few weak cardinality conditions then $X$ can be given a compact metric topology with respect to which $T$ is a homeomorphism (we give the precise statement of the result in the next section). In proving this classification of homeomorphisms, many different construction techniques were utilized and many exotic dynamical systems were constructed.

In this paper, motivated by the classification theorem of homeomorphisms on compact metric spaces, we address the following question:

Does there exist a single compact metric space $X$ of size $\kappa$ which admits every possible homeomorphism allowed on the class of compact metric spaces of size $\kappa$?

That is to say in the language of this paper, is there a compact metric space that is universal for homeomorphisms?

The answer depends, not surprisingly, on the size of $\kappa$. If $\kappa$ is finite, then the answer is trivially ‘yes’ and the only possible compact metric topology, the discrete topology, is the example. It is not hard to show that if $\kappa$ is countably infinite ($\aleph_0$) then the answer is ‘no.’ The main result of this paper is that if $\kappa = \mathfrak{c}$, the cardinality of the continuum, then there is a single compact metric space $X$ with $\mathfrak{c}$-many points such that $X$ admits every possible homeomorphism allowed on the class of compact metric spaces of size $\mathfrak{c}$. Surprisingly, $X$ cannot be a Cantor space or even a subset of a Cantor space, for instance $X$ cannot be a subshift. It is known that any such compact metric universal space $X$ must contain non-degenerate connected components, [10], and it is not hard to see that it must also have isolated points (see [10] for a classification of homeomorphisms on Cantor space).

In [5], Good and Greenwood give a classification theorem for all homeomorphisms on separable metric spaces (we state the theorem in the next section). We use this theorem to address the related question of the existence of a separable metric space $X$ with size $\mathfrak{c}$ which is universal for homeomorphisms. Assuming a statement weaker than the Continuum Hypothesis, we construct a separable metric space with size $\mathfrak{c}$ which is universal for homeomorphisms. Under the assumption of the Continuum Hypothesis, we show that the space made up of the irrational numbers together with a single convergent sequence of isolated points is also such a universal space.

In the next section we give definitions and preliminary results that are useful throughout the paper. In Section 3 we show that there is no compact metric space $X$ with $\aleph_0$-many points which is universal for homeomorphisms. In Section 4 we construct a compact metric space with $\mathfrak{c}$-many points that is universal for homeomorphisms. In the last section we prove (with some set theoretic assumptions) that there is a separable metric space with size $\mathfrak{c}$ which is universal for homeomorphisms.

2. Preliminary definitions and results

By an abstract dynamical system we mean a function $T : X \to X$ on a set $X$ that has no specified structure. The relation $\sim$ on $X$, defined by $x \sim y$ if and only if there exist $m, n \in \mathbb{N}$ with $T^m(x) = T^n(y)$, is an equivalence relation, whose equivalence classes are the orbits of $T$. If $O$ is an orbit of $T$, then we say that:
(1) $O$ is an $n$-cycle, for some $n \in \mathbb{N}$, if there are distinct points $x_0, \ldots, x_{n-1}$ in $O$ such that $T(x_{j+1}) = x_j$, where $j$ is taken modulo $n$;

(2) $O$ is a $\mathbb{Z}$-orbit if there are distinct points $\{x_j : j \in \mathbb{Z}\} \subseteq O$ such that $T(x_{j+1}) = x_j$ for all $j \in \mathbb{Z}$;

(3) $O$ is an $\mathbb{N}$-orbit if it is neither an $n$-cycle for some $n \in \mathbb{N}$, nor a $\mathbb{Z}$-orbit.

The cycle type or orbit spectrum of $T$ is, then, the sequence

$$\sigma(T) = (\nu, \zeta, \sigma_1, \sigma_2, \sigma_3, \ldots)$$

of cardinals, where $\nu$ is the number of $\mathbb{N}$-orbits, $\zeta$ is the number of $\mathbb{Z}$-orbits and $\sigma_n$ is the number of $n$-cycles. In this paper we will be dealing solely with bijections, so $\nu$ will always be 0. We denote the cardinality of the continuum by $\mathfrak{c}$ and regard cardinals as initial ordinals (so that $\omega = \aleph_0$).

**Definition 2.1.** Let $\mathcal{P}$ be a topological property. We say that a space $X$ is universal for homeomorphisms of spaces with property $\mathcal{P}$ if $X$ has $\mathcal{P}$ and whenever a space $Y$ with $\mathcal{P}$, has a homeomorphism with orbit spectrum $\sigma = (0, \zeta, \sigma_1, \sigma_2, \ldots)$, then there is a homeomorphism of $X$ with orbit spectrum $\sigma$.

Given an abstract dynamical system, one might naturally ask whether one can impose a structure on $X$ with respect to which $T$ has some property. In particular, if $\mathcal{P}$ is a topological property, one can ask whether one can endow $X$ with a topology that satisfies $\mathcal{P}$ and with respect to which $T$ is continuous.

In [6], the existence of a compact Hausdorff topology on $X$ with respect to which $T$ is continuous is characterized in terms of the orbit structure of $T$. The existence of a compact metric topology on $X$ making a bijection $T$ continuous is also characterized in [6].

**Theorem 2.2.** (See [6].) Let $T : X \to X$ be a bijection and let $\sigma(T) = (0, \zeta, \sigma_1, \sigma_2, \ldots)$ be the orbit spectrum of $T$. The following are equivalent:

(1) There is a compact metrizable topology on $X$ with respect to which $T$ is a homeomorphism.

(2) Either

(a) $X$ is finite; or

(b) $X$ is countably infinite and $T$ has both a $\mathbb{Z}$-orbit and a cycle; or

(c) $X$ is countably infinite and there are $n_i$-cycles, for each $i \leq k$, such that if $T$ has an $n$-cycle, then $n$ is divisible by $n_i$ for some $i \leq k$; or

(d) $X$ has the cardinality of the continuum and the number of $\mathbb{Z}$-orbits and the number of $n$-cycles, for each $n \in \mathbb{N}$, is finite, countably infinite, or has the cardinality of the continuum.

(3) $\zeta$ and each $\sigma_i$ are finite, equal to $\omega$ or equal to $\mathfrak{c}$ and either:

(a) $\zeta = 0$ and $\sum_{n \in \mathbb{N}} \sigma_n < \omega$; or

(b) $\zeta \neq 0$ and $\sum_{n \in \mathbb{N}} \sigma_i \neq 0$; or

(c) there are $n_i$ such that $\sigma_{n_i} \neq 0$, for each $i \leq k$, and, if $\sigma_n \neq 0$, then $n$ is divisible by some $n_i$; or

(d) $\zeta + \sum_{n \in \mathbb{N}} \sigma_n = \mathfrak{c}$.
In [5], it is shown that if \( T \) is a function from the set \( X \) to itself, then there is a metric on \( X \) with respect to which \( X \) is separable and \( T \) is continuous if and only if the cardinality of \( X \) is no greater than that of the continuum. The next theorem follows immediately from Theorem 1.8 in [5].

**Theorem 2.3.** (See [5].) Let \( T : X \to X \) be a bijection and let \( \sigma(T) = (0, \zeta, \sigma_1, \sigma_2, \ldots) \) be the orbit spectrum of \( T \). The following are equivalent:

1. \( |X| \leq c \).
2. \( X \) can be identified with a subset of the Cantor set in such a way that \( T \) is a homeomorphism.
3. There is a (zero-dimensional) separable metric topology on \( X \) with respect to which \( T \) is a homeomorphism.
4. \( \zeta + \sum_{n \in \mathbb{N}} \sigma_n \leq c \).

