Spreads and the symmetric topos

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Abstract

We introduce a new intrinsic notion of spread for toposes and geometric morphisms, and use it to give a "topological" characterization of Lawvere distributions on a topos. In the process, we relate spreads to zero-dimensional locales, and establish two new pure/spread factorizations for geometric morphisms. Our results are then applied to the study of the symmetric topos as a generalized lower power locale. In particular, we show that the symmetric topos is part of a Kock–Zöberlein 2-monad on toposes, give a new construction of bicomma squares in which the "lower" leg is an essential geometric morphism, characterize local connectedness in terms of the symmetric topos, and relate the symmetric and bagdomain constructions via the classifier of probability distributions.


0. Introduction

The notion of a distribution on a topos was introduced by Lawvere [20, 21] as a generalization of the classical notion for topological spaces. Recall that for toposes \( \mathcal{E} \) and \( \mathcal{G} \) over a base topos \( \mathcal{Y} \), a \( \mathcal{G} \)-valued distribution on \( \mathcal{E} \) is an \( \mathcal{Y} \)-cocontinuous functor \( \mathcal{E} \to \mathcal{G} \), i.e., a functor over \( \mathcal{Y} \) which preserves all \( \mathcal{Y} \)-colimits.

Distributions on an \( \mathcal{Y} \)-topos \( \mathcal{E} \) may be viewed as \( \mathcal{Y} \)-valued cosheaves on a site of definition for \( \mathcal{E} \), and as such, Bunge [4] showed that they are classified by a topos \( \Sigma \mathcal{E} \). Subsequently, an "algebraic", rather than "logical", construction of \( \Sigma \mathcal{E} \) was given by Bunge and Carboni [5], by analogy with the symmetric algebra construction.

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Just as a sheaf on a topological space, or locale, admits a topological description as an étale space, a cosheaf on a locale was shown by Funk [10] to correspond to a complete spread with locally connected domain, this notion a generalization to locales of the original one introduced in 1957 by Fox [9]. Although the notion of a complete spread over a locale [10] is "enough" to describe cosheaves on a site, provided one incorporates the action of a localic groupoid [4], an intrinsic description is possible and useful, as shown here.

In Section 1 we formulate the notion of spread for geometric morphisms over a base topos in terms of definable morphisms in the sense of Barr and Paré [2]. We show that (geometric) spreads are, as in topology, fiberwise zero-dimensional. Here, zero-dimensionality is defined using the relatively complemented elements of a frame, a notion due to Jibladze and studied by Kock and Reyes [19].

In Section 2 we introduce fiberwise versions of density and purity for geometric morphisms over a base topos. We establish the existence and uniqueness of a pure surjection/spread factorization in Top$. It generalizes the factorization for continuous maps of Collins and Dyckhoff [8], and for the case of a Boolean topos $\mathcal{F}$, it has been considered already by Johnstone [13]. Related to this factorization is the pure dense/complete spread factorization of a geometric morphism (but now uniqueness is subject to a proviso on the middle topos). The factorization theorems are obtained as corollaries of the characterization of distributions on a topos $\mathcal{E}$ as complete spreads over $\mathcal{E}$ with locally connected domain. This extends the localic version [10], and is used in Section 3.

There is an analogy between the symmetric topos and the lower power locale [6]. Both have similar "algebraic" constructions as coinverters [5]. Also, the pure dense/complete spread factorization of Section 2 parallels the strongly dense/weakly closed factorization of Johnstone [14] for sublocales, and whereas the lower power locale $P_L(X)$ classifies weakly closed sublocales of $X$ with open domain [6], $\Sigma\mathcal{E}$ classifies complete spreads with locally connected domain. Further reasons for viewing the symmetric topos as a generalized lower power locale are given in Section 3. We begin Section 3 by exhibiting $\Sigma$ as part of a Kock–Zöberlein 2-monad [17, 28] on Top$, obtained by mating from the corresponding op-KZ monad on $A$, the 2-category of locally presentable categories, as discussed in [5]. We denote the symmetric construction on Top$ by $M$ (for generalized measures) and the unit of the corresponding KZ-monad on Top$ by $\delta$ (for the Dirac delta).

The characterization of distributions established in Section 2 can be used to construct the bicomma square of a diagram of toposes in which the "lower" leg is an essential geometric morphism, first done by Pitts [26]. This construction is linked to an interpretation of the equivalence between points of the symmetric topos and complete spreads with locally connected domain as being given by the bicomma square in which the lower leg is the unit of the symmetric monad. The localic analogue of this result is Vermeulen's proof [30] that the characterization of the points of the lower power locale [6] can be similarly obtained.
We show that a topos $\mathcal{E}$ over $\mathcal{S}$ is locally connected if and only if the representable fibration on $\text{Top}_\mathcal{S}$ associated with $M\mathcal{E}$ has a terminal object (equivalently, $M\mathcal{E}$ is totally connected [1, 18]). This is analogous to the characterization of openness for locales in terms of the lower power locale given by Vickers [32].

The bagdomain topos $\mathcal{B}_L\mathcal{E}$, defined by Vickers [31] and elaborated upon by Johnstone [16], can also be regarded as a generalized lower power locale, for reasons other than the ones given above for the symmetric topos. Therefore, it seems natural to seek a connection between $M$ and $\mathcal{B}_L$. Beyond those remarks already made in [6] to that effect, we prove here that

$$M\mathcal{E} \cong \mathcal{B}_L(C(\mathcal{E})),$$

where $C\mathcal{E}$ is the topos classifier of probability distributions on $\mathcal{E}$, and which we construct. A distribution $\mathcal{E} \rightarrow \mathcal{B}$ is said to be a probability distribution if it preserves the terminal object. In terms of complete spreads, probability distributions correspond to those with connected locally connected domain. In order to establish (1), we show that the fibration of complete spreads is the free coproduct completion of the fibration of complete spreads with connected domain.

