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A sum theorem for (FPV) operators and normal cones

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ABSTRACT

In his 2008 book "From Hahn–Banach to Monotonicity", S. Simons mentions that the proof of Lemma 41.3, which was presented in the previous edition of his book "Minimax and Monotonicity" in 1998, is incorrect, and one does not know if this lemma and its consequences are true. The aim of this short note is not only to give a proof to the mentioned lemma but also to improve upon it by relaxing one of its assumptions.

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1. Introduction

The well-known problem concerning the maximality of a sum of two maximal monotone operators defined in a non-reflexive Banach space and under the classical Rockafellar constraint is currently open. Under additional assumptions positive answers have been given to this problem, the main argument being recently the use of convex analysis and convex representations associated to a maximal monotone operator.

In chronological order some of the main results concerning the maximal monotonicity of the sum of two multi-valued maximal monotone operators under non-reflexive settings are: the 1976 result of Rockafellar for subdifferentials (see e.g. [18, Theorem 2.8.7, p. 126]), the 1996 result of Taa [9, Theorem 3.1] involving monotone non-maximal operators, the 1996 result of Heisler for full space domain operators (see e.g. [7, Theorem 40.4, p. 156]), the 1998 result of Phelps and Simons [5, Theorem 7.2] for single-valued linear operators, the 2000 result of Verona and Verona [11, Theorem 2.8] about regular operators, and the 2005 result of Groh [3, Theorem 1.6] involving the method of monotone forms. In 2006 a new and more usable characterization of maximal monotonicity in a general locally convex space context has been provided (see [13, Theorem 2.3] or [17, Theorem 1(ii)]) doubled by a new method to prove the maximal monotonicity of an operator based on the so-called negative-infimum type operators (see the next section for more details).

A good source of information on monotone operators is the 2008 book [8] with the remark that some of the results concerning the sum theorem presented in this monograph have no proper citation or are limited in comparison with the results found in the literature prior to the publication of [8]. For example [8, Theorem 24.1, p. 105] first appeared in 2006 in [13,12] and is a particular case of the results in [15,12], [8, Theorem 46.3, p. 182] was first published in 2006 in [14, Theorem 1] or [12, Theorem 5.13(β)], and [8, Theorem 51.1, p. 197] is part of [12, Theorem 5.10] or [16, Theorem 5.10].

In 2008 Simons in his book [7] claims the following.

Theorem 1. *(See Simons [7, Lemma 41.3, p. 158].) Let C be a closed convex subset of a Banach space X, let $A : X \rightrightarrows X^*$ be a multi-valued operator of type (FPV), and* $\operatorname{cen} D(A) \cap \operatorname{int} C \neq \emptyset$. *Then $A + N_C$ is maximal monotone.*

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However, on p. 199 of [8] the author states that "the proof of [99, Lemma 41.3] is incorrect, and we do not know whether it, [99, Theorem 41.5] and [99, Theorem 41.6] are true". Ref. [99] in the quoted text is our reference [7].

A quick inspection of the proof of [7, Lemma 41.3, p. 158] reveals that it fails due to an incorrect use of the (FPV) condition (see e.g. [7, Definition 25.4, p. 99] or [8, Definition 36.7, p. 150]).

The goal of the present note is to prove a stronger version of the mentioned lemma obtained by relaxing one of its assumptions. Based on our general method developed in [13] (see also [12, Remark 3.5]) we are able to provide a simple and short proof for an improved version of Theorem 1. We mention that our proof is not modeled on the general ideas of the proof in [7, Lemma 41.3, p. 158]; in fact our proof is completely different.

The plan of the paper is as follows. In the next section the main notions and notation are presented together with the new class of operators we have in view. Also, accompanied by some comments, our method used in general to prove the maximality of an operator is reviewed here. Section 3 contains our main result with its proof followed by a comparison with a recent result on the same topic.

2. Preliminaries

Recall that for a Banach space X with topological dual X^* a multi-valued operator $A : D(A) \subset X \rightrightarrows X^*$ is called *monotone* if $\langle x_1 - x_2, x_1^* - x_2^* \rangle \geqslant 0$, for every $x_1, x_2 \in D(A)$, $x_1^* \in Ax_1$, $x_2^* \in Ax_2$. Here $c(x, x^*) = \langle x, x^* \rangle = x^*(x)$, $(x, x^*) \in X \times X^*$, stands for the coupling function of $X \times X^*$.