A closely related question arises in permutation group theory: given a subgroup \( G \) of the full permutation group on a set \( X \), when is there a structure on \( X \) that gives elements of \( G \) some meaning? Neumann [9], Mekler [8] and Truss [11] consider this question in the case that \( X \) is countable (see also [1]).

**Theorem 2.4.** (See Mekler [8] and Truss [11].) Let \( X \) be a countable set and \( G \) be a countable subgroup of the full symmetric group on \( X \). There is a topology on \( X \) with respect to which \( X \) is homeomorphic to the rationals and each \( g \in G \) is a homeomorphism of \( X \) if and only if, for every finite \( F \subseteq G \),

\[
\bigcap_{g \in F} \{ x : g(x) \neq x \}
\]

is empty or infinite.

**Theorem 2.5.** (See Neumann [9] and Truss [11].) Let \( T : X \to X \) be a bijection of the countable set \( X \) with spectrum \( \sigma(T) = (0, \zeta, \sigma_1, \sigma_2, \ldots) \). The following are equivalent:

1. There is a topology on \( X \) with respect to which \( X \) is homeomorphic to \( \mathbb{Q} \) and \( T \) is a homeomorphism.
2. \( \bigcap_{n \in F} \{ x : T^n(x) \neq x \} \) is empty or infinite for all finite \( F \subseteq \mathbb{N} \).
3. Either:
   (a) \( \zeta \neq 0 \); or
   (b) \( \{ n : \sigma_n \neq 0 \} \) is infinite; or
   (c) whenever \( \sigma_k \neq 0 \), there is some \( m \) such that \( k \) divides \( m \) and \( \sigma_m = \omega \).

So in a strong sense, the rationals act as a universal space for groups of permutations of a countable set with the property that, for finite \( F \), \( \bigcap_{g \in F} \{ x : g(x) \neq x \} \) is either empty or infinite (which corresponds to the fact that non-empty open subsets of \( \mathbb{Q} \) are infinite). (On the other hand, Truss [11] points out that not all bijections can be realized as homeomorphisms of \( \mathbb{Q} \), for example when \( T \) is such that \( \nu = \zeta = 0, \sigma_2 = \sigma_8 = \omega, \sigma_3 = 1 \) and \( \sigma_n = 0 \) otherwise.)

In a private communication, W.N. Everitt asked how specific the spaces constructed in the proof of Theorem 2.2 of this result were to the function \( T \). This question and the other results
mentioned above raise the question of the existence of universal spaces for certain types of abstract dynamical system. The characterization in Theorem 2.4 for the rationals is in terms of the behaviour of the fixed point sets, the characterization in 2.3 for separable metric spaces is in terms of the cardinality $X$, and the characterization in 2.2 for metric compacta is in terms of the orbit structure of the function. All three of these, for bijections at least, can be expressed in terms of the orbit spectrum of the function, and so we focus on universal spaces for the set of admissible spectra of homeomorphisms.

3. Universal spaces for homeomorphisms of small compact metric spaces

It is well known (see [4]) that a compact metric spaces is either countable or has cardinality of the continuum. If $X$ is a set of cardinality $k$ for some $k \in \mathbb{N}$, then the discrete topology is the unique topology with respect to which $X$ is compact metric and any bijection of $X$ will then be a homeomorphism, so $X$ is then universal for homeomorphisms of metric compacta of cardinality $k$. A more interesting situation occurs when $X$ is countable. We have the following theorem.

**Theorem 3.1.** No space is universal for homeomorphisms of countably infinite metric compacta.

**Proof.** Recall that the Cantor–Bendixson derivative of a subset $A$ of a topological space $X$ is the set $A'$ of limit points of $A$. Let $X^{(0)} = X$. Given $X^{(\alpha)}$, for any ordinal $\alpha$, define $X^{(\alpha+1)} = X^{(\alpha)}'$. Given $X^{(\beta)}$, for all ordinals $\beta < \lambda$, and $\lambda$ a limit ordinal, define $X^{(\lambda)} = \bigcap_{\beta < \lambda} X^{(\beta)}$.

Now suppose that $X$ is countable, compact metric space. It is a well-known fact (see [4]), that a countable compact metric space is homeomorphic to a compact subset of the rationals with their usual topology. Such spaces are scattered, which is equivalent to saying, at least in the compact case, that for some ordinal $\alpha$, $X^{(\alpha)}$ is finite and $X^{(\alpha+1)}$ is empty. (In the case of countable, metric compacta, $\alpha$ will be a countable ordinal.) Let us suppose that $X^{(\alpha)}$ has $k$ points.

Since being a limit point is preserved by homeomorphism, any homeomorphism $T$ of $X$ will permute $X^{(\alpha)}$. But this implies that $T$ must have an $m$-cycle for some $m \leq k$. On the other hand there are homeomorphisms of countable metric compacta which do not have $m$-cycles for any $m \leq k$. See for example Lemma 4.3. Thus $X$ is not universal. \[ \square \]

4. A universal space for homeomorphisms of metric compacta with cardinality $c$

In this section we construct a universal compact metric space $X$ for bijections that can be realized as homeomorphisms of metric compacta. The space $X$ is a compact subset of a family of concentric circles in the complex plane $\mathbb{C}$. We will express it as the union of two compact subsets of $\mathbb{C}$, $Y$ and $Z$, where $Y \cap Z$ is the unit circle centered at the origin.

The following notation will be helpful in what follows. Recall that a subset $A$ of $X$ is said to be invariant under $T : X \to X$ if and only if $T(A) \subseteq A$.

**Definition 4.1.** Let $T : X \to X$ and $S : Y \to Y$ be functions such that $X - Y$ is invariant under $T$, $Y - X$ is invariant under $S$, $X \cap Y$ is invariant under both $S$ and $T$, and $T(x) = S(x)$ for all $x \in X \cap Y$. We define

$$ T \lor S(x) = \begin{cases} T(x) & \text{if } x \in X, \\ T(x) = S(x) & \text{if } x \in X \cap Y, \\ S(x) & \text{if } x \in Y. \end{cases} $$
The proof of the next lemma is straightforward.

**Lemma 4.2.** Let \( T : X \to X \) and \( S : Y \to Y \) be functions such that \( X - Y \) is invariant under \( T \), \( Y - X \) is invariant under \( S \), \( X \cap Y \) is invariant under both \( S \) and \( T \), and \( T(x) = S(x) \) for all \( x \in X \cap Y \). Suppose that

1. \( \sigma(T \mid_{X - Y}) = (v^1, \xi^1, \sigma_1^1, \sigma_2^1, \ldots) \),
2. \( \sigma(T \mid_{X \cap Y}) = \sigma(S \mid_{X \cap Y}) = (v^2, \xi^2, \sigma_1^2, \sigma_2^2, \ldots) \),
3. \( \sigma(S \mid_{Y - X}) = (v^3, \xi^3, \sigma_1^3, \sigma_2^3, \ldots) \).