The topos $\mathcal{B}_L\mathcal{E}$ classifies a certain type of distributions on $\mathcal{E}$, namely those which preserve pullbacks, i.e., those which are partial points of $\mathcal{E}$. But also, as shown in [16, 31], $\mathcal{B}_L\mathcal{E}$ classifies bags of points of $\mathcal{E}$, and these correspond to what we call complete spreads whose domains have totally connected components. We end Section 3 by posing a question concerning a related construction which involves the finite connected limit completion of a small category.

1. Spreads and zero-dimensional locales

The notion of a spread is due to Fox [9]. In topology, a spread is a map $Y \rightarrow X$, with $Y$ locally connected, having the property that the components of open sets $f^{-1}U$, for $U$ open, form a base of $Y$. We begin this section by formulating this idea for geometric morphisms in terms of the notion of a definable morphism [2]. Then we show that (geometric) spreads are, as in topology, fiberwise zero-dimensional, where we define zero-dimensionality in terms of relatively complemented opens [19].

Recall that a geometric morphism $\mathcal{F} \rightarrow \mathcal{F}'$ is said to be bounded [11, 24] if there is a $K \in \mathcal{F}'$ and a morphism $D \rightarrow \gamma^*K$ in $\mathcal{F}$, said to be a generating family (for $\mathcal{F}$ over $\mathcal{F}'$), such that for every $X \in \mathcal{F}$, there is a morphism $I \rightarrow K$ and an epimorphism $A \rightarrow X$, where

$$\begin{array}{ccc}
A & \rightarrow & D \\
\downarrow & & \downarrow \\
\gamma^*I & \rightarrow & \gamma^*K
\end{array}$$

\[\gamma_a\]
is a pullback. The geometric morphism $\gamma$ is said to be localic if the family $D \to \gamma^*K$ can be taken to be the left side of the pullback

$$
\begin{array}{c}
D \\
\downarrow \\
\gamma^*\Omega_I \\
\downarrow \\
\Omega_J
\end{array} 
$$

Let $\textbf{Top}$ denote the 2-category of elementary toposes, bounded geometric morphisms and natural transformations (between inverse image functors of geometric morphisms). If $\mathcal{S}$ is an elementary topos, let $\textbf{Top}_\mathcal{S}$ denote the corresponding 2-category of (bounded) toposes and geometric morphisms over $\mathcal{S}$.

Let $\mathcal{E} \xrightarrow{e} \mathcal{S}$ denote an arbitrary geometric morphism in $\textbf{Top}$. A morphism $E \xrightarrow{b} E'$ in $\mathcal{E}$ is called $\mathcal{S}$-definable [2], or just definable, if it arises as a pullback

$$
\begin{array}{c}
E \\
\downarrow \\
e^*I \\
\downarrow \\
e^*J
\end{array} \xrightarrow{e^*a} 
\begin{array}{c}
E' \\
\downarrow \\
e^*J
\end{array}
$$

for some morphism $I \xrightarrow{a} J$ in $\mathcal{S}$. The pullback of a definable morphism is again definable, but the composite of two definable morphisms need not be so.

**Definition 1.1.** In a diagram

$$
\begin{array}{c}
\mathcal{E} \\
\downarrow \phi \\
\mathcal{S}
\end{array} 
\xleftarrow{f} 
\begin{array}{c}
\mathcal{F} \\
\downarrow \\
\mathcal{S}
\end{array}
$$

in $\textbf{Top}$, $\phi$ is said to be a spread over $\mathcal{S}$ if there is a generating family $E \to \phi^*F$ for $\mathcal{E}$ over $\mathcal{F}$ which is $\mathcal{S}$-definable.

We emphasize that in Definition 1.1, since the morphism $E \to \phi^*F$ is required to be definable, the notion of a spread is given relative to a base topos. Nevertheless, we omit reference to the base topos when this is clear from the context. Spreads are, by definition, bounded. Proposition 1.3 below shows that in fact they are localic.

The following facts are immediate consequences of the definitions.

**Proposition 1.2.** 1. For any topos $\mathcal{E}$ over $\mathcal{S}$, the identity $\mathcal{E} \xrightarrow{1} \mathcal{E}$ is a spread over $\mathcal{S}$.

2. If $\mathcal{E} \xrightarrow{\phi} \mathcal{F}$ is any spread over $\mathcal{S}$, and $\mathcal{F} \xrightarrow{i} \mathcal{E}$ is any subtopos, then the composite $\phi \cdot i$ is a spread. In particular, any inclusion is a spread.
3. If a composite \( \delta \rightarrow \gamma \rightarrow \mathcal{F} \) is a spread, then \( \psi \) is a spread.

4. If \( \mathcal{X} \xrightarrow{\varphi} \mathcal{Y} \xrightarrow{\psi} \mathcal{F} \) are spreads, and if definable morphisms compose in \( \mathcal{X} \), then the composite \( \psi \cdot \varphi \) is a spread.

