The behavior of solutions of second order delay differential equations

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Abstract
In this paper, we study the behavior of solutions of second order delay differential equation

\[ y''(t) = p_1 y'(t) + p_2 y'(t - \tau) + q_1 y(t) + q_2 y(t - \tau), \]

where \( p_1, p_2, q_1, q_2 \) are real numbers, \( \tau \) is positive real number. A basic theorem on the behavior of solutions is established. As a consequence of this theorem, a stability criterion is obtained.

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1. Introduction and preliminaries

Let us consider initial value problem for second order delay differential equation

\[
y''(t) = p_1 y'(t) + p_2 y'(t - \tau) + q_1 y(t) + q_2 y(t - \tau), \quad t \geq 0, \\
y(t) = \phi(t), \quad -\tau \leq t \leq 0,
\]

where \( p_1, p_2, q_1, q_2 \) are real numbers, \( \tau \) is positive real number and \( \phi(t) \) is a given continuously differentiable initial function on the interval \([-\tau, 0]\).

In many fields of the contemporary science and technology systems with delaying links are often met and the dynamical processes in these are described by systems of delay differential equations [1,4,5]. The delay appears in complicated systems with logical and computing devices,
where certain time for information processing is needed. The theory of linear delay differential equations has been developed in the fundamental monographs [1,4–6,8].

The equation of form of (1) is of interest in biology in explaining self-balancing of the human body and in robotics in constructing biped robots (see [10,12]). These are illustrations of inverted pendulum problems. A typical example is the balancing of a stick (see [13]). Equation of form of (1) can be used as test of equations for numerical methods (see [7,14]).

In [3], it has been established the boundedness under the conditions
\[ b > 0, \quad k > 0, \quad p > -k \quad \text{and} \quad |q| + |r|p < b, \]
on \([0, \infty)\) of the solution of the second order equation
\[ y''(t) + by'(t) + qy(t-r) + ky(t) + py(t-r) = 0, \quad \text{for} \ t \geq 0, \]
where \( r > 0 \), together with a given continuously differentiable initial function
\[ y(t) = \phi(t) \quad \text{on} \ [-r, 0]. \]

Recently, Cahlon and Schmidt et al. [2] have established the stability criteria for a second order delay differential equation of form (1) with \( p_1, p_2 \geq 0 \) and \( q_1, q_2 < 0 \).

This paper deals with the stability of the trivial solution for a second order linear delay differential equation with constant delay. An estimate of the solutions is established. The sufficient conditions for the stability, the asymptotic stability and instability of the trivial solution are given. Our results are derived by the use of real roots (with an appropriate property) of the corresponding (in a sense) characteristic equations. The techniques applied in obtaining our results are originated in a combination of the methods used in [9] and [11].

As usual, a twice continuously differentiable real-valued function \( y \) defined on the interval \([-\tau, \infty)\) is said to be a solution of the initial value problem (1) and (2) if \( y \) satisfies (1) for all \( t \geq 0 \) and (2) for all \(-\tau \leq t \leq 0\).

It is known that (see, for example, [4]), for any given initial function \( \phi \), there exists a unique solution of the initial problem (1)–(2) or, more briefly, the solution of (1)–(2).

If we look for a solution of (1) of the form \( y(t) = e^{\lambda t} \) for \( t \in \mathbb{R} \), we see that \( \lambda \) is a root of the characteristic equation
\[ \lambda^2 = p_1 \lambda + p_2 \lambda e^{-\lambda \tau} + q_1 + q_2 e^{-\lambda \tau}. \] (3)

Before closing this section, we will give two well-known definitions (see, for example, [5]). The trivial solution of (1) is said to be “stable” (at 0) if for every \( \varepsilon > 0 \), there exists a number \( \ell = \ell(\varepsilon) > 0 \) such that, for any initial function \( \phi \) with
\[ \|\phi\| = \max_{-\tau \leq t \leq 0} |\phi(t)| < \ell, \]
the solution \( y \) of (1)–(2) satisfies
\[ |y(t)| < \varepsilon, \quad \text{for all} \ t \in [-\tau, \infty). \]
Otherwise, the trivial solution of (1) is said to be “unstable” (at 0). Moreover, the trivial solution of (1) is called “asymptotically stable” (at 0) if it is stable in the above sense and in addition there exists a number \( \ell_0 > 0 \) such that, for any initial function \( \phi \) with \( \|\phi\| < \ell_0 \), the solution \( y \) of (1)–(2) satisfies
\[ \lim_{t \to \infty} y(t) = 0. \]
2. Statement of the main results and comments