Then

\[
\sigma(T \lor S) = \left((v^1 + v^2 + v^3), (\xi^1 + \xi^2 + \xi^3), (\sigma_1^1 + \sigma_1^2 + \sigma_1^3), \ldots\right).
\]

The space \( Y \) is constructed to consist of a sequence of concentric circles converging to the unit circle. \( Y \) will have the following property: if \( \sigma = (0, \xi, \sigma_1, \sigma_2, \ldots) \) is a sequence of cardinals such that \( \xi \) and each \( \sigma_n \) are either 0 or \( c \), then there is a homeomorphism of \( Y \) with \( \sigma(T) = \sigma \).

The space \( Z \) is constructed to consist of the unit circle together with a countable collection of isolated copies of the convergent sequence \( \{0\} \cup \{1/n: 1 \leq n \in \mathbb{N}\} \) whose union is dense in \( Z \). \( Z \) will have the following property: if \( \sigma = (0, \xi, \sigma_1, \sigma_2, \ldots) \) is a sequence of cardinals with exactly one of \( \xi \) or \( \sigma_n \), \( n \in \mathbb{N} \), equal to \( c \) and all others finite or equal to \( \omega \), then there is a homeomorphism \( T \) of \( Z \) with \( \sigma(T) = \sigma \).

The space \( X \) is the space formed by taking the union of \( Y \) and \( Z \), applying Lemma 4.2. It is then easy to see that \( X \) then has the following property: if \( \sigma = (0, \xi, \sigma_1, \sigma_2, \ldots) \) is a sequence of cardinals such that \( \xi \) and each \( \sigma_n \) are either finite, equal \( \omega \) or equal to \( c \) and \( \xi + \sum_{n \in \mathbb{N}} \sigma_n = c \), then there is a homeomorphism \( T \) of \( X \) with \( \sigma(T) = \sigma \).

For each \( d > 0 \), let \( S_d = \{ z \in \mathbb{C}: |z| = d \} \) and, for each \( n \geq 1 \), let

\[
E_n = S_{1+1/2^n} \cup \left( \bigcup_{j > n+1} S_{1+1/2^{n+1}+1/2^j} \right).
\]

For \( z \in S_1 \), let \( L_z \) be the ray passing from the origin through \( z \). Let \( D = \{d_n: 1 \leq n \} \) be a countable dense subset of \( S_1 \).

For \( R = \{r\} \cup \{r_n: 1 \leq n \} \), where \( \{r_n: 1 \leq n \} \) is an increasing sequence of positive reals converging to \( r \), we define

\[
Y_R = S_r \cup \left( \bigcup_{n \in \mathbb{N}} S_{r_n} \right).
\]

In particular, we define

\[
Y = Y_{\{1\} \cup \{1-1/2^n: 1 \leq n \}}.
\]

Clearly \( Y \) is a compact subset of \( \mathbb{C} \).
For a sequence $\mathcal{F} = \{F_n: n \geq 1\}$ of subsets of $S_1$, let

$$Z_{\mathcal{F}} = S_1 \cup \bigcup_{n \geq 1} \bigcup_{z \in F_n} E_n \cap L_z.$$  

In particular, we define

$$Z = Z_D$$

where $D = \{[d_n]: d_n \in D\}$. Notice that if $z \in S_1$, then $L_z \cap E_n$ consists of the sequence of points $\{(1 + 1/2^n + 1/2^j)z: n + 1 < j\}$ together with its limit point $(1 + 1/2^n)z$. Hence $Z$ consists of the unit circle and an isolated sequence of copies of the convergent sequence $\{0\} \cup \{1/n: 0 < n\}$ whose union is dense in the space. Since, for any $\epsilon > 0$, only finitely many copies of the convergent sequence contain points of $Z$ that lie distance more than $\epsilon$ from $S_1$, $Z$ is a compact subset of $\mathbb{C}$.

The space $X$, then is defined by

$$X = Y \cup Z = S_1 \cup \bigcup_{1 \leq n} S_{1-\frac{1}{2^n}} \cup \bigcup_{1 \leq n} \{(1 + 1/2^n + 1/2^j)d_n: n + 1 < j\} \cup \{(1 + 1/2^n)d_n\}.$$  

Since $Y$ and $Z$ are both compact, $X$ is also compact.

Let $W = \{0\} \cup \{1/n: 1 \leq n\}$ inherit the usual Euclidean topology from $\mathbb{R}$, and, for each $n \in \mathbb{N}$, let $n = \{0, 1, \ldots, n - 1\}$ have the discrete topology. Let $W_n = W \times n$ have the Tychonoff product topology so that it consists of $n$ disjoint copies of the convergent sequence $W$.

**Lemma 4.3.**

1. There is a homeomorphism $T$ of $W_n$ such that each point of $W_n$ lies in an $n$-cycle.
2. For each $1 \leq k \leq \omega$, there is a homeomorphism $T$ of $W_n$ such that $\{(0, j): 0 \leq j < n\}$ forms an $n$-cycle and all other points lie in one of $k$ many $\mathbb{Z}$-orbits.

**Proof.** For (1), we write $W_n = \{(0, j): 0 \leq j < n\} \cup \{(1/k, j): 1 \leq k, 0 \leq j < n\}$. The map $T : W_n \to W_n$ defined by $T(x, j) = (x, j + 1)$, where $j + 1$ is taken modulo $n$, is clearly a homeomorphism of $W_n$ whose only orbits are $n$-cycles.

For (2) we express $W_n$ in a slightly different, but equivalent, form:

$$W_n = \{(0, j): 0 \leq j < n\} \cup \{(l, m, j): l \in \mathbb{Z}, m \in \mathbb{N}, 0 \leq j < n\}.$$  

Points of the form $(l, m, j)$ are isolated and basic open neighbourhoods of $(0, j)$ take the form $\{(0, j)\} \cup \{(l, m, j): l \in \mathbb{Z} - F \text{ and } m \in \mathbb{N} - G \text{ where } F \text{ and } G \text{ are finite}\}$. Define $T : W_n \to W_n$ by $T(0, j) = (0, j + 1)$, where $j + 1$ is taken modulo $n$, $T(l, m, j) = (l, m, j + 1)$, where $j < n - 1$, and $T(l, m, n - 1) = (l + 1, m, 0)$. Then $T$ is a homeomorphism of $W_n$ consisting of one $n$-cycle, $\{(0, j): 0 \leq j < n\}$, and $\omega$ many $\mathbb{Z}$-orbits, one for each $m \in \mathbb{N}$. Restricting $T$ to $W'_n = \{(0, j): 0 \leq j < n\} \cup \{(l, m, j): l \in \mathbb{Z}, 0 \leq m < k, 0 \leq j < n\}$, we again obtain a $\omega$-many $\mathbb{Z}$-orbits, etc.
which is homeomorphic to \( W_n \), yields a homeomorphism with one \( n \)-cycle and \( k \) many \( \mathbb{Z} \)-orbits. \( \square \)

The proof of the following lemma is obvious.

**Lemma 4.4.** Suppose that \( R' \) and \( R = \{ r \} \cup \{ r_n: 0 \leq n \} \) are sets of positive reals, that \( R' \) is finite and that \( r_n \to r \) as \( n \to \infty \). Then \( Y_R \cup \bigcup_{r \in R'} S_r \) is homeomorphic to \( Y \).