Consider the morphism, labeled \( \tau \), defined by the pullback

\[
\begin{array}{ccc}
1 & \xrightarrow{1} & 1 \\
\downarrow{\tau} & & \downarrow{T} \\
e^*\Omega_x & \xrightarrow{\tau} & \Omega_h.
\end{array}
\]

If the characteristic map of a monomorphism \( S \hookrightarrow E \) factors through \( \tau \), then \( S \hookrightarrow E \) is a definable morphism. Conversely, if \( S \hookrightarrow E \) is definable, then it is definable by a monomorphism \( I \hookrightarrow J \) in \( \mathcal{S} \), and hence, the characteristic map of \( S \hookrightarrow E \) must factor through \( \tau \) (though this factorization may not be unique). A definable monomorphism will also be referred to as a definable subobject.

**Proposition 1.3.** For \( \delta \xrightarrow{\varphi} \mathcal{F} \) over \( \mathcal{S} \), the following are equivalent:

1. \( \varphi \) is a spread.
2. For every \( X \in \mathcal{S} \), there is \( F' \in \mathcal{F} \) and a diagram

\[
\begin{array}{ccc}
C & \xrightarrow{c} & X \\
\downarrow{\varphi} & & \downarrow{\varphi}\cdot F'
\end{array}
\]

where \( c \) is definable.

3. For every \( X \in \mathcal{S} \), there is \( F' \in \mathcal{F} \) and a diagram

\[
\begin{array}{ccc}
C & \xrightarrow{c} & X \\
\downarrow{\varphi} & & \downarrow{\varphi}F'
\end{array}
\]

where \( c \) is a definable subobject. In particular, spreads are localic.

4. The left side of the pullback

\[
\begin{array}{ccc}
S & \xrightarrow{e^*1} & e^*\mathcal{X} \\
\downarrow{\varphi} & \downarrow{e^*\tau_x} & \downarrow{e^*\Omega_x} \\
\varphi^*\varphi, e^*\Omega_x & \xrightarrow{e^*\Omega_x} & e^*\Omega_x
\end{array}
\]

generates \( \delta \) over \( \mathcal{S} \).

**Proof.** 1 \( \Rightarrow \) 2: This follows immediately from the definitions and the fact that definable morphisms are pullback stable.
2 \Rightarrow 3: Suppose that the pullback

\[
\begin{array}{ccc}
C & \rightarrow & \phi^*F' \\
\downarrow a & & \downarrow b \\
e^*I & \rightarrow & e^*J
\end{array}
\]

witnesses that \( c \) is definable. Then

\[
\begin{array}{ccc}
C & \xrightarrow{(a,c)} & e^*I \times \phi^*F' \\
\downarrow a & & \downarrow 1 \times b \\
e^*I & \xrightarrow{(1,e^*x)} & e^*I \times e^*J
\end{array}
\]

is a pullback, and therefore the monomorphism \((a,c)\) is definable.

3 \Rightarrow 4: For any \( X \in \mathcal{E}, \) there is a pullback

\[
\begin{array}{ccc}
C & \rightarrow & e^*1 \\
\downarrow & & \downarrow e^*\top \\
\phi^*F' & \rightarrow & e^*\Omega_x
\end{array}
\]

and an epimorphism \( C \rightarrow X, \) and this square factors as

\[
\begin{array}{ccc}
C & \rightarrow & S & \rightarrow & e^*1 \\
\downarrow & & \downarrow & & \downarrow e^*\top \\
\phi^*F' & \phi^* \rightarrow & \phi_* e^*\Omega_x & \rightarrow & e^*\Omega_x
\end{array}
\]

which gives the desired conclusion.

4 \Rightarrow 1: \( S \rightarrow \phi^* \phi_* e^*\Omega_\mathcal{E} \) is definable. \( \square \)

If \( M \) is a complete join-semilattice in a topos \( \mathcal{F}, \) then a morphism \( K \rightarrow M \) sup-generates \( M \) when \( \Omega^K \rightarrow \Omega^M \rightarrow M \) is an epimorphism. This condition is equivalent to being a generating family in the sense of indexed categories [24]. If \( M \) is a frame \( \mathcal{E}(X) \) and \( \mathcal{S}h(X) \xrightarrow{\phi^*} \mathcal{F} \) is its topos of sheaves, then \( K \rightarrow \mathcal{E}(X) = \phi_* \Omega_X \) sup-generates iff the subobject \( F \hookrightarrow \phi^*K \) corresponding to the transpose \( \phi^* K \rightarrow \Omega_X \) is a generating family for \( \mathcal{S}h(X) \) over \( \mathcal{F}. \)

**Proposition 1.4.** A geometric morphism \( \mathcal{E} \xrightarrow{\phi^*} \mathcal{F} \) over \( \mathcal{F} \) is a spread iff \( \phi \) is localic and the morphism

\[
\phi_* \tau : \phi_* e^*\Omega_\mathcal{E} \rightarrow \phi_* \Omega_\mathcal{E}
\]

sup-generates the frame \( \phi_* \Omega_\mathcal{E}. \)
Proof. See Proposition 1.3(4), and the preceding remarks. □

If in the domain topos the composite of two definable subobjects is again definable, then spreads can also be described as follows. For example, this condition holds when the topos is locally connected, or when the base topos is Boolean (see [2]).