Let now \( y \) be the solution of (1)–(2). Define
\[
x(t) = e^{-\lambda_0 t} y(t), \quad \text{for } t \in [-\tau, \infty),
\]
where \( \lambda_0 \) is a real root of the characteristic equation (3). Then, for every \( t \geq 0 \), we have
\[
x''(t) + 2\lambda_0 x'(t) + \lambda_0^2 x(t) = p_1 x'(t) + p_1 \lambda_0 x(t) + p_2 e^{-\lambda_0 \tau} x'(t - \tau) + p_2 \lambda_0 e^{-\lambda_0 \tau} x(t - \tau) + q_1 x(t) + q_2 e^{-\lambda_0 \tau} x(t - \tau)
\]
or
\[
\left[ x'(t) + (2\lambda_0 - p_1) x(t) - p_2 e^{-\lambda_0 \tau} x(t - \tau) \right]'
= \left( p_1 \lambda_0 + q_1 - \lambda_0^2 \right) x(t) + (p_2 \lambda_0 + q_2) e^{-\lambda_0 \tau} x(t - \tau).
\]
Moreover, the initial condition (2) can be equivalently written
\[
x(t) = e^{-\lambda_0 t} \phi(t), \quad \text{for } t \in [-\tau, 0].
\]
Furthermore, by using the fact that \( \lambda_0 \) is a root of (3) and taking into account (5), we can verify that (4) is equivalent to
\[
x'(t) = (p_1 - 2\lambda_0) x(t) + p_2 e^{-\lambda_0 \tau} x(t - \tau) + (p_1 \lambda_0 + q_1 - \lambda_0^2) \int_0^t x(s) \, ds + (p_2 \lambda_0 + q_2) e^{-\lambda_0 \tau} \int_0^t x(s - \tau) \, ds,
\]
\[
x'(t) = (p_1 - 2\lambda_0) x(t) + p_2 e^{-\lambda_0 \tau} x(t - \tau) + \left( p_2 \lambda_0 + q_2 \right) e^{-\lambda_0 \tau} \int_{-\tau}^t x(s) \, ds + \phi'(0) + (2\lambda_0 - p_1) \phi(0) - p_2 \phi(-\tau),
\]
\[
x'(t) = (p_1 - 2\lambda_0) x(t) + p_2 e^{-\lambda_0 \tau} x(t - \tau) + \left( p_2 \lambda_0 + q_2 \right) e^{-\lambda_0 \tau} \int_0^t x(s) \, ds + L(\lambda_0; \phi),
\]
\[
x'(t) = (p_1 - 2\lambda_0) x(t) + p_2 e^{-\lambda_0 \tau} x(t - \tau) - \left( p_2 \lambda_0 + q_2 \right) e^{-\lambda_0 \tau} \int_0^t x(s) \, ds + L(\lambda_0; \phi),
\]
\[
x'(t) = (p_1 - 2\lambda_0) x(t) + p_2 e^{-\lambda_0 \tau} x(t - \tau) + \left( p_2 \lambda_0 + q_2 \right) e^{-\lambda_0 \tau} \int_0^t x(s) \, ds + L(\lambda_0; \phi).
\]
\[ x'(t) = (p_1 - 2\lambda_0)x(t) + p_2 e^{-\lambda_0 t} x(t - \tau) - (p_2 \lambda_0 + q_2) e^{-\lambda_0 t} \int_{t-\tau}^{t} x(s) \, ds + L(\lambda_0; \phi), \]  

where

\[ L(\lambda_0; \phi) = \phi'(0) + (2\lambda_0 - p_1)\phi(0) - p_2\phi(-\tau) + (p_2 \lambda_0 + q_2) e^{-\lambda_0 \tau} \int_{-\tau}^{0} e^{-\lambda_0 s} \phi(s) \, ds. \]  

(7)

Let

\[ \beta_{\lambda_0} \equiv (p_2 \lambda_0 + q_2) \tau e^{-\lambda_0 \tau} + 2\lambda_0 - p_1 - p_2 e^{-\lambda_0 \tau} \neq 0 \]

and we define

\[ z(t) = x(t) - \frac{L(\lambda_0; \phi)}{\beta_{\lambda_0}}, \quad \text{for } t \geq -\tau. \]

Then we can see that (6) reduces to the following equivalent equation

\[ z'(t) = (p_1 - 2\lambda_0)z(t) + p_2 e^{-\lambda_0 \tau} z(t - \tau) - (p_2 \lambda_0 + q_2) e^{-\lambda_0 \tau} \int_{t-\tau}^{t} z(s) \, ds. \]

(8)

If we look for a solution of (8) of the form \( z(t) = e^{\delta t} \) for \( t \in \mathbb{R} \), we see that \( \delta \) is a root of the second characteristic equation

\[ \delta = p_1 - 2\lambda_0 + p_2 e^{-\lambda_0 \tau} e^{-\delta \tau} - \delta^{-1}(1 - e^{-\delta \tau})(p_2 \lambda_0 + q_2)e^{-\lambda_0 \tau}. \]

(9)

On the other hand, the initial condition (5) can be equivalently written

\[ z(t) = \phi(t)e^{-\lambda_0 t} - \frac{L(\lambda_0; \phi)}{\beta_{\lambda_0}}, \quad t \in [-\tau, 0]. \]

(10)

Let \( F(\delta) \) denote the characteristic function of (9), i.e.,

\[ F(\delta) = \delta - p_1 + 2\lambda_0 - p_2 e^{-\lambda_0 \tau} e^{-\delta \tau} + \delta^{-1}(1 - e^{-\delta \tau})(p_2 \lambda_0 + q_2) e^{-\lambda_0 \tau}. \]

Since \( \delta = 0 \) is a removable singularity of \( F(\delta) \), we can regard \( F(\delta) \) as an entire function with

\[ F(0) = (p_2 \lambda_0 + q_2) \tau e^{-\lambda_0 \tau} + 2\lambda_0 - p_1 - p_2 e^{-\lambda_0 \tau} \equiv \beta_{\lambda_0}. \]

But, by the definition of \( \beta_{\lambda_0} \neq 0 \), a root of the characteristic equation (9) must become \( \delta_0 \neq 0 \).