**Lemma 4.5.** Suppose that:

1. \( \mathcal{F} = \{ F_n: 1 \leq n \} \) is a sequence of subsets of \( S_1 \) such that each \( F_n \) is finite and \( \bigcup_{1 \leq n} F_n \) is dense in \( S_1 \).
2. \( G = \{ z_n \in \mathbb{C}: n \in \mathbb{N} \} \) is a (possibly finite) sequence of points such that \( G \) is disjoint from \( Z_{\mathcal{F}} \), \( 1 < |z_n| \) for each \( n \in \mathbb{N} \) but that, for each \( \epsilon > 0 \), \( \{ n \in \mathbb{N}: 1 + \epsilon < |z_n| \} \) is finite.
3. \( N = \{ n_j \in \mathbb{N}: j \leq k \} \) is a finite subset of \( \mathbb{N} \), that is possibly empty. For each \( j \leq k \), let \( W(j) \) be a distinct copy of \( W_n \) in \( \mathbb{C} \) disjoint from \( Z_{\mathcal{F}} \cup G \).

Then the free union \( Z_{\mathcal{F}} \cup G \cup \bigcup_{j \leq k} W(j) \) is homeomorphic to \( Z \).

**Proof.** It is routine to show that \( Z \) is homeomorphic to a free union of itself together with a finite number of distinct points and a finite number of distinct copies of the convergent sequence \( W \). Hence we may assume that \( G \) is infinite and \( N \) is empty.

**Claim 1.** There is a homeomorphism \( f \) of \( \mathbb{C} \) that maps \( Z_{\mathcal{F}} \) onto \( Z \).

To see this, we first note that \( S_1 \) is countably dense homogeneous [2], that is to say, that for any two countable dense subsets \( A \) and \( B \) of \( S_1 \), there is a homeomorphism of \( S_1 \) which maps the set \( A \) to the set \( B \). Let \( f \) be such a homeomorphism of \( S_1 \) taking \( \bigcup_{1 \leq n} F_n \) to \( D \). The function \( f \) extends to a homeomorphism \( f^* \) of the complex plane by defining \( f^*(r z) = rf(z) \) for each non-negative real \( r \). Note that for all \( z \in \mathbb{C} \), \( |z| = |f^*(z)| \).

The image \( f^*(Z_{\mathcal{F}}) \) of \( Z_{\mathcal{F}} \) has the following property: for \( z \notin D \), \( L_z \cap f^*(Z_{\mathcal{F}}) = L_z \cap S_1 = \{ z \} \), whereas, for \( z \in D \), \( L_z \cap f^*(Z_{\mathcal{F}}) \) consists of the point \( z \) itself and a single copy of a convergent sequence together with its limit point. Specifically, if \( z = d_m \in D \), then there exists an \( n_m \in \mathbb{N} \) and \( w \in F_{n_m} \) such that \( d_m = f^*(w) \), and

\[
L_{d_m} \cap f^*(Z_{\mathcal{F}}) = \{ d_m \} \cup \left\{ (1 + 1/2^n + 1/2^j)d_m: n_m + 1 < j \right\} \cup \left\{ (1 + 1/2^n)d_m \right\}.
\]

On the other hand,

\[
L_{d_m} \cap Z = \{ d_m \} \cup \left\{ (1 + 1/2^n + 1/2^j)d_m: m + 1 < j \right\} \cup \left\{ (1 + 1/2^n)d_m \right\}.
\]

To complete the proof of the claim, we construct a homeomorphism \( h : f^*(Z_{\mathcal{F}}) \to Z \) that ‘slides’ each \( L_{d_m} \cap f^*(Z_{\mathcal{F}}) \) along the ray \( L_{d_m} \) so that it is mapped onto \( L_{d_m} \cap Z \). To begin, let \( h \) be the bijection that leaves \( S_1 \) fixed and maps
\[(1 + 1/2^n)d_m \to (1 + 1/2^n)d_m, \]
\[(1 + 1/2^n + 1/2^j)d_m \to (1 + 1/2^n + 1/2^k)d_m, \]

where \(k = m + (j - n_m)\) for all \(j > n_m + 1\).

To see that \(h\) is continuous, let \(\{a_i: i \in \mathbb{N}\}\) be a convergent sequence of points in \(f^*(Z\mathcal{F})\) with \(a_i \to a\) as \(i \to \infty\). For each \(i\), let \(m_i\) and possibly \(j_i\) be chosen such that \(a_i = h(d_{m_i}) = (1 + 1/2^{m_i})d_{m_i}\) or \((1 + 1/2^{m_i} + 1/2^j)d_{m_i}\). If all but finitely many of the \(a_i\)'s are in \(S_1\) then so is \(a\). Since \(h\) leaves these points fixed, \(h(a_i) \to h(a)\). If instead, infinitely many of the \(a_i\)'s are not in \(S_1\) but \(a \in S_1\), then still \(h(a) = a\), and it must be the case that the associated \(d_{m_i}\)'s converge to \(a\) in \(S_1\). Moreover, the \(m_i\)'s (and thus the associated \(n_{m_i}\)'s) must go to infinity as \(i \to \infty\) since the \(a_i\)'s converge to a point in \(S_1\). Since \(n_{m_i} \to \infty\) it must also be the case that \(m_i \to \infty\). If there are infinitely many \(j_i\)'s also defined then since \(j_i > n_{m_i}\), the associated \(k_i = m_i + (j_i - n_{m_i})\) also has \(k_i \to \infty\). Therefore it is again the case that \(h(a_i) \to h(a)\). Finally, if \(a \notin S_1\) then \(a\) must be of the form \((1 + 1/2^n)d_t\) for some \(t\) as the other points in \(f^*(Z\mathcal{F}) - S_1\) are isolated. Since \(a_i \to a\), all but finitely many of the \(a_i\)'s are of the form \((1 + 1/2^{m_i} + 1/2^j)d_{m_i}\) with \(m_i = t, m_i + 1 < j_i\), and \(j_i \to \infty\). By the definition of \(h, a\) is mapped to \((1 + 1/2^t)d_t\) and all (except perhaps finitely many) of the \(a_i\)'s are mapped to points of the form \((1 + 1/2^t + 1/2^k)d_t\). Again, since \(j_i \to \infty\), \(k_i \to \infty\). Therefore, \(h(a_i) \to h(a)\). This proves that \(h\) is a continuous bijection between compact metric spaces, and hence \(h\) is a homeomorphism. The claim is proved by setting \(f = h \circ f^*\).

Now consider the image \(Z \cup f(G)\) of \(Z\mathcal{F} \cup G\) under \(f\). That \(G' = f(G)\), satisfies condition (2) of the lemma with respect to \(Z\) is routine. The following claim will, therefore, complete the proof of the lemma.

**Claim 2.** \(Z \cup G'\) is homeomorphic to \(Z\).

Let \(G' = \{z'_{n}; n \in \mathbb{N}\}\). Recall that for each \(d_m \in D, Z \cap L_{d_m}\) consists of a convergent sequence of points together with the point \(d_m\) lying on the unit circle. Note that for each \(1 \leq m\) and each \(d_m \in D, if w \in E_m \cap L_{d_m}\), then \(|w| < 1 + 1/2^{m-1}\). For each \(z'_n \in G'\), we inductively choose \(m_n \in \mathbb{N}\) as follows:

(a) \(m_j < m_n\) for all \(j < n\);
(b) \(1 + 1/2^{m_n-1} < |z'_n|\);
(c) \(|\arg(z'_n) - \arg(d_{m_n})| < |\arg(z'_{n-1}) - \arg(d_{m_{n-1}})| < 1/2^{n-1}\).