Proposition 1.5. For \( \mathcal{E} \xrightarrow{\varphi} \mathcal{F} \) over \( \mathcal{S} \), assume that the composite of two definable subobjects in \( \mathcal{E} \) is again definable. Let \( D \xrightarrow{f} K \) be an arbitrary generating family for \( \mathcal{F} \) over \( \mathcal{S} \). Then \( \varphi \) is a spread if for every \( X \in \mathcal{E} \), there is \( I \in \mathcal{S} \) and a diagram

\[
\begin{array}{ccc}
C & \longrightarrow & X \\
\downarrow c & & \downarrow e^*I \times \varphi^*D, \\
\end{array}
\]

where \( c \) is a definable subobject.

Proof. Assume that \( \varphi \) is a spread. Then for a given \( X \in \mathcal{E} \), there is an object \( F' \in \mathcal{F} \), and a diagram

\[
\begin{array}{ccc}
C & \longrightarrow & X \\
\downarrow s & & \downarrow \varphi^*F', \\
\end{array}
\]

where \( s \) is a definable subobject. For this \( F' \), there is in \( \mathcal{F} \) a diagram

\[
\begin{array}{ccc}
f^*I & \xleftarrow{d} & A & \xrightarrow{b} & F' \\
\downarrow f^*\chi & & \downarrow a & & \downarrow \varphi^*D, \\
\end{array}
\]

where the square is a pullback. Apply \( \varphi^* \) to this diagram, and let \( c \) be the composite of the pullback of \( s \) along \( \varphi^*b \), which is definable, and \((\varphi^*d, \varphi^*a)\), also definable (see the proof of Proposition 1.3, 2 \( \Rightarrow \) 3).

The converse is immediate since \( e^*I \times \varphi^*D \simeq \varphi^*(f^*I \times D) \). □

Proposition 1.5 says that when definable subobjects compose in the domain topos, a spread can be (informally) described as a geometric morphism \( \varphi \) with the property that for any generating family \( D \xrightarrow{f} K \) for \( \mathcal{F} \), the collection

\[
\{ A \mid \text{there is a } k \in K \text{ and a definable monomorphism } A \hookrightarrow \varphi^*F_k, F_k \in D \}
\]
generates \( \mathcal{E} \) over \( \mathcal{S} \). Furthermore, we can describe in these terms what is meant by a complete spread. To do this, let \((\mathcal{D}, J_f)\) denote a site for \( \mathcal{F} \) over \( \mathcal{S} \), and let \( X \) denote
a locale in \( \mathcal{F} \). Then \( \mathcal{E} = Sh_{\mathcal{F}}(X) \xrightarrow{\varphi} \mathcal{F} \) has a site description over \( \mathcal{F} \) as a continuous fibration \([23]\)

\[
P : D \ni \mathcal{O}(X) \to D; (F,A) \mapsto F.
\]

The objects of \( D \ni \mathcal{O}(X) \) are pairs \((F,A)\), where \( F \in D \) and \( A \hookrightarrow \varphi^*F \) is a subobject. The morphisms \((F,A) \to (G,B)\) are pairs \((x,a)\) such that

\[
\begin{array}{ccc}
A & \xrightarrow{a} & B \\
\downarrow & & \downarrow \\
\varphi^*F & \xrightarrow{\varphi^*x} & \varphi^*G
\end{array}
\]

commutes in \( \mathcal{E} \). Covers are collections \(\{(F_i,A_i) \to (F,A)\}\) such that \(\{A_i \to A\}\) is an epimorphic family in \( \mathcal{E} \). Let \( C \) denote the full subcategory of \( D \ni \mathcal{O}(X) \) determined by those \((F,A)\), such that \( A \hookrightarrow \varphi^*F \) is a definable subobject. This category becomes an \( \mathcal{F} \)-site \((C,J)\) by declaring that a family in \( C \) is a covering iff it is a covering family when regarded in \( D \ni \mathcal{O}(X) \). There is induced a continuous fibration

\[
P : C \to D,
\]

and in the case that definable subobjects compose in \( \mathcal{E} \), \( \varphi \) is a spread iff the topos of sheaves on this site gives \( \mathcal{E} \). Let \((C,J_\delta)\) denote the site with underlying category \( C \), but equipped with the Grothendieck topology generated by those collections \(\{(F_i,A_i) \to (F,A)\}\) such that \(\{F_i \to F\} \in J_f\), and such that for each \( i \) the diagram

\[
\begin{array}{ccc}
A_i & \xrightarrow{a} & A \\
\downarrow & & \downarrow \\
\varphi^*F_i & \xrightarrow{\varphi^*a} & \varphi^*F
\end{array}
\]

is a pullback. For such a collection, \(\{A_i \to A\}\) is an epimorphic family in \( \mathcal{E} \), so that the identity functor on \( C \) is the underlying functor of a site morphism \((C,J_\delta) \to (C,J_\epsilon)\). The term "complete" was used in \([9]\) for the topological version of the following, hence we adopt it as well.

**Definition 1.6.** In the above notation, \( \mathcal{E} \xrightarrow{\varphi} \mathcal{F} \) will be said to be a **complete for** \((D,J_f)\) if \((C,J_\delta) \to (C,J_\epsilon)\) is a site isomorphism, i.e., if \(Sh(C,J_\delta)\) is equivalent to \(Sh(C,J_\epsilon)\).

**Example 1.7.** Let \( Y \) denote the subspace \(\{(0,y) | -1 \leq y \leq 1\} \cup \{(x, \sin 1/x) | x > 0\}\) of the real plane. Let \( X \) denote the subspace \(\{(x, \sin 1/x) | x > 0\}\). It is clear that
$X \rightarrow Y$ is a spread, but note that it is complete, i.e., that the collection of "cogersms" above any point $y \in Y$ is in bijection with its fiber. A cogerm above $y \in Y$ is a consistent choice of components of inverse images of opens of $X$ which contain $y$ [10].