Let now \( z \) be the solution of (8)–(10) and \( \delta_0 \) be a real root of the characteristic equation (9). Define for \( \delta_0 \neq 0 \)

\[ v(t) = e^{-\delta_0 t} z(t), \quad \text{for all } t \in [-\tau, \infty). \]

Then, for every \( t \geq 0 \), we have

\[ v'(t) = (p_1 - 2\lambda_0 - \delta_0)v(t) + p_2 e^{\lambda_0 \tau \delta_0} v(t - \tau) - (p_2 \lambda_0 + q_2) e^{-\lambda_0 \tau} \int_{0}^{\tau} e^{-\delta_0 s} v(t - s) \, ds. \]

(11)
Moreover, the initial condition (10) can be equivalently written

\[ v(t) = \phi(t)e^{-(\lambda_0 + \delta_0)t} - e^{-\delta_0 t} \frac{L(\lambda_0; \phi)}{\beta_{\lambda_0}}, \quad \text{for } t \in [-\tau, 0]. \]  

(12)

Furthermore, by using the fact that \( \delta_0 \neq 0 \) is a real root of (9) and taking into account (12), we can verify that (11) is equivalent to

\[
\begin{align*}
v(t) &= v(0) + (p_1 - 2\lambda_0 - \delta_0) \int_0^t v(s) \, ds + p_2 e^{-(\lambda_0 + \delta_0)\tau} \int_0^t v(s - \tau) \, ds \\
&\quad - (p_2\lambda_0 + q_2)e^{-\lambda_0 \tau} \int_0^{\tau} e^{-\delta_0 s} \left\{ \int_0^t v(u - s) \, du \right\} \, ds,
\end{align*}
\]

\[
\begin{align*}
v(t) &= \phi(0) - \frac{L(\lambda_0; \phi)}{\beta_{\lambda_0}} + (p_1 - 2\lambda_0 - \delta_0) \int_0^t v(s) \, ds + p_2 e^{-(\lambda_0 + \delta_0)\tau} \int_{-\tau}^t v(s) \, ds \\
&\quad - (p_2\lambda_0 + q_2)e^{-\lambda_0 \tau} \int_0^{\tau} e^{-\delta_0 s} \left\{ \int_{-s}^{t-s} v(u) \, du \right\} \, ds,
\end{align*}
\]

\[
\begin{align*}
v(t) &= (p_1 - 2\lambda_0 - \delta_0) \int_0^t v(s) \, ds \\
&\quad + p_2 e^{-(\lambda_0 + \delta_0)\tau} \int_0^t e^{-\delta_0 s} \left( e^{-\lambda_0 s} \phi(s) - \frac{L(\lambda_0; \phi)}{\beta_{\lambda_0}} \right) \, ds + \int_0^{t-\tau} v(s) \, ds \\
&\quad + \phi(0) - \frac{L(\lambda_0; \phi)}{\beta_{\lambda_0}} - (p_2\lambda_0 + q_2)e^{-\lambda_0 \tau} \\
&\quad \times \int_0^{\tau} e^{-\delta_0 s} \left\{ \int_0^{-s} e^{-\lambda_0 u} \phi(u) - \frac{L(\lambda_0; \phi)}{\beta_{\lambda_0}} \right\} \, du + \int_0^{t-s} v(u) \, du \right\} \, ds,
\end{align*}
\]

\[
\begin{align*}
v(t) &= R(\lambda_0, \delta_0; \phi) + \left( \delta_0^{-1} (1 - e^{-\delta_0 t}) (p_2\lambda_0 + q_2)e^{-\lambda_0 t} - p_2 e^{-(\lambda_0 + \delta_0)\tau} \right) \int_0^t v(s) \, ds \\
&\quad + p_2 e^{-(\lambda_0 + \delta_0)\tau} \int_0^{t-\tau} v(s) \, ds - (p_2\lambda_0 + q_2)e^{-\lambda_0 \tau} \int_0^{\tau} e^{-\delta_0 s} \left\{ \int_0^{t-s} v(u) \, du \right\} \, ds,
\end{align*}
\]

\[
\begin{align*}
v(t) &= R(\lambda_0, \delta_0; \phi) + \left( (p_2\lambda_0 + q_2)e^{-\lambda_0 \tau} \int_0^\tau e^{-\delta_0 s} \, ds - p_2 e^{-(\lambda_0 + \delta_0)\tau} \right) \int_0^t v(s) \, ds \\
&\quad + p_2 e^{-(\lambda_0 + \delta_0)\tau} \int_0^{t-\tau} v(s) \, ds - (p_2\lambda_0 + q_2)e^{-\lambda_0 \tau} \int_0^{\tau} e^{-\delta_0 s} \left\{ \int_0^{t-s} v(u) \, du \right\} \, ds,
\end{align*}
\]
On the other hand, the initial condition (12) can be equivalently written

\[ v(t) = R(\lambda_0, \delta_0; \phi) - p_2e^{-(\lambda_0 + \delta_0)t} \int_{t-\tau}^{t} v(s) \, ds + (p_2\lambda_0 + q_2)e^{-\lambda_0\tau} \int_{0}^{\tau} e^{-\delta_0s} \left\{ \int_{t-s}^{t} v(u) \, du \right\} \, ds, \]

where

\[ R(\lambda_0, \delta_0; \phi) = \phi(0) - \frac{L(\lambda_0; \phi)}{\beta_{\lambda_0}} + p_2e^{-(\lambda_0 + \delta_0)t} \int_{-\tau}^{0} e^{-\delta_0s} \left( e^{-\lambda_0s} \phi(s) - \frac{L(\lambda_0; \phi)}{\beta_{\lambda_0}} \right) \, ds \]

\[ - (p_2\lambda_0 + q_2)e^{-\lambda_0\tau} \int_{0}^{\tau} e^{-\delta_0s} \left\{ \int_{-s}^{0} e^{-\lambda_0u} \phi(u) - \frac{L(\lambda_0; \phi)}{\beta_{\lambda_0}} \right\} \, ds. \]