To prove the claim, we first perturb each \(z'_n\) in \(G'\) to a point \(|z'_n|d_{m_n}\) lying on the ray \(L_{d_{m_n}}\), \(d_{m_n} \in D\), so that it lies just further from the origin than the points of the convergent sequence. Then we shuffle the points of the convergent sequence along the ray to accommodate the point \(|z'_n|d_{m_n}\), much in the same way that the map \(1/n \mapsto 1/(n + 1)\) shuffles \(\{0\} \cup \{1/n; 1 \leq n\}\) into \(\{0\} \cup \{1/n; 1 \leq n\}\).

Define the map \(h' : Z \cup G' \to \mathbb{C}\) by

\[
h'(z) = \begin{cases} 
  z & \text{if } z \in Z, \\
  |z'_n|d_{m_n} & \text{if } z = z'_n.
\end{cases}
\]

Since, for any \(\epsilon > 0\), \(h'\) moves only finitely many points a distance more than \(\epsilon\), and the set of points that are moved by \(h'\) converges to points in \(S_1\), a compact subset of \(\mathbb{C}\), \(h'\) extends to a
homeomorphism of $\mathbb{C}$. Let $Z' = h'(Z \cup G')$. Notice that for each $m \in \mathbb{N}$, if $m \neq m_n$, for any $n$, then $Z' \cap L_{d_m} = Z \cap L_{d_m}$. On the other hand, if $m = m_n$, for some $n \in \mathbb{N}$, then

$$
Z' \cap L_{d_m} = (Z \cap L_{d_m}) \cup \{|z_n|d_{m_n}\}
$$

where

$$
Z \cap L_{d_m} = (S_1 \cup E_m) \cap L_{d_m} = \{d_m\} \cup \{(1 + 1/2^m + 1/2^j)d_m: m + 1 < j\} \cup \{(1 + 1/2^m)d_m\}.
$$

Note, in particular, that $Z' = Z \cup \{|z_n|d_{m_n}: 1 \leq n\}$, where $1 + 1/2^{m_n-1} < |z_n||d_{m_n}|$. This accomplishes our first goal of perturbing each $z$ the ray $L_{d_m}$ to a homeomorphism of $\mathbb{C}$ stated above, it is easy to see that $Z$ is a rotation with $c$.

Now let $Z'' = Z \cup \{(1 + 1/2^m + 1/2^{m_n+1})d_{m_n}: 1 \leq n\}$. It is easy to see that there is a homeomorphism, $h''$, from $Z''$ to $Z$. Define $h^*: Z' \to Z''$ by

$$
h^*(z) = \begin{cases} z & \text{if } z \in Z, \\ (1 + 1/2^m + 1/2^{m_n+1})d_{m_n} & \text{if } z = |z_n|d_{m_n}. \end{cases}
$$

Again, since, for any $\epsilon > 0$, $h^*$ moves only finitely many points a distance more than $\epsilon$, and the set of points that are moved by $h^*$ converges to points in $S_1$, a compact subset of $\mathbb{C}$, $h^*$ extends to a homeomorphism of $\mathbb{C}$. This accomplishes our second goal of sliding each point a little along the ray $L_{d_m}$. It follows that there is a homeomorphism of $\mathbb{C}$ that maps $Z \cup G'$ to $Z''$, and, as stated above, it is easy to see that $Z''$ is homeomorphic with $Z$. This completes the proof of the claim and hence the lemma. 

**Definition 4.6.** Let $u_x : \mathbb{C} \to \mathbb{C}$, for $x \in \mathbb{R}$, and $s : \mathbb{C} \to \mathbb{C}$ be defined by

$$
u_x : re^{i\theta} \mapsto re^{i(\theta + 2\pi x)},
$$

$$
u_s : re^{i\theta} \mapsto re^{i(\theta + 2\pi r)}.
$$

**Lemma 4.7.** Both $u_x$, for any $x \in \mathbb{R}$, and $s$ are homeomorphisms of $\mathbb{C}$ that preserve each circle $S_r$, $0 \leq r$, and fix the origin.

The orbit of a point $z \neq 0$ under $u_x$ is an $n$-cycle if $x = m/n$ is a rational expressed in lowest terms and is a $\mathbb{Z}$-orbit if $x$ is irrational.

The orbit of a point $re^{i\theta} \neq 0$ under $s$ is an $n$-cycle if $r = m/n$ is a rational expressed in lowest terms and is a $\mathbb{Z}$-orbit if $r$ is irrational.

**Lemma 4.8.** Let $\sigma = (0, \xi, \sigma_1, \sigma_2, \ldots)$ be a sequence of cardinals each of which is either $0$ or $\mathfrak{c}$. There is a homeomorphism $T$ of $Y$ such that:

1. $\sigma(T) = \sigma$; and
2. such that the orbits of points in $S_1$ under $T$ are $\mathbb{Z}$-orbits if $\xi = \mathfrak{c}$ and $n$-cycles if $\sigma_n = \mathfrak{c}$ and $\xi = \sigma_k = 0$ for all $k < n$.

**Proof.** Suppose that $J = \{j: \sigma_j \neq 0\}$ is finite. For each $j \in J$, the map $u_{1/j}$ restricted to $S_{1/j}$ or restricted to $Y$ is a rotation consisting of $\mathfrak{c}$ many $j$-cycles. For any irrational $x$, $u_x$ restricted to $Y$ is a rotation with $\mathfrak{c}$ many $\mathbb{Z}$-orbits. The result now follows trivially by Lemmas 4.2 and 4.4.
Suppose then that \( J = \{ j : \sigma_j \neq 0 \} \) is infinite. There are two cases. \( \zeta = c \); or for some \( 1 \leq n \), \( \sigma_n = c \) and \( \zeta = \sigma_j = 0 \) for all \( j < n \). In the first case choose \( r \) to be a positive irrational. In the second case choose \( r = 1/n \). Index \( J \) (in the first case), or \( J - \{ n \} \) (in the second case) as \( \{ j_k : k \in \mathbb{N} \} \), so that \( j_k < j_m \) whenever \( k < m \). We can choose natural numbers \( p_k \) such that \( p_k \) and \( j_k \) are co-prime and \( |r - p_k/j_k| \) is minimal. Then \( p_k/j_k \to r \) as \( k \to \infty \). Let \( R = \{ r \} \cup \{ p_k/j_k : k \in \mathbb{N} \} \) and let \( T \) be the restriction of the homeomorphism \( s \) to \( Y_R \). Clearly \( \sigma(T) = \sigma \).

By Lemma 4.4, \( Y_R \) is homeomorphic to \( Y \) and we are done. \( \square \)

**Lemma 4.9.** Let \( \sigma = (0, \zeta, \sigma_1, \sigma_2, \ldots) \) be a sequence of cardinals such that either:

1. \( \zeta = c \) and each \( \sigma_n \) is finite; or
2. \( \zeta = 0, \sigma_n = c \) for some \( n \) and \( \sigma_m \) is finite for all \( m \neq n \).

There is a homeomorphism \( T \) of \( Y' = S_1 \cup \bigcup_{1 \leq m} S_{1+1/2^m} \) and a compact subspace \( S_1 \cup G \) of \( Y' \) such that:

(a) \( G = \bigcup_{1 \leq m} G_m \);
(b) for each \( 1 \leq m \), \( G_m \subseteq S_{1+1/2^m} \) consists of \( \sigma_m \) many \( m \)-cycles under \( T \);
(c) the orbits of points in \( S_1 \) under \( T \) are \( \mathbb{Z} \)-orbits if (1) holds and \( n \)-cycles if (2) holds;
(d) \( \sigma(T \upharpoonright S_1 \cup G) = \sigma \);
(e) for any \( \epsilon > 0 \), \( \{ z \in G : 1 + \epsilon < |z| \} \) is finite; and
(f) \( \{ z/|z| : z \in G \} \) is dense in \( S_1 \).