**Proposition 1.8.** Let $\mathcal{E} \xrightarrow{\varphi} \mathcal{F}$ be localic, and let $(D, J_f)$ be a site for $\mathcal{F}$ over $\mathcal{S}$ with finite limits. Then there is a factorization

$$
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{\varphi} & \mathcal{F} \\
\rho \downarrow & & \downarrow \psi \\
\text{Sh}(C, J_f) & \xrightarrow{i} & \text{Sh}(C, J_\mathcal{S}).
\end{array}
$$

**Proof.** If $D$ has finite limits, then so do $C$ and $D \bowtie \mathcal{O}(X)$, and there is a finite limit cover preserving functor i.e., there is a site morphism, $C \twoheadrightarrow D \bowtie \mathcal{O}(X)$.

*Note:* $C$ has binary products since the intersection of two definable subobjects is definable. The geometric morphism $\rho$ is induced by this site morphism, and $\psi \cdot i$ is induced by the continuous fibration (1).

In Section 2 we will investigate the above factorization in detail, and with different methods, when the topos $\mathcal{E}$ is locally connected. We will see that in this case the factorization does not depend on the choice of presentation for $\mathcal{F}$ over $\mathcal{S}$ (nor does completeness – see Definition 1.6).

In the remainder of this section we show that spreads are zero-dimensional, in a sense relative to the base topos. In order to define zero-dimensionality we recall the notion of a relatively complemented element of a frame [19]. Consider our setting of a diagram

$$
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{\varphi} & \mathcal{F} \\
\mathcal{E} & \xrightarrow{\varphi} & \mathcal{F} \\
\mathcal{S} & \xrightarrow{f} & \mathcal{S}
\end{array}
$$

in Top. We assume that $\varphi$ is localic, i.e., that $\mathcal{E}$ is equivalent to the topos of sheaves over $\mathcal{F}$ on the frame $\varphi_* \Omega_\mathcal{E}$. There is the composite morphism

$$
f^* \Omega_\mathcal{S} \xrightarrow{\varphi_*} \Omega_\mathcal{F} \xrightarrow{\varphi_*} \varphi_* \Omega_\mathcal{E},
$$

where the second morphism is the unique frame map. Referring to this morphism, we have the object of $\mathcal{S}$-complemented elements of $\varphi_* \Omega_\mathcal{E}$:

$$
Clp_{\mathcal{S}}(\varphi_* \Omega_\mathcal{E}) = \{ U \in \varphi_* \Omega_\mathcal{E} \mid \top = \bigvee \{ U \Rightarrow \omega \mid \omega \in f^* \Omega_\mathcal{S} \}\}.
$$
Our aim is to show that if \( \varphi \) is a spread, then \( \varphi \) is zero-dimensional in the following sense.

**Definition 1.9.** A localic geometric morphism \( \mathcal{E} \xrightarrow{\varphi} \mathcal{F} \) over \( \mathcal{S} \) will be said to be **zero-dimensional over** \( \mathcal{S} \) if the frame \( \varphi_* \Omega_\mathcal{E} \) is sup-generated by its sublattice \( Clp_\mathcal{S}(\varphi_* \Omega_\mathcal{E}) \).

We begin our analysis with a description of the objects \( \varphi_* \varphi^* F \) in terms of "flat functions".

**Definition 1.10.** For any locale \( X \) in a topos \( \mathcal{F} \), a morphism \( F \xrightarrow{\varphi} \mathcal{O}(X) \) in \( \mathcal{F} \) will be referred to as **flat**, or as a **flat function**, if its unique sup-lattice extension \( \alpha^* \):

\[
\begin{array}{c}
F \\
\downarrow \varphi \\
\mathcal{S} \\
\downarrow \alpha^* \\
\mathcal{O}(X)
\end{array}
\]

is a frame homomorphism (i.e., if \( \alpha^* \) preserves finite infima).

Flatness has the following characterization.

**Proposition 1.11.** A morphism \( F \xrightarrow{\varphi} \mathcal{O}(X) \) is flat iff

1. \( TX = \bigvee \{ \alpha(x) \mid x \in F \} \).
2. \( \forall x, y \in F, \alpha(x) \land \alpha(y) = \bigvee \{ V \mid V = \alpha(x) \land x = y \} \).

**Proof.** The sup-lattice extension of \( \alpha \) is given, for \( S \in \Omega_\mathcal{S}^F \), as

\[
\alpha^* S = \bigvee \{ \alpha(x) \mid x \in S \}.
\]

Clearly \( \alpha^* \) preserves \( \top \) iff \( \alpha \) satisfies 1. The preservation of binary infima by \( \alpha^* \) reduces to the condition

\[
\forall x, y \in F, \alpha^* \{ x \} \land \alpha^* \{ y \} = \alpha^*(\{ x \} \cap \{ y \}),
\]

which, in turn, reduces to

\[
\forall x, y \in F, \alpha(x) \land \alpha(y) = \bigvee \{ \alpha(z) \mid z \in \{ x \} \cap \{ y \} \}.
\]

This condition translates directly into condition 2. \( \square \)