For the sake of notational simplicity we identify operators with their graphs and write $(x, x^*) \in A \Leftrightarrow x \in D(A)$, $x^* \in Ax$. A monotone operator is *maximal monotone* if it is maximal in the sense of inclusion in $X \times X^*$.

In the sequel the *Fitzpatrick function* of a set $S \subset X \times X^*$ is denoted by $\varphi_S(x, x^*) = \sup\{\langle x - s, s^* \rangle + \langle s, x^* \rangle \mid (s, s^*) \in S\}$, $(x, x^*) \in X \times X^*$. With these notations the fact that $z = (x, x^*)$ is monotonically related (m.r. for short) to S comes to $z \in [\varphi_S \leqslant c] := \{w \in X \times X^* \mid \varphi_S(w) \leqslant c(w)\}$.

An operator $A : X \rightrightarrows X^*$ is called of type *negative infimum* (NI for short) in the sense of Voisei if $\varphi_A(z) \geqslant c(z)$, for every $z \in X \times X^*$ (see e.g. [12, Remark 3.5]). Note that this notion differs from the notion of negative infimum introduced in $X^* \times X^{**}$ by Simons (see [7, Definition 25.5, p. 99] or [8, Definition 36.2, p. 148]). Parenthetically said, the NI-operators in the sense of Simons should be called of dense-type (D-type for short) in the sense of Gossez (see [2]) since recently it has been proved that these two classes coincide (see [4]) and the dense-type property is stronger and has been introduced prior to the NI class in the sense of Simons.

Recall the following general characterization of maximal monotone operators:

Theorem 2. *(See Voisei [13, Theorem 2.3], [12, Remark 3.5].) An operator is maximal monotone if and only if it is representable and of NI type.*

While the representability of a sum of representable operators is easily provided by a Rockafellar type constraint (see [12, Section 5]) the weight of proving the maximal monotonicity for the sum shifts to the NI-type part of the argument.

In the sequel, a very useful method (first developed in [13, Theorem 1.1]) to prove that an operator $A : X \rightrightarrows X^*$ is of NI type is

$$\big(z = \big(x, x^*\big) \text{ is m.r. to } A \Rightarrow x \in D(A)\big) \quad \Rightarrow \quad A \text{ is NI,} \tag{1}$$

or equivalently

$$\big(\Pr_X[\varphi_A \leqslant c] \subset D(A)\big) \quad \Rightarrow \quad A \text{ is NI.}$$

Here $\Pr_X$ is the projection of $X \times X^*$ onto X.

Indeed, $(z = (x, x^*)$ is m.r. to $A \Rightarrow x \in D(A))$ is equivalent to $[\varphi_A \leqslant c] \subset D(A) \times X^*$ which implies, according to [12, Proposition 2.1(iv)], that A is NI and $\Pr_X[\varphi_A = c] \subset D(A)$.

If, in addition, A is representable then

$$A \text{ is maximal monotone} \quad \Leftrightarrow \quad \big(z = \big(x, x^*\big) \text{ is m.r. to } A \Rightarrow x \in D(A)\big). \tag{2}$$

In the case of a Minkowski sum $S := A + B$ of two maximal monotone operators $A, B : X \rightrightarrows X^*$ for which the weakest type of constraint holds (and this assures, according to [12, Section 5], that S is representable) our method translates into:

- S is maximal monotone if and only if $z = (x, x^*)$ is m.r. to $A + B$ implies that $x \in D(A) \cap D(B)$.