**Proof.** For case (1), by Lemma 4.8, there is a homeomorphism \( T' \) of \( Y \) and, therefore, a homeomorphism \( T \) of \( Y' \), such that \( \sigma(T) = \sigma(T') = (0, c, c, c, \ldots) \), points of \( S_1 \subseteq Y' \) lie in \( \mathbb{Z} \)-orbits under \( T \) and points of \( S_{1+1/2^m} \) lie in \( m \)-cycles. For each \( \sigma_m \neq 0 \), choose a subset \( G_m \) of \( S_{1+1/2^m} \) consisting of \( \sigma_m \) many \( m \)-cycles. Otherwise, let \( G_m \) be empty. Let \( G = \bigcup_{1 \leq m} G_m \). Clearly one may choose \( G \) so that \( \{ z/|z| : z \in G \} \) is dense in \( S_1 \).

For case (2), there is a homeomorphism \( T' \) of \( Y' \) such that points of \( S_1 \) lie in \( n \)-cycles and points of \( S_{1+1/2^m} \) lie in \( m \)-cycles, for all \( 1 \leq m \). Again we can choose \( G_m \subseteq S_{1+1/2^m} \) to consist of \( \sigma_m \) many \( m \)-cycles and so that \( \{ z/|z| : z \in G \} \) is dense. \( \square \)

**Lemma 4.10.** Let \( \sigma = (0, \zeta, \sigma_1, \sigma_2, \ldots) \) be a sequence of cardinals all of which are countable except for precisely one, which is equal to \( c \). There is a homeomorphism \( T \) of \( Z \) such that \( \sigma(T) = \sigma \) and each point of \( S_1 \) has the same orbit type under the action of \( T \).

**Proof.** Suppose that \( 0 < \zeta < c \), \( \sigma_n = c \) and that there is a homeomorphism \( S \) of \( Z \) such that \( \sigma(S) = (0, 0, \sigma_1, \sigma_2, \ldots) \). By Lemma 4.3, there is a homeomorphism \( U \) of \( W_n \) with exactly one \( n \)-cycle and \( \zeta \) many \( \mathbb{Z} \)-orbits. By Lemma 4.2, \( S \vee U \) is a homeomorphism of \( Z \cup W_n \) with \( \sigma(S \vee U) = \sigma \). By Lemma 4.5, \( Z \cup W_n \) is homeomorphic to \( Z \). Therefore there is a homeomorphism \( T \) of \( Z \) with \( \sigma(T) = \sigma \) and we may assume without loss of generality that either \( \zeta = c \) or \( \zeta = 0 \).

Suppose that \( J = \{ n : \sigma_n = \omega \} \) is non-empty but finite. Let \( \sigma' = (0, \zeta, \sigma'_1, \sigma'_2, \ldots) \), where

\[
\sigma'_n = \begin{cases} 
0 & \text{if } \sigma_n = \omega, \\
\sigma_n & \text{otherwise}.
\end{cases}
\]
Suppose that there is a homeomorphism $S$ of $C$ such that $\sigma(S) = \sigma'$. Again, by Lemma 4.3, there is a homeomorphism $U_n$ of $W_n$ that consists of precisely countably many $n$-cycles. It follows by Lemma 4.2 that there is a homeomorphism $T'$ of $Z \cup \bigcup_{n < j} W_n$ such that $\sigma(T') = \sigma$. Therefore, by Lemma 4.5, there is a homeomorphism $T$ of $Z$ with $\sigma(T) = \sigma$. Hence we may assume that \{n: $\sigma_n = \omega$\} is either empty or infinite. Exactly the same argument allows us to assume that if $\sigma_n = c$ for some $n \in \mathbb{N}$, then $\{k < n: \sigma_k = \omega\}$ is empty.

Now let $\sigma' = (0, \zeta, \sigma'_1, \sigma'_2, \ldots)$, where

$$\sigma'_n = \begin{cases} 0 & \text{if } \sigma_n < \omega, \\ \sigma_n & \text{otherwise,} \end{cases}$$

and let $\sigma'' = (0, \zeta, \sigma''_1, \sigma''_2, \ldots)$, where

$$\sigma''_n = \begin{cases} 0 & \text{if } \sigma_n = \omega, \\ \sigma_n & \text{otherwise.} \end{cases}$$

By Lemma 4.9, there are a compact subset $S_1 \cup G$ of $Y$ and a homeomorphism $U$ of $S_1 \cup G$ such that $\sigma(U) = \sigma''$ and points of $S_1$ lie in $\mathbb{Z}$-orbits of $U$ if $\zeta = \epsilon$ or $n$-cycles of $U$ if $\sigma_n = \sigma_n'' = \epsilon$. Suppose that there is a homeomorphism $S$ of $Z$ such that $\sigma(S) = \sigma'$ and such that the action of $S$ on $S_1$ consists of either $\mathbb{Z}$-orbits, if $\zeta = \epsilon$, or $n$-cycles, if $\sigma_n = \sigma'_n = \sigma''_n = \epsilon$. Hence, by Lemma 4.2, $S \vee U$ is a homeomorphism of $Z \cup G$ with $\sigma(S \vee U) = \sigma$. But by Lemma 4.5, $Z \cup G$ is homeomorphic to $Z$, so that there is a homeomorphism $T$ of $Z$ with $\sigma(T) = \sigma$ and without loss of generality we may assume that $\{n: 0 < \sigma_n < \omega\}$ is empty.

We are left with four cases to consider:

(1) $\zeta = \epsilon$ and $\sigma_j = 0$ for all $1 \leq j$;

(2) $\zeta = \epsilon$ and $\{j: \sigma_j = \omega\}$ is infinite;

(3) $\zeta = \sigma_j = 0$, for $j \neq n$, and $\sigma_n = \epsilon$ and

(4) $\zeta = \sigma_j = 0$, for $j < n$, $\sigma_n = \epsilon$ and $\{j: n < j, \sigma_j = \omega\}$ is infinite.

We deal with these in order of simplicity. For case (3), let $u_{1/n}$ be the homeomorphism defined in Lemma 4.7 and let

$$Z' = Z \cup u_{1/n}(Z) \cup \cdots \cup u_{1/n}^{n-1}(Z).$$

Clearly $Z'$ is a compact subset of $\mathbb{C}$ with the property that all orbits under the action of $u_{1/n}$ are $n$-cycles. On the other hand, it is easy to see that $Z' = Z_F$, where $F = \{F_n: 1 \leq n\}$ and $F_n = \{d_n, u_{1/n}(d_n), \ldots, u_{1/n}^{n-1}(d_n)\}$, which is homeomorphic to $Z$.