Next, we define

\[ w(t) = v(t) - \frac{R(\lambda_0, \delta_0; \phi)}{\eta_{\lambda_0, \delta_0}}, \quad \text{for } t \geq -\tau, \]

where

\[ \eta_{\lambda_0, \delta_0} \equiv 1 + p_2e^{-(\lambda_0 + \delta_0)t} \tau - \delta_0^{-2}(1 - e^{-\delta_0\tau} - \delta_0\tau e^{-\delta_0\tau})(p_2\lambda_0 + q_2)e^{-\lambda_0\tau}. \]

Then we can see that (13) reduces to the following equivalent equation

\[ w(t) = -p_2e^{-(\lambda_0 + \delta_0)t} \int_{t-\tau}^{t} w(s) \, ds + (p_2\lambda_0 + q_2)e^{-\lambda_0\tau} \int_{0}^{\tau} e^{-\delta_0s} \left\{ \int_{t-s}^{t} w(u) \, du \right\} \, ds. \]

On the other hand, the initial condition (12) can be equivalently written

\[ w(t) = \phi(t)e^{-(\lambda_0 + \delta_0)t} - \frac{L(\lambda_0; \phi)}{\beta_{\lambda_0}} e^{-\delta_0t} - \frac{R(\lambda_0, \delta_0; \phi)}{\eta_{\lambda_0, \delta_0}} \quad \text{for } t \in [-\tau, 0]. \]

We have the following basic theorem.

**Theorem 1.** Let \( \lambda_0 \) and \( \delta_0 \) (\( \delta_0 \neq 0 \)) be real roots of the characteristic equations (3) and (9). Assume that the roots \( \lambda_0 \) and \( \delta_0 \) have the following property

\[ \mu_{\lambda_0, \delta_0} \equiv |p_2|e^{-(\lambda_0 + \delta_0)\tau} + \delta_0^{-2}(1 - e^{-\delta_0\tau} - \delta_0\tau e^{-\delta_0\tau})(p_2\lambda_0 + q_2)e^{-\lambda_0\tau} < 1 \]

and

\[ \beta_{\lambda_0} \equiv (p_2\lambda_0 + q_2)e^{-\lambda_0\tau} + 2\lambda_0 - p_1 - p_2e^{-\lambda_0\tau} \neq 0. \]

Then, for any \( \phi \in C([-\tau, 0], IR) \), the solution \( y \) of (1)–(2) satisfies

\[ \left| e^{-(\lambda_0 + \delta_0)t} y(t) - \frac{L(\lambda_0; \phi)}{\beta_{\lambda_0}} e^{-\delta_0t} - \frac{R(\lambda_0, \delta_0; \phi)}{\eta_{\lambda_0, \delta_0}} \right| \leq M(\lambda_0, \delta_0; \phi)\mu_{\lambda_0, \delta_0}, \quad \text{for all } t \geq 0, \]

\[ (20) \]
where \(L(\lambda; \phi), \ R(\lambda, \delta_0; \phi)\) and \(\eta_{\lambda_0, \delta_0}\) were given in (7), (14) and (16), respectively and

\[
M(\lambda, \delta_0; \phi) = \max_{-\tau \leq t \leq 0} \left| e^{-(\lambda_0 + \delta_0)t} \phi(t) - \frac{L(\lambda_0; \phi)}{\beta_{\lambda_0}} - \frac{R(\lambda_0, \delta_0; \phi)}{\eta_{\lambda_0, \delta_0}} \right|.
\]  

(21)

**Proof.** It is easy to see that property (19) guarantees that \(\eta_{\lambda_0, \delta_0} > 0\). Applying the definitions of \(x, z, v\) and \(w\) we can obtain that (20) is equivalent to

\[
|w(t)| \leq M(\lambda, \delta_0; \phi)\mu_{\lambda_0, \delta_0} \quad \forall t \geq 0.
\]  

(22)

So, we will prove (22).

From (18) and (21) it follows that

\[
|w(t)| \leq M(\lambda, \delta_0; \phi), \quad \text{for } t \in [-\tau, 0].
\]  

(23)

We will show that \(M(\lambda, \delta_0; \phi)\) is a bound of \(w\) on the whole interval \([-\tau, \infty)\). Namely

\[
|w(t)| \leq M(\lambda, \delta_0; \phi), \quad \text{for all } t \in [-\tau, \infty).
\]  

(24)

To this end, let us consider an arbitrary number \(\varepsilon > 0\). We claim that

\[
|w(t)| < M(\lambda, \delta_0; \phi) + \varepsilon, \quad \text{for every } t \in [-\tau, \infty).
\]  

(25)

Otherwise, by (23), there exists a \(t^* > 0\) such that

\[
|w(t)| < M(\lambda, \delta_0; \phi) + \varepsilon, \quad \text{for } t < t^* \quad \text{and} \quad |w(t^*)| = M(\lambda, \delta_0; \phi) + \varepsilon.
\]