**Proposition 1.12.** For any \( F \in \mathcal{F} \), \( \varphi_* \varphi^* F \) is canonically isomorphic to \( \{ \alpha \in (\varphi_* \Omega_\mathcal{E})^F \mid \alpha \text{ is flat} \} \).
Proof. For any $E \in \mathcal{F}$, we have natural bijections

| morphisms $E \rightarrow \phi \ast \varphi \ast F$ | morphisms $\phi \ast E \rightarrow \varphi \ast F$ |
| geo. morphisms $\mathcal{E} / \phi \ast E \rightarrow \mathcal{E} / \varphi \ast F$ over $\mathcal{E}$ | geo. morphisms $\mathcal{E} / \phi \ast E \rightarrow \mathcal{F} / \varphi \ast F$ over $\mathcal{F}$ |
| frame morphisms $\Omega^F \rightarrow \varphi \ast (\Omega \phi \ast \mathcal{E})^E = (\phi \ast \Omega) \mathcal{E}^E$ | flat morphisms $F \rightarrow (\varphi \ast \Omega) \mathcal{E}^E$ |
| morphisms $E \rightarrow \{ \alpha \in (\phi \ast \Omega) \mathcal{E}^E | \alpha \text{ is flat } \}$. |

We are specially interested in the case where $F$ is $f \ast \Omega_{\mathcal{X}}$ in Proposition 1.12. In this case the following additional information is available. For convenience, we abbreviate $\Omega_{\mathcal{X}}$ to $\Omega$.

Lemma 1.13. Any flat function $f \ast \Omega \xrightarrow{\pi} \mathcal{O}(X) \in \mathcal{F}$ satisfies

1. $\pi(\top) = \bigvee \{ \omega \land \pi(\omega) \mid \omega \in f \ast \Omega \}$, where $\omega \in f \ast \Omega$ is regarded in $\mathcal{O}(X)$ via the morphism $f \ast \Omega \xrightarrow{\pi} \Omega_{\mathcal{X}} \rightarrow \mathcal{O}(X)$, and

2. $\forall \omega \in f \ast \Omega, \pi(\omega) \leq (\pi(\top) \Leftrightarrow \omega)$. 

Proof. 1. In the frame $\Omega_{\mathcal{X}} f \ast \Omega$ we have

$$\{ \top \} = \bigcup \{ \omega \cap \{ \omega \} \mid \omega \in f \ast \Omega \},$$

where $\{ \cdot \} : f \ast \Omega \rightarrow \Omega_{\mathcal{X}} f \ast \Omega$. Indeed, referring to the pullback

$$
\begin{array}{c}
1 \\
\downarrow \pi \\
\top \\
\downarrow \\
f \ast \Omega \\
\downarrow \\
\Omega_{\mathcal{X}}
\end{array}
$$

for any $\lambda \in f \ast \Omega$, we have $\lambda \in \omega \cap \{ \omega \}$ for some $\omega \in f \ast \Omega$ iff $\pi(\omega) = \top$ and $\lambda = \omega$ for some $\omega$ iff $\omega = \top$ and $\lambda = \omega$ for some $\omega$ iff $\lambda \in \{ \top \}$. The conclusion now follows by applying the unique frame extension $\pi$ to (2), noting that the map

$$f \ast \Omega \xrightarrow{\pi} \Omega_{\mathcal{X}} \rightarrow \Omega_{\mathcal{X}} f \ast \Omega \xrightarrow{\pi} \mathcal{O}(X)$$

is equal to $f \ast \Omega \xrightarrow{\pi} \Omega_{\mathcal{X}} \rightarrow \mathcal{O}(X)$.

2. Fix $\omega \in f \ast \Omega$. By 1, we have $\pi(\omega) \leq \omega \Rightarrow \pi(\top)$. On the other hand, by Proposition 1.11,

$$\pi(\omega) \land \pi(\top) = \bigvee \{ V \mid V = \pi(\omega) \land \omega = \top \} \leq \bigvee \{ V \mid \omega = \top \} = \omega.$$

Therefore, $\pi(\omega) \leq \pi(\top) \Rightarrow \omega$. □
Proposition 1.14. Under the identification made in Proposition 1.12, the morphism \( \varphi_\tau : \varphi_\ast e^*\Omega \to \varphi_\ast \Omega_\mathcal{E} \) sends a flat function \( f^*\Omega \xrightarrow{\varsigma} \varphi_\ast \Omega_\mathcal{E} \) to \( \varsigma(\tau) \).

Proof. A flat function \( \varsigma \) determines an atlas \([19]\) for a global section of \( e^*\Omega \) regarded as a sheaf on the frame \( \varphi_\ast \Omega_\mathcal{E} \), namely the family

\[
\{(\varsigma(\omega), \omega) \mid \omega \in f^*\Omega\}.
\]

That this family is an atlas can be seen using Proposition 1.11. Then, following \([19]\), \( \varphi_\ast \tau \) associates with this atlas the element

\[
\bigvee \{\omega \land \varsigma(\omega) \mid \omega \in f^*\Omega\},
\]

which, by Lemma 1.13(1) is equal to \( \varsigma(\tau) \). \( \square \)

Proposition 1.15. For any localic geometric morphism \( \mathcal{E} \xrightarrow{\varphi} \mathcal{F} \) over \( \mathcal{S} \), the morphism \( \varphi_\ast \tau : \varphi_\ast e^*\Omega \to \varphi_\ast \Omega_\mathcal{E} \) factors through \( Clp_{\mathcal{F}}(\varphi_\ast \Omega_\mathcal{E}) \).