Cases (2) and (4) are similar. We prove case (2). By Lemma 4.9, there are a compact subset of $\mathbb{C}$, $Z' = S_1 \cup \bigcup_{1 < m} G_m$ and a homeomorphism $T'$ of $Z'$ such that $S_1$ consists of $Z$-orbits of $T'$, $G_m \subseteq S_{1+1/2^m}$ and consists of one $m$-cycle of $T'$ for each $m$ with $\sigma_m = \omega$, and $\{z/|z|: z \in G_m, 1 \leq m\}$ is dense in $S_1$. We now add $\omega$-many $m$-cycles to $Z'$. For each $m$ with $\sigma_m = \omega$, let $F_m = \{z/|z|: z \in G_m\}$ and let $F = \{F_m: 1 < m\}$. Recall that

$$Z_F = S_1 \cup \bigcup_{\sigma_m = \omega} \bigcup_{z \in F_m} E_m \cap L_z,$$
where

\[ E_m \cap \mathcal{L}_z = \left\{(1 + 1/2^m)z\right\} \cup \left\{(1 + 1/2^m + 1/2^j)z : m + 1 < j\right\}. \]

Define \( T : \mathcal{Z}_\mathcal{F} \to \mathcal{Z}_\mathcal{F} \) by

\[
T(z) = \begin{cases} 
T'(z), & \text{if } z \in S_1, \\
(1 + 1/2^m)\frac{T'(w)}{|T'(w)|}, & \text{if } z = (1 + 1/2^m)\frac{w}{|w|}, \ w \in G_m, \\
(1 + 1/2^m + 1/2^j)\frac{T'(w)}{|T'(w)|}, & \text{if } z = (1 + 1/2^m + 1/2^j)\frac{w}{|w|}, \ w \in G_m.
\end{cases}
\]

We have \( \sigma(T) = \sigma \) and every point on \( S_1 \) is on a \( \mathbb{Z} \)-orbit.

Finally, for case (1), let \( p \) be an irrational, let \( u_p \) be the homeomorphism defined in Lemma 4.7 and let \( \{z_n : n \in \mathbb{Z}\} \) be a \( \mathbb{Z} \)-orbit of the action of \( u_p \) on \( S_1 \), indexed so that \( u_p(z_n) = z_{n+1} \), for each \( n \in \mathbb{Z} \). Let \( F_0 = \{z_0\} \) and, for each \( 0 < n \), let \( F_n = \{z_{-n}, z_n\} \). Let \( \mathcal{F} = \{F_n : 1 \leq n\} \) and let

\[
Z' = \mathcal{Z}_\mathcal{F} \cup (E_0 \cap L_{z_0}) \\
= S_1 \cup (E_0 \cap L_{z_0}) \cup \bigcup_{0 < n} E_n \cap (L_{z_{-n}} \cup L_{z_n}).
\]

Since \( \{z_n : n \in \mathbb{Z}\} \) is dense in \( S_1 \), by Lemma 4.10, \( Z' \) is homeomorphic to \( Z \). Note that

\[
E_0 \cap L_{z_0} = \left\{(1 + 1/2^0)z_0\right\} \cup \left\{(1 + 1/2^0 + 1/2^j)z_0 : 1 < j\right\}
\]

and, for \( 0 < n \),

\[
E_n \cap L_{z_{-n}} = \left\{(1 + 1/2^n)z_{-n}\right\} \cup \left\{(1 + 1/2^n + 1/2^j)z_{-n} : n + 1 < j\right\},
\]

\[
E_n \cap L_{z_n} = \left\{(1 + 1/2^n)z_n\right\} \cup \left\{(1 + 1/2^n + 1/2^j)z_n : n + 1 < j\right\}.
\]

Define \( U : Z' \to Z \) by

\[
U(z) = \begin{cases} 
\frac{u_p(z)}{1 + 2^m}z_{-n+1} & \text{if } z \in S_1, \\
(1 + 1/2^m + 1/2^j)z_{-n+1} & \text{if } z = (1 + 1/2^m + 1/2^j)z_{-n}, \ 0 < n, \ n + 1 < j, \\
(1 + 1/2^m + 1/2^j)z_{n+1} & \text{if } z = (1 + 1/2^m + 1/2^j)z_n, \ 0 \leq n, \ n + 1 < j.
\end{cases}
\]

It is routine to check that \( U \) is a homeomorphism of \( Z' \) each orbit of which is a \( \mathbb{Z} \)-orbit. \( \square \)

**Theorem 4.11.** Let \( \sigma = (0, \xi, \sigma_1, \sigma_2, \ldots) \) be a sequence of cardinals. There is a homeomorphism \( T \) of \( X \) such that \( \sigma(T) = \sigma \) if and only if

1. \( \xi \) and each \( \sigma_n, \ 1 \leq n, \) are either finite, equal to \( \omega \) or equal to \( \varsigma, \) and
2. at least one of \( \xi \) or \( \sigma_n, \ 1 \leq n, \) is equal to \( \varsigma. \)
Proof. That $\sigma(T)$ satisfies conditions (1) and (2) follows immediately from Theorem 2.2. Conversely, given $\sigma$, let $\sigma^Y = (0, \xi, \sigma_1^Y, \sigma_2^Y, \ldots)$, where

$$\xi = \begin{cases} 0 & \text{if } \xi \leq \omega, \\ c & \text{if } \xi = c, \end{cases}$$

and, for each $1 \leq n$,

$$\sigma_n^Y = \begin{cases} 0 & \text{if } \sigma_n \leq \omega, \\ c & \text{if } \sigma_n = c, \end{cases}$$

and let $\sigma^Z = (0, \xi, \sigma_1^Z, \sigma_2^Z, \ldots)$, where $\xi = \xi$ and

$$\sigma_n^Z = \begin{cases} c & \text{if } \sigma_n = c, \xi \neq c \text{ and } \sigma_k \neq c, \text{ for all } k < n, \\ \sigma_n & \text{if } \sigma_n \leq \omega, \\ 0 & \text{if } \sigma_n = c, \text{ and either } \xi = c \text{ or } \sigma_k = c, \text{ for any } k < n. \end{cases}$$

So $\sigma^Y$ is a sequence of cardinals each of which is equal to either 0 or $c$. $\sigma^Z$ is a sequence of cardinals each of which is countable except for one element of the sequence which is equal to $c$. Moreover, if $\xi = c$, then $\xi = \xi = \xi = c$, and if $\sigma_n^Z = c$, then $\sigma_n^Y = \sigma_n^Z = \sigma_n = c$ and this is the first term in the sequence $\sigma$ that is equal to $c$.

By Lemmas 4.8 and 4.10, there are homeomorphisms $U$ of $Y$ and $S$ of $Z$ such that $U$ and $S$ agree on $S_1$, $\sigma(U) = \sigma^Y$, and $\sigma(S) = \sigma^Z$. By Lemma 4.2, $T = U \lor S$ is a homeomorphism of $X = Y \cup Z$ such that $\sigma(T) = \sigma$.

5. A universal space for homeomorphisms of separable metric spaces

In the previous sections, we used the fact that the set of points of a particular orbit type in a compact metric space is a Borel set, and therefore must have countable cardinality or size $c$. This allowed us to interpret the restriction

$$\zeta + \sum \sigma_n = c$$

as $\sigma_n \in \omega \cup \{\omega\} \cup \{c\}$ for each $n$, $\xi \in \omega \cup \{\omega\} \cup \{c\}$ as well, and at least one of those must be $c$. In the preceding sections we demonstrated that all orbit spectra meeting this criteria are realizable as homeomorphisms on $X$; hence $X$ is universal for compact metric spaces of size $c$.