Then using (17), we obtain

\[
M(\lambda, \delta_0; \phi) + \varepsilon
\]  

\[
= |w(t^*)|
\]

\[
\leq |p_2|e^{-(\lambda_0 + \delta_0)\tau} \int_{t^*-\tau}^{t^*} |w(s)| \, ds + |p_2\lambda_0 + q_2|e^{-\lambda_0\tau} \int_0^{t^*} e^{-\delta_0s} \left\{ \int_{t^*-s}^{t^*} |w(u)| \, du \right\} \, ds
\]

\[
\leq \left\{ |p_2|e^{-(\lambda_0 + \delta_0)\tau} + \delta_0^{-2}(1 - e^{-\delta_0\tau} - \delta_0\tau e^{-\delta_0\tau})|p_2\lambda_0 + q_2|e^{-\lambda_0\tau}\right\} [M(\lambda, \delta_0; \phi) + \varepsilon]
\]

\[
< [M(\lambda, \delta_0; \phi) + \varepsilon],
\]

which, by view of (19), leads to a contradiction. So, our claim is true. Since (25) holds for every \(\varepsilon > 0\), it follows that (24) is always satisfied. By using (24) and (17), we derive

\[
|w(t)| \leq |p_2|e^{-(\lambda_0 + \delta_0)\tau} \int_{t-\tau}^{t} |w(s)| \, ds + |p_2\lambda_0 + q_2|e^{-\lambda_0\tau} \int_0^{t} e^{-\delta_0s} \left\{ \int_{t-s}^{t} |w(u)| \, du \right\} \, ds
\]

\[
\leq \left\{ |p_2|e^{-(\lambda_0 + \delta_0)\tau} + \delta_0^{-2}(1 - e^{-\delta_0\tau} - \delta_0\tau e^{-\delta_0\tau})|p_2\lambda_0 + q_2|e^{-\lambda_0\tau}\right\} M(\lambda, \delta_0; \phi)
\]

\[
= M(\lambda, \delta_0; \phi)\mu_{\lambda_0, \delta_0},
\]

for all \(t \geq 0\). That means (22) holds. \(\square\)

**Theorem 2.** Let \(\lambda_0\) and \(\delta_0\) (\(\delta_0 \neq 0\)) be real roots of the characteristic equations (3) and (9). Consider \(\beta_{\lambda_0}\) as in Theorem 1. Then, for any \(\phi \in C([-\tau, 0], IR)\), the solution \(y\) of (1)–(2) satisfies

\[
\lim_{t \to \infty} \left\{ e^{-(\lambda_0 + \delta_0)t} y(t) - \frac{L(\lambda_0; \phi)}{\beta_{\lambda_0}} e^{-\delta_0t} \right\} = \frac{R(\lambda_0, \delta_0; \phi)}{\eta_{\lambda_0, \delta_0}},
\]

where \(L(\lambda_0; \phi), \ R(\lambda_0, \delta_0; \phi)\) and \(\eta_{\lambda_0, \delta_0}\) were given in (7), (14) and (16), respectively.
Proof. By the definitions of $x$, $z$, $v$ and $w$, we have to prove that

$$\lim_{t \to \infty} w(t) = 0.$$  \hfill (26)

In the end of the proof we will establish (26). By using (17) and taking into account (22) and (24), one can show, by an easy induction, that $w$ satisfies

$$|w(t)| \leq (\mu_{\lambda_0, \delta_0})^n M(\lambda_0, \delta_0; \phi), \quad \text{for all } t \geq n \tau - \tau \ (n = 0, 1, \ldots).$$  \hfill (27)

But, (19) guarantees that $0 < \mu_{\lambda_0, \delta_0} < 1$. Thus, from (27) it follows immediately that $w$ tends to zero as $t \to \infty$, i.e. (26) holds.

The proof of Theorem 2 is completed. \hfill \Box

**Theorem 3.** Let $\lambda_0$ and $\delta_0$ ($\delta_0 \neq 0$) be real roots of the characteristic equations (3) and (9) and also the conditions in Theorem 1 $\beta_{\lambda_0}$ and $\mu_{\lambda_0, \delta_0}$ be provided. Then, for any $\phi \in C([-\tau, 0], IR)$, the solution $y$ of (1)–(2) satisfies for all $t \geq 0$

$$|y(t)| \leq \frac{k_{\lambda_0}}{|\beta_{\lambda_0}|} N(\lambda_0, \delta_0; \phi) e^{\lambda_0 t}$$

$$+ \left[ \frac{h_{\lambda_0, \delta_0}}{\eta_{\lambda_0, \delta_0}} + \left( 1 + \frac{k_{\lambda_0} e^{\delta_0}}{|\beta_{\lambda_0}|} + \frac{h_{\lambda_0, \delta_0}}{|\beta_{\lambda_0}|} \right) \mu_{\lambda_0, \delta_0} \right] N(\lambda_0, \delta_0; \phi) e^{(\lambda_0 + \delta_0) t},$$  \hfill (28)

where

$$\eta_{\lambda_0, \delta_0} \quad \text{was given in (16)},$$

$$k_{\lambda_0} = 1 + |2\lambda_0 - p_1| + |p_2| e^{-\lambda_0 \tau} + |p_2 \lambda_0 + q_2| e^{-\lambda_0 \tau} \tau,$$  \hfill (29)

$$h_{\lambda_0, \delta_0} = 1 + \frac{k_{\lambda_0}}{|\beta_{\lambda_0}|} \delta_0^{-1} (1 - e^{-\delta_0 \tau}) |p_2| e^{-(\lambda_0 + \delta_0) \tau} \left( 1 + \frac{k_{\lambda_0}}{|\beta_{\lambda_0}|} \right)$$