Proof. By Proposition 1.11(1) and Lemma 1.13(2), for any flat function \( f^*\Omega \xrightarrow{\varsigma} \varphi_\ast \Omega_\mathcal{E} \), we have

\[
\tau = \bigvee \{\varsigma(\omega) \mid \omega \in f^*\Omega\} \leq \bigvee \{\varsigma(\tau) \iff \omega \mid \omega \in f^*\Omega\}.
\]

The result now follows by Proposition 1.14. \( \square \)

Theorem 1.16. Any spread over \( \mathcal{S} \) is zero-dimensional over \( \mathcal{S} \).

Proof. Propositions 1.4 and 1.15. \( \square \)

In the terminology of \([15]\), a localic geometric morphism \( \mathcal{E} \xrightarrow{\varphi} \mathcal{F} \) is weakly zero-dimensional if \( \varphi_\ast \Omega_\mathcal{E} \) is generated by its weakly closed elements (with respect to the unique frame map \( \Omega_\mathcal{S} \to \varphi_\ast \Omega_\mathcal{E} \) – see \([14]\)). G.E. Reyes has observed that for any locale \( X \) and any morphism \( I \to m \) \( \mathcal{O}(X) \), an arbitrary \( m \)-complemented element of \( \mathcal{O}(X) \) is weakly closed for \( m \). Thus, zero-dimensional in the sense of Definition 1.9 implies weakly zero-dimensional so that, by Theorem 1.16, spreads are weakly zero-dimensional for \( f^*\Omega \to \varphi_\ast \Omega_\mathcal{E} \).

We have the following partial converse to Theorem 1.16. Recall that a geometric morphism \( \mathcal{F} \xrightarrow{f} \mathcal{S} \) is said to be subopen \([12]\) if \( f^*\Omega \xrightarrow{f} \Omega_\mathcal{S} \) is a monomorphism.

Proposition 1.17. Assume that \( \mathcal{F} \xrightarrow{f} \mathcal{S} \) is subopen, and that \( \mathcal{E} \xrightarrow{\varphi} \mathcal{F} \) is open and localic. If \( \varphi \) is zero-dimensional over \( \mathcal{S} \), then \( \varphi \) is a spread.

Proof. If \( \varphi \) is open, then the factorization of \( \varphi_\ast e^*\Omega \) through \( Clp_{\mathcal{F}}(\varphi_\ast \Omega_\mathcal{E}) \) is an epimorphism. In fact, if \( U \in Clp_{\mathcal{F}}(\varphi_\ast \Omega_\mathcal{E}) \), then the function \( \omega \mapsto (U \iff \omega) \) is flat, and its
value at \( T \) is \( U \). It is flat as condition 1 of Proposition 1.11 is immediately satisfied. For condition 2, observe that for any \( \omega, \omega' \in f^*\Omega \), we have

\[
(U \leftrightarrow \omega) \wedge (U \leftrightarrow \omega') = (U \leftrightarrow \omega) \wedge (\omega \leftrightarrow \omega'),
\]

which by the openness of \( \varphi \) is equal to

\[
(U \leftrightarrow \omega) \wedge \bigvee \{ W \in \mathcal{O}(X) \mid \tau \omega = \tau \omega' \} = \bigvee \{ W \wedge (U \leftrightarrow \omega) \mid \omega = \omega' \},
\]

where \( f^*\Omega \rightarrow \Omega \mathcal{O} \) is by assumption a monomorphism (subopenness of \( f \)). But clearly

\[
\bigvee \{ W \wedge (U \leftrightarrow \omega) \mid \omega = \omega' \} \leq \bigvee \{ V \mid (V = (U \leftrightarrow \omega)) \wedge (\omega = \omega') \}. \quad \square
\]

2. The pure/spread factorizations on \( \text{Top}_\mathcal{O} \)

In this section we will establish two factorizations, the pure surjection/spread and the pure dense/complete spread, for geometric morphisms in \( \text{Top}_\mathcal{O} \) with locally connected domain. In the process, we characterize the category of distributions on a topos [20] (see also the Introduction) in terms of complete spreads, a result we will use also in Section 3.

To prepare for the factorization theorems, we define the notions of pure and dense for geometric morphisms and establish some of their basic properties. We are specially interested in the case when the domain topos is locally connected over the base topos. The definitions that follow refer to the diagram

\[
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{\varphi} & \mathcal{F} \\
\downarrow{\mathcal{E}} & & \downarrow{f} \\
\mathcal{F} & & \mathcal{F}
\end{array}
\]

diagram of geometric morphisms.

Recall that a locale morphism \( X \xrightarrow{g} Y \) is said to be strongly dense [14] if for all \( \omega \rightarrow 1 \), the unit \( \omega \leq g^*g^*\omega \) is an equality. Suppose that the geometric morphism \( \varphi \) has the analogous property that for all \( \omega \rightarrow 1 \) in \( \mathcal{F} \), \( f^*\omega \rightarrow \varphi_*\varphi^*f^*\omega \) is an isomorphism, and consider, for \( I \in \mathcal{F} \), the unit \( f^*I \rightarrow \varphi_*\varphi^*f^*I \). In a topos, coproducts are disjoint, i.e., two elements \( i, j \in \bigsqcup I \) are distinct to the extent of their pullback over \( f^*I \).

From our assumption we deduce that \( \varphi_*\varphi^*I \) and \( \varphi_*\varphi^*j \) are distinct in \( \varphi_*\varphi^*f^*I \) to the extent that \( i \) are \( j \) are distinct. This says that \( f^*I \rightarrow \varphi_*\varphi^*f^*I \) is a monomorphism, and it motivates the following.