If we consider though the class of separable metric spaces, then it is possible that some of the sets of a particular orbit type might have cardinality between $\omega$ and $c$. Consider, for example, a model of set theory in which $\aleph_1 < c$. An irrational rotation of the circle has $c$ many $\mathbb{Z}$ orbits. The restriction of this rotation to any $\aleph_1$ orbits is a homeomorphism of a separable metric space with precisely $\aleph_1$ many $\mathbb{Z}$ orbits. In any model where $\aleph_1 < c$, the space $X$ constructed above will fail to be universal for separable metric spaces.

Under the continuum hypothesis, of course, the space $X$ is universal for homeomorphisms of separable metric spaces of size $c$. However, we can do better. If instead of assuming the full strength of the continuum hypothesis ($c = \aleph_1$) we simply assume that $c < \aleph_1$ then we can construct a universal space for separable metric spaces.
Theorem 5.1 ($\epsilon < \aleph_{\omega_1}$). There is a universal space for homeomorphisms of separable metric spaces of cardinality $\epsilon$.

Proof. Let $\mathbb{R}$ have its usual topology, and let $\mathbb{Z}$ have the discrete topology. Since $\epsilon < \aleph_{\omega_1}, \epsilon = \aleph_{\gamma}$ for some $\gamma < \omega_1$. Let $R_0 = \{0\}$ and, for each $0 < \alpha \leq \gamma$, choose a subspace $R_\alpha$ of $\mathbb{R}$ of cardinality $\aleph_\alpha$ so that $R_\alpha$ and $R_\beta$ are disjoint when $\alpha \neq \beta$. Then, for each $\alpha \leq \gamma$, $X_\alpha = R_\alpha \times \mathbb{Z}$ is a separable metric space.

Let

$$X = \bigcup_{\alpha \leq \gamma} X_\alpha$$

be the free topological sum of the spaces $X_\alpha$, so that $X$ has cardinality $\epsilon$.

Let $\pi$ be a permutation of $\mathbb{Z}$. If $\sigma(\pi) = (0, \zeta_\pi, \sigma_\pi, \sigma_\pi, \ldots)$, then, for each $0 < \alpha$, the map $T_{\alpha, \pi} : X_\alpha \rightarrow X_\alpha$ defined by $T_{\alpha, \pi}(x, n) = (x, \pi(n))$ is a homeomorphism of $X_\alpha$ with orbit spectrum $\sigma(T_{\alpha, \pi}) = (0, \zeta, \sigma_1, \sigma_2, \ldots)$ where

$$\zeta = \begin{cases} \aleph_\alpha & \text{if } \zeta_\pi \neq 0, \\ 0 & \text{if } \zeta_\pi = 0, \end{cases} \text{ and, for } 1 \leq n, \quad \sigma_n = \begin{cases} \aleph_\alpha & \text{if } \sigma_{\pi n} \neq 0, \\ 0 & \text{if } \sigma_{\pi n} = 0. \end{cases}$$

Given $\sigma = (0, \zeta, \sigma_1, \sigma_2, \ldots)$, there exists a (not necessarily unique) corresponding permutation $\pi$ of $\mathbb{Z}$ with $\sigma(\pi) = (0, \zeta_\pi, \sigma_\pi, \sigma_\pi, \ldots)$ such that $\zeta_\pi \neq 0$ if and only if $\zeta \neq 0$ and $\sigma_{\pi n} \neq 0$ if and only if $\sigma_n \neq 0$. So given a sequence $\sigma = (0, \zeta, \sigma_1, \sigma_2, \ldots)$ not all $0$’s where $\zeta$ and each $\sigma_n$ are either $0$ or $\aleph_\alpha$, we can choose a corresponding permutation $\pi$ of $\mathbb{Z}$, so that $T_{\alpha, \pi} : X_\alpha \rightarrow X_\alpha$ is a homeomorphism with $\sigma(T_{\alpha, \pi}) = \sigma$.

Let $\sigma = (0, \zeta, \sigma_1, \sigma_2, \ldots)$ be a sequence of cardinals such that $\zeta + \sum_{1 \leq n} \sigma_n = \epsilon$. Since the cofinality of $\epsilon$ is uncountable [7], at least one term of $\sigma$ is equal to $\epsilon$. For each $\alpha \leq \gamma$, define $\tau_\alpha = (0, \zeta_\alpha, \sigma_\alpha_1, \sigma_\alpha_2, \ldots)$ where

$$\zeta_\alpha = \begin{cases} \zeta & \text{if } \alpha = 0 \text{ and } \zeta \leq \omega, \\ 0 & \text{if } \alpha \neq 0 \text{ and } \zeta = \aleph_\alpha, \end{cases} \text{ and, for } 1 \leq n, \quad \sigma_{\alpha n} = \begin{cases} \sigma_n & \text{if } \alpha = 0 \text{ and } \sigma_n \leq \omega, \\ 0 & \text{if } \alpha \neq 0 \text{ and } \sigma_n = \aleph_\alpha, \end{cases}$$

Let $A = \{\alpha < \gamma : \tau_\alpha \neq (0, 0, 0, \ldots)\}$. By the above, for each $\alpha \in A$, there is a homeomorphism $T_\alpha$ of $X_\alpha$ such that $\sigma(T_\alpha) = \tau_\alpha$. Let $X_1 = \bigcup_{\alpha \in A} X_\alpha$ and define $T^1 : X_1 \rightarrow X_1$ in the natural way, so that $T^1(x) = T_\alpha(x)$, when $x \in X_\alpha$. Then $\sigma(T^1) = (\epsilon, \zeta(T^1), \sigma_1(T^1), \sigma_2(T^1), \ldots)$, where $\zeta(T^1) = \zeta$ if and only if $\zeta < \epsilon$ and $\sigma_n(T^1) = \sigma_n$ if and only if $\sigma_n < \epsilon$. Since at least one term of $\sigma$ is equal to $\epsilon$, $\tau_\alpha \neq (0, 0, 0, \ldots)$ and, again by the above, there is a homeomorphism $T_\gamma$ of $X_\gamma$ such that $\sigma(T_\gamma) = \tau_\gamma$. For each $\alpha \not\in A \cup \{\gamma\}$ let $T_\alpha$ be a homeomorphism on $X_\alpha$ such that $\zeta(T_\alpha) = 0$ if and only if $\zeta = 0$ and $\sigma_n(T_\alpha) = 0$ if and only if $\sigma_n = 0$, which we can again do by choosing an appropriate permutation of $\mathbb{Z}$. Letting $X_2 = X_\gamma \cup \bigcup_{\alpha \not\in A} X_\alpha$, and defining $T^2 : X_2 \rightarrow X_2$ similarly to the way we defined $T^1$, we have $\zeta(T^2) = 0$ if and only if $\zeta = 0$ and $\sigma_n(T^2) = 0$ if and only if $\sigma_n = 0$. Let $X = X_1 \cup X_2$ and define $T : X \rightarrow X$ by $T(x) = T^i(x)$, for $x \in X_i$. By Lemma 4.2, $T$ is a homeomorphism of $X$ with $\sigma(T) = \sigma$. □

Applying the above proof in the case $\epsilon = \aleph_1$ and noting that $(\mathbb{R} - \mathbb{Q}) \cap (0, 1)$ is homeomorphic to $\mathbb{R} - \mathbb{Q}$, so that $\mathbb{R} - \mathbb{Q}$ is homeomorphic to countably many copies of itself, we have the following corollary.
Corollary 5.2 (CH). \((\mathbb{R} \setminus \mathbb{Q}) \cup \mathbb{N}\) is universal for homeomorphisms of separable metric spaces of cardinality \(c\).

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References