$$+ \delta_0^{-2} (\delta_0 \tau + e^{-\delta_0 \tau} - 1) |p_2 \lambda_0 + q_2| e^{-\lambda_0 \tau} \left( 1 + \frac{k_{\lambda_0}}{|\beta_{\lambda_0}|} \right),$$  \hfill (30)

$$e_{\delta_0} = \max_{-\tau \leq t \leq 0} \left| e^{-\delta_0 t} \right|$$  \hfill (31)

and

$$N(\lambda_0, \delta_0; \phi)$$

$$= \max \left\{ \max_{-\tau \leq t \leq 0} \left| e^{-\lambda_0 t} \phi(t) \right|, \max_{-\tau \leq t \leq 0} \left| e^{-(\lambda_0 + \delta_0) t} \phi(t) \right|, \max_{-\tau \leq t \leq 0} \left| \phi'(t) \right|, \max_{-\tau \leq t \leq 0} \left| \phi(t) \right| \right\}. $$  \hfill (32)

Moreover, the trivial solution of (1) is stable if $\lambda_0 \leq 0$, $\lambda_0 + \delta_0 \leq 0$, it is asymptotically stable if $\lambda_0 < 0$, $\lambda_0 + \delta_0 < 0$ and it is unstable if $\delta_0 > 0$, $\lambda_0 + \delta_0 > 0$.

Proof. By Theorem 1, (20) is satisfied, where $L(\lambda_0; \phi)$ and $M(\lambda_0, \delta_0; \phi)$ are defined by (7) and (21), respectively. From (20) it follows that

$$e^{-(\lambda_0 + \delta_0) t} |y(t)| \leq \frac{|L(\lambda_0; \phi)|}{|\beta_{\lambda_0}|} e^{-\delta_0 t} + \frac{|R(\lambda_0, \delta_0; \phi)|}{\eta_{\lambda_0, \delta_0}} + M(\lambda_0, \delta_0; \phi) \mu_{\lambda_0, \delta_0}.$$  \hfill (33)

Furthermore, by using (29)–(32), from (7), (14) and (21), we obtain
\[ |L(\lambda_0; \phi)| \leq |\phi'(0)| + |2\lambda_0 - p_1||\phi(0)| + |p_2||\phi(-\tau)| + |p_2\lambda_0 + q_2e^{-\lambda_0\tau} \int_{-\tau}^{0} e^{-\lambda_0s} |\phi(s)| \, ds \]
\[
\leq (1 + |2\lambda_0 - p_1| + |p_2|e^{-\lambda_0\tau} + |p_2\lambda_0 + q_2e^{-\lambda_0\tau}|)N(\lambda_0, \delta_0; \phi) = k_{\lambda_0}N(\lambda_0, \delta_0; \phi),
\]
\[ |R(\lambda_0, \delta_0; \phi)| \leq |\phi(0)| + \frac{|L(\lambda_0; \phi)|}{|\beta_{\lambda_0}|} + |p_2|e^{-(\lambda_0+\delta_0)\tau} \int_{-\tau}^{0} e^{-\delta_0s} \left( e^{-\lambda_0u} |\phi(u)| + \frac{|L(\lambda_0; \phi)|}{|\beta_{\lambda_0}|} \right) \, du \, ds \]
\[ \leq \left[ 1 + \frac{k_{\lambda_0}}{|\beta_{\lambda_0}|} + \delta_0^{-1}(1 - e^{-\delta_0\tau})|p_2|e^{-(\lambda_0+\delta_0)\tau} \left( 1 + \frac{k_{\lambda_0}}{|\beta_{\lambda_0}|} \right) + \delta_0^{-2}(\delta_0\tau + e^{-\delta_0\tau} - 1)|p_2\lambda_0 + q_2e^{-\lambda_0\tau} \left( 1 + \frac{k_{\lambda_0}}{|\beta_{\lambda_0}|} \right) \right] N(\lambda_0, \delta_0; \phi) = h_{\lambda_0,\delta_0}N(\lambda_0, \delta_0; \phi), \]
\[ M(\lambda_0, \delta_0; \phi) \leq \max_{-\tau \leq t \leq 0} \left\{ e^{-(\lambda_0+\delta_0)t} |\phi(t)| \right\} + \frac{|L(\lambda_0; \phi)|}{|\beta_{\lambda_0}|} \max_{-\tau \leq t \leq 0} \left\{ e^{-\delta_0t} \right\} + \frac{|R(\lambda_0, \delta_0; \phi)|}{\eta_{\lambda_0,\delta_0}} \]
\[ \leq \left\{ 1 + \frac{k_{\lambda_0}e^{\delta_0}}{|\beta_{\lambda_0}|} + \frac{h_{\lambda_0,\delta_0}}{\eta_{\lambda_0,\delta_0}} \right\} N(\lambda_0, \delta_0; \phi). \]

Hence, from (33), we conclude that for all \( t \geq 0, \)
\[ e^{-(\lambda_0+\delta_0)t} |y(t)| \leq \frac{k_{\lambda_0}}{|\beta_{\lambda_0}|} N(\lambda_0, \delta_0; \phi)e^{-\delta_0t} + \frac{h_{\lambda_0,\delta_0}}{\eta_{\lambda_0,\delta_0}} N(\lambda_0, \delta_0; \phi) + \left( 1 + \frac{k_{\lambda_0}e^{\delta_0}}{|\beta_{\lambda_0}|} + \frac{h_{\lambda_0,\delta_0}}{\eta_{\lambda_0,\delta_0}} \right) N(\lambda_0, \delta_0; \phi) \mu_{\lambda_0,\delta_0}, \]
and consequently, (28) holds.