Recall that $A : X \rightrightarrows X^*$ is of type (*FPV*) if for every open convex set $V \subset X$ with $V \cap D(A) \neq \emptyset$ if $z = (x, x^*)$ is m.r. to $A|_V$ and $x \in V$ then $z \in A$ or equivalently if $z = (x, x^*) \notin A$ and $x \in V$ then there is $(a, a^*) \in A|_V$ such that $\langle x - a, x^* - a^* \rangle < 0$ (see e.g. [10, p. 268], [8, Definition 36.7, p. 150]). Here for $V \subset X$ the operator $A|_V : X \rightrightarrows X^*$ is defined by $(x, x^*) \in A|_V$ if $x \in V$ and $(x, x^*) \in A$. In other words a monotone operator A is of type (FPV) if, for every open convex set $V \subset X$ with $V \cap D(A) \neq \emptyset$, $A|_V$ is maximal monotone in $V \times X^*$.

It is easily seen that a monotone (FPV) operator is maximal monotone (take $V = X$ in the previous definition). In reflexive spaces the sum theorem holds (see e.g. [6]). As a consequence, under reflexivity settings, every maximal monotone operator is of type (FPV) (see e.g. [8, Theorem 44.1, p. 170]). The situation is not completely known in non-reflexive Banach spaces.

Let us introduce a new class of operators: $A : X \rightrightarrows X^*$ is called of type *weak-(FPV)* if for every open convex set $V \subset X$ with $V \cap D(A) \neq \emptyset$ if $z = (x, x^*)$ is m.r. to $A|_V$ and $x \in V$ then $x \in D(A)$ or equivalently for every $z = (x, x^*) \in [V \setminus D(A)] \times X^*$ there is $(a, a^*) \in A|_V$ such that $\langle x - a, x^* - a^*\rangle < 0$. In other words a monotone operator A is of type weak-(FPV) if for every open convex set $V \subset X$ with $V \cap D(A) \neq \emptyset$, $A|_V$ cannot be extended outside $D(A) \cap V$, as a monotone operator in $V \times X^*$.

It is clear that every (FPV) operator is weak-(FPV). The converse is not true even if we consider monotone operators. For example $\{0\} \times (\mathbb{R} \setminus \{0\})$ is monotone weak-(FPV) (by a straightforward verification) but is not (FPV) since it is not maximal. This proves that among monotone operators the weak-(FPV) notion is weaker than the (FPV) notion. The situation is not known for maximal monotone operators with possibly non-empty domain center (see the next definition).

The *center* of a set $D \subset X$, denoted by $\operatorname{cen} D$, is defined by $x \in \operatorname{cen} D$ if the segment $[x, y] := \{tx + (1-t)y \mid 0 \leqslant t \leqslant 1\} \subset D$, for every $y \in D$.

3. The main result

Theorem 3. *Let X be a Banach space, let C be a closed convex subset of X, and let $A : X \rightrightarrows X^*$ be maximal monotone and of type weak-(FPV) with $\operatorname{cen} D(A) \cap \operatorname{int} C \neq \emptyset$. Then $A + N_C$ is maximal monotone.*

Proof. Without loss of generality we may assume that $0 \in \operatorname{cen} D(A) \cap \operatorname{int} C$, $0 \in A0$, and for some $r > 0$, $rU \subset C$, where U denotes the unit open ball in X. Since $A + N_C$ is representable (see [12, Corollary 5.6]) it remains to prove that $A + N_C$ is NI (see Theorem 2). Assume by contradiction that $A + N_C$ is not NI, that is, there is $z = (x, y^*) \in [\varphi_{A+N_C} < c] := \{w \in X \times X^* \mid \varphi_{A+N_C}(w) < c(w)\}$. Since for every $y \in C$, $N_C(y)$ is a cone, note that

$$\bar z = (\bar x, \bar x^*) \text{ is m.r. to } A + N_C \quad \Leftrightarrow \quad \bar z \text{ is m.r. to } A|_C \quad \text{and} \quad \langle \bar x - a, x^*\rangle \leqslant 0, \quad \forall a \in D(A) \cap C,\ \forall x^* \in N_C(a). \tag{3}$$

Therefore z is m.r. to $A|_C$ and

$$\langle x - a, x^*\rangle \leqslant 0, \quad \forall a \in D(A) \cap C,\ \forall x^* \in N_C(a). \tag{4}$$