**Definition 2.1.** We will say that \( \varphi \) is dense over \( S \), or simply dense, if for every \( I \in \mathcal{F} \), the unit \( f^*I \rightarrow \varphi_*\varphi^*f^*I \) is a monomorphism.
If, for example, \( \phi \) is a surjection, then \( \phi \) is dense over \( \mathcal{S} \). Another class of geometric morphisms which are dense in this sense are those \( \phi \) for which the direct image functor \( \phi_* \) preserves \( \mathcal{S} \)-coproducts as an \( \mathcal{S} \)-indexed functor. This means that for all morphisms \( I \rightarrow J \) in \( \mathcal{S} \),

\[
\begin{array}{ccc}
\mathcal{S}/e^*I & \xrightarrow{\Sigma} & \mathcal{S}/e^*J \\
\mathcal{F}/f^*I & \xrightarrow{\Sigma} & \mathcal{F}/f^*J \\
\phi_! & \downarrow & \phi_! \\
\end{array}
\]

commutes. It is easy to see that \( \phi_* \) preserves \( \mathcal{S} \)-coproducts iff for every \( I \in \mathcal{S} \), the unit \( f^*I \rightarrow \phi_*\phi^*f^*\Omega \) is an isomorphism.

**Proposition 2.2.** \( \phi \) is dense over \( \mathcal{S} \) iff \( f^*\Omega \rightarrow \phi_*\phi^*f^*\Omega \) is a monomorphism.

**Proof.** Assume that \( f^*\Omega \rightarrow \phi_*\phi^*f^*\Omega \) is a monomorphism. Then for any \( A \in \mathcal{F} \), the map

\[
\phi^*: \mathcal{F}(A, f^*\Omega) \rightarrow \mathcal{F}(A, \phi_*\phi^*\Omega) \cong \mathcal{E}(\phi^*A, e^*\Omega)
\]

is injective. Fix \( I \in \mathcal{S} \). To show that the unit \( f^*I \rightarrow \phi_*\phi^*f^*I \) is a monomorphism, let \( A \xrightarrow{a} f^*I \) and \( A \xrightarrow{b} f^*I \) be two morphisms equalized by the unit. The equalizer of \( a \) and \( b \) is a definable subobject as it can be obtained as the pullback

\[
\begin{array}{ccc}
E & \xrightarrow{(a,b)} & A \\
\downarrow & & \downarrow \\
f^*I & \xrightarrow{f^*\delta} & f^*(I \times I)
\end{array}
\]

where \( \delta \) is the diagonal. Let \( m \) denote a factorization of the characteristic map of this subobject through \( f^*\Omega \). We have \( \phi^*E \cong \phi^*A \) since the unit is assumed to equalize \( a \) and \( b \). Therefore, \( \phi^*(m) \) is equal to the morphism \( \phi^*A \rightarrow 1 \xrightarrow{1^*} e^*\Omega \), and so \( m \) is equal to \( A \rightarrow 1 \xrightarrow{1^*} f^*\Omega \). Thus, \( E \cong A \), whence \( a = b \). \( \Box \)

Collins and Dyckhoff [8] refer to a map of topological spaces \( f: X \rightarrow Y \) as hereditarily 2-extendable if for every open set \( U \) of \( Y \), every clopen subset of \( f^{-1}U \) is the inverse image of a clopen subset of \( U \). For geometric morphisms, we formulate this property as follows (cf. [13]).

**Definition 2.3.** A geometric morphism \( \mathcal{E} \xrightarrow{\phi} \mathcal{F} \) will be said to be pure over \( \mathcal{S} \), or simply pure, if the unit \( f^*\Omega \rightarrow \phi_*\phi^*f^*\Omega \) is an epimorphism, where \( \Omega \) is the subobject classifier of \( \mathcal{S} \).
Proposition 2.4. 1. A geometric morphism $\mathcal{E} \xrightarrow{\varphi} \mathcal{F}$ over $\mathcal{S}$ is pure dense iff $f^*\Omega \rightarrow \varphi_\ast f^*\Omega$ is an isomorphism.

2. The composite of two pure dense geometric morphisms is pure dense. If $\varphi \cdot \psi$ and $\psi$ are both pure dense, then so is $\varphi$.

3. A pure dense spread is an inclusion. A pure surjection which is a spread is an equivalence.

Proof. 1. By Proposition 2.2.

2. These facts follow easily from 1.

3. Let $\mathcal{E} \xrightarrow{\varphi} \mathcal{F}$ be an arbitrary geometric morphism over $\mathcal{S}$. If $\varphi$ is pure dense, then the top arrow in the commutative square

$$
\begin{array}{ccc}
f^*\Omega & \xrightarrow{\varphi_\ast e^*\Omega} & \varphi_\ast\eta_\mathcal{F}^* \Omega \\
\downarrow \varphi_\ast & & \downarrow \varphi_\ast e^* \\
\Omega_\mathcal{F} & \xrightarrow{\varphi_\ast \Omega_\mathcal{F}} & \varphi_\ast \Omega_\mathcal{F} \\
\end{array}
$$

is an isomorphism. The bottom arrow is the unique frame morphism. If in addition $\varphi$ is a spread, then ($\varphi$ is localic and) $\varphi_\ast \tau_{\mathcal{E}}$ sup-generates $\varphi_\ast \Omega_\mathcal{E}$ (Proposition 1.4). Therefore, $\Omega_\mathcal{F}$ sup-generates $\varphi_\ast \Omega_\mathcal{