Now, let us assume that \( \lambda_0 \leq 0 \) and \( \lambda_0 + \delta_0 \leq 0. \) Define \( \|\phi\| \equiv \max_{-\tau \leq t \leq 0} |\phi(t)|. \) It follows that \( \|\phi\| \leq N(\lambda_0, \delta_0; \phi). \) From (34), it follows that
\[ |y(t)| \leq \left\{ \frac{k_{\lambda_0}}{|\beta_{\lambda_0}|} + \left( 1 + \frac{k_{\lambda_0}e^{\delta_0}}{|\beta_{\lambda_0}|} \right) \mu_{\lambda_0,\delta_0} + (1 + \mu_{\lambda_0,\delta_0}) \frac{h_{\lambda_0,\delta_0}}{\eta_{\lambda_0,\delta_0}} \right\} N(\lambda_0, \delta_0; \phi), \]
for every \( t \geq 0. \)

Since \( \frac{k_{\lambda_0}}{|\beta_{\lambda_0}|} > 1, \) by taking into account the fact that
\[ \frac{k_{\lambda_0}}{|\beta_{\lambda_0}|} + \left( 1 + \frac{k_{\lambda_0}e^{\delta_0}}{|\beta_{\lambda_0}|} \right) \mu_{\lambda_0,\delta_0} + (1 + \mu_{\lambda_0,\delta_0}) \frac{h_{\lambda_0,\delta_0}}{\eta_{\lambda_0,\delta_0}} > 1, \]
we have
\[ |y(t)| \leq \left\{ \frac{k_{\lambda_0}}{|\beta_{\lambda_0}|} + \left( 1 + \frac{k_{\lambda_0} e^{\delta_0}}{|\beta_{\lambda_0}|} \right) \mu_{\lambda_0, \delta_0} + (1 + \mu_{\lambda_0, \delta_0} \frac{h_{\lambda_0, \delta_0}}{\eta_{\lambda_0, \delta_0}}) \right\} N(\lambda_0, \delta_0; \phi), \]
for every \( t \in [-\tau, \infty) \),
which means that the trivial solution of (1) is stable (at 0).

Next, if \( \lambda_0 < 0 \) and \( \lambda_0 + \delta_0 < 0 \), then (28) guarantees that
\[
\lim_{t \to \infty} y(t) = 0
\]
and so the trivial solution of (1) is asymptotically stable (at 0).

Finally, if \( \delta_0 > 0, \lambda_0 + \delta_0 > 0 \), then the trivial solution of (1) is unstable (at 0). Otherwise, there exists a number \( \ell \equiv \ell(1) > 0 \) such that, for any \( \phi \in C([-\tau, 0], IR) \) with \( \|\phi\| < \ell \), the solution \( y \) of problem (1)–(2) satisfies
\[
|y(t)| < 1 \quad \text{for all } t \geq -\tau.
\]
(35)

Define
\[
\phi_0(t) = e^{(\lambda_0 + \delta_0)t} - e^{\lambda_0 t} \quad \text{for } t \in [-\tau, 0].
\]
Furthermore, by the definition of \( L(\lambda_0; \phi) \) and \( R(\lambda_0, \delta_0; \phi) \), by using (9), we have
\[
L(\lambda_0; \phi_0) = \delta_0 - p_2 e^{-(\lambda_0 + \delta_0)t} + p_2 e^{-\lambda_0 t}
\]
\[+ (p_2 \lambda_0 + q_2) e^{-\lambda_0 t} \int_{-\tau}^{0} e^{\delta_0 s} \, ds - (p_2 \lambda_0 + q_2) e^{-\lambda_0 t} \tau = \delta_0 + p_1 - 2 \lambda_0 - \delta_0 + p_2 e^{-\lambda_0 t} - (p_2 \lambda_0 + q_2)e^{-\lambda_0 t} \tau \equiv -\beta_{\lambda_0},
\]
\[
R(\lambda_0, \delta_0; \phi_0) = 1 + p_2 e^{-(\lambda_0 + \delta_0)t} \int_{-\tau}^{0} e^{-\delta_0 s} \left( e^{-\lambda_0 s} (e^{(\lambda_0 + \delta_0)s} - e^{\lambda_0 s}) + 1 \right) \, ds
\]
\[- (p_2 \lambda_0 + q_2) e^{-\lambda_0 t} \int_{0}^{\tau} e^{-\delta_0 s} \left\{ \int_{-s}^{0} e^{-\delta_0 u} (e^{-\lambda_0 u} (e^{(\lambda_0 + \delta_0)u} - e^{\lambda_0 u}) + 1) \, du \right\} \, ds
\]
\[= 1 + p_2 e^{-(\lambda_0 + \delta_0)t} \tau - (p_2 \lambda_0 + q_2) e^{-\lambda_0 t} \int_{0}^{\tau} e^{-\delta_0 s} \, ds \equiv \eta_{\lambda_0, \delta_0} > 0.
\]

Let \( \phi \in C([-\tau, 0], IR) \) be defined by
\[
\phi = \frac{\ell_1}{\|\phi_0\|} \phi_0,
\]
where \( \ell_1 \) is a number with \( 0 < \ell_1 < \ell \). Moreover, let \( y \) be the solution of (1)–(2). From Theorem 2 it follows that \( y \) satisfies
\[
\lim_{t \to \infty} \left\{ e^{-(\lambda_0 + \delta_0)t} y(t) - \frac{L(\lambda_0; \phi)}{\beta_{\lambda_0}} e^{-\delta_0 t} \right\} = \lim_{t \to \infty} \left\{ e^{-(\lambda_0 + \delta_0)t} y(t) + \frac{\ell_1}{\|\phi_0\|} e^{-\delta_0 t} \right\} \\
= R(\lambda_0, \delta_0; \phi) \\
= \frac{(\ell_1/\|\phi_0\|) R(\lambda_0, \delta_0; \phi_0)}{\|\phi_0\|} = \frac{\ell_1}{\|\phi_0\|} > 0.
\]