Assume that $x \in D(A)$. Recall that, according to [12, Proposition 2.1(iv)], $(D(A) \cap C) \times X^* \subset [\varphi_{A+N_C} \geqslant c]$. Since $z \in [\varphi_{A+N_C} < c]$ this yields that $x \notin C$. Therefore, there is $\mu \in (0, 1)$ such that $\mu x \in D(A) \cap \operatorname{bd} C$ (recall that $0 \in \operatorname{cen} D(A) \cap \operatorname{int} C$). Take $u^* \in N_C(\mu x)$ such that $\langle \mu x - y, u^*\rangle > 0$, for every $y \in \operatorname{int} C$; whence $\langle x, u^*\rangle > 0$, because $0 \in \operatorname{int} C$ and $\mu > 0$. From (4) applied for $a = \mu x$ and since $\mu < 1$ one gets the contradiction $\langle x, u^*\rangle \leqslant 0$.

Therefore $x \notin D(A)$. For $n \geqslant 1$, let $V_n := [0, x] + \frac{1}{n}U$. Notice that V_n is open convex, $V_n \cap D(A) \neq \emptyset$, and $x \in V_n$, for every $n \geqslant 1$. Since A is weak-(FPV), for every $n \geqslant 1$, there is $z_n = (a_n, a_n^*) \in A$ such that $a_n \in V_n$ and $c(z - z_n) < 0$. This implies that $a_n \in D(A) \setminus C$, because z is m.r. to $A|_C$. Hence there is $t_n \in (0, 1)$ such that $x_n = t_n a_n \in \operatorname{bd} C \cap D(A)$, since $0 \in \operatorname{cen} D(A) \cap \operatorname{int} C$. Let $x_n^* \in N_C(x_n)$, $\|x_n^*\| = 1$, $n \geqslant 1$. Because $x_n \in V_n$ there is $\lambda_n \in [0, 1]$ such that $\|x_n - \lambda_n x\| \leqslant \frac{1}{n}$, for every $n \geqslant 1$. On a subnet, denoted by the same index for simplicity, we may assume that $\lambda_n \to \lambda \in [0, 1]$, $x_n \to \lambda x \in \operatorname{bd} C$, strongly in X, $x_n^* \to x^* \in N_C(\lambda x)$, weakly-star in X^* as $n \to \infty$. Note that $\lambda > 0$ because $\lambda x \in \operatorname{bd} C$ and $0 \in \operatorname{int} C$.

By the monotonicity of N_C for $0 \in N_C(ru)$, $\|u\| < 1$, we get $\langle x_n - ru, x_n^*\rangle \geqslant 0$ or $\langle x_n, x_n^*\rangle \geqslant r$, for every $n \geqslant 1$. Let $n \to \infty$ to find $\langle x, x^*\rangle \geqslant r/\lambda > 0$. From (4) applied for $a = x_n$ and $x_n^* \in N_C(x_n)$ we have $\langle x - x_n, x_n^*\rangle \leqslant 0$, and after we pass to limit, we get $(1 - \lambda)\langle x, x^*\rangle \leqslant 0$, whence $\lambda = 1$, and so $x \in \operatorname{bd} C$.

Consider $f(t) = (\varphi_{A+N_C} - c)(tz)$, $t \in \mathbb{R}$; f is continuous on its domain (an interval) with $f(0) = 0$ and $f(1) < 0$. Therefore there is $0 < t < 1$ such that $f(t) < 0$. This implies that tz is m.r. to $A + N_C$ (in particular, according to (3), tz is m.r. to $A|_C$) with $tx \in \operatorname{int} C$, so $tx \in D(A)$, since A is weak-(FPV). From $tx \in D(A) \cap C$ we get the contradiction $f(t) \geqslant 0$.

This contradiction occurred due to the consideration of the assumption that $A + N_C$ is not NI. Hence $A + N_C$ is NI and consequently maximal monotone. □

Therefore [7, Lemma 41.3, p. 158] and its consequence [7, Theorem 41.5, p. 159] are true. We mention that the multi-valued version of [7, Theorem 41.5, p. 159] which has been first proved in [1, Theorem 3.1] is also a particular case of our Theorem 3 since every linear multi-valued maximal monotone operator is of type (FPV) (see e.g. [8, Theorem 46.1(b), p. 180]) and its domain coincide with its domain center.

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