But, we have \( \|\phi\| = \ell_1 < \ell \) and hence from (35) and conditions \( \delta_0 > 0, \lambda_0 + \delta_0 > 0 \) it follows that
\[
\lim_{t \to \infty} \left\{ e^{-(\lambda_0 + \delta_0)t} y(t) - \frac{L(\lambda_0; \phi)}{\beta_{\lambda_0}} e^{-\delta_0 t} \right\} = 0.
\]

This is a contradiction.

The proof of Theorem 3 is completed. \( \square \)

3. Examples

**Example 1.** Consider
\[
y''(t) = -4y'(t) + \frac{1}{e} y\left(t - \frac{1}{2}\right) - 3y(t) + \frac{1}{e} y\left(t - \frac{1}{2}\right), \quad t \geq 0,
\]
y(t) = \phi(t), \quad -\frac{1}{2} \leq t \leq 0,
\]
where \( \phi(t) \) is an arbitrary continuously differentiable initial function on the interval \([-\frac{1}{2}, 0]\). In this example we apply the characteristic equations (3) and (9). That is, the characteristic equation (3) is
\[
\lambda^2 = -4\lambda + \frac{1}{e^2} \lambda e^{-\lambda \frac{1}{2}} - 3 + \frac{1}{e} e^{-\lambda \frac{1}{2}},
\]
and we see that \( \lambda = -1 \) is a root of (37). Then, for \( \lambda_0 = -1 \) the characteristic equation (9) is
\[
\delta = -2 + e^{-\frac{1}{2}(1+\delta)}.
\]
Therefore, \( \delta = \delta_0 = -1 \) is a root, and the conditions of Theorem 3 are satisfied. That is,
\[
\mu_{\lambda_0, \delta_0} = \mu_{-1, -1} = \frac{1}{2} < 1 \quad \text{and} \quad \beta_{\lambda_0} = \beta_{-1} = 2 - \frac{1}{\sqrt{e}} \neq 0.
\]
Since \( \lambda_0 = -1 < 0 \) and \( \lambda_0 + \delta_0 = -2 < 0 \), the zero solution of (36) is asymptotically stable.

**Example 2.** Consider
\[
y''(t) = -\frac{e}{2} y'(t) - \frac{1}{2} y'(t - 1) + y(t) - y(t - 1), \quad t \geq 0,
\]
y(t) = \phi(t), \quad -1 \leq t \leq 0,
\]
where \( \phi(t) \) is an arbitrary continuously differentiable initial function on \([-1, 0]\). The characteristic equation (3) is
\[
\lambda^2 = -\frac{e}{2} \lambda - \frac{1}{2} \lambda e^{-\lambda} + 1 - e^{-\lambda},
\]

(39)
and we see easily that $\lambda = 0$ is a root of (39). Taking $\lambda_0 = 0$, the characteristic equation (9) is
\[
\delta^2 = -\frac{e}{2} - \frac{1}{2} e^{-\delta} + 1 - e^{-\delta}.
\]

Therefore, we find that $\delta = \delta_0 = -1$ is a root. The condition of Theorem 3 is $\mu_{\lambda_0, \delta_0} = \mu_{0, -1} > 1$. Thus, Theorem 3 is not applicable. But, $\lambda = -1$ is another root of (39). Then, for $\lambda_0 = -1$ we have that
\[
\delta^2 = \left(-\frac{e}{2} + 2\right) \delta - \frac{1}{2} e^{1-\delta} + \frac{1}{2} e^{1-\delta},
\]
and we find $\delta = \delta_0 = 1$. Corresponding to the roots $\lambda_0 = -1$ and $\delta_0 = 1$, the conditions of Theorem 3 are satisfied. Since $\lambda_0 = -1 < 0$ and $\lambda_0 + \delta_0 = 0$, the zero solution of (38) is stable.

**Example 3.** Consider
\[
y''(t) = 3y'(t) - y\left(t - \frac{\pi}{2}\right) - 2y(t) + y\left(t - \frac{\pi}{2}\right), \quad t \geq 0,
\]
\[
y(t) = \phi(t), \quad -\frac{\pi}{2} \leq t \leq 0,
\]
where $\phi(t)$ is an arbitrary continuously differentiable initial function on $[-\frac{\pi}{2}, 0]$. The characteristic equation (3) is
\[
\lambda^2 = 3\lambda - \lambda e^{-\frac{\pi}{2}} - 2 + e^{-\frac{\pi}{2}},
\]
and we see easily that $\lambda = 1$ is a root of (41). Taking $\lambda_0 = 1$, the characteristic equation (9) is
\[
\delta = 1 - e^{-\frac{\pi}{2}(\delta+1)}.
\]

Therefore, we find that $\delta = \delta_0 \cong 0.95351$ is a root. Corresponding to the roots $\lambda_0 = 1$ and $\delta_0 = 0.95351$, the conditions of Theorem 3 are satisfied. Since $\delta_0 > 0$ and $\lambda_0 + \delta_0 > 0$, the zero solution of (40) is unstable.

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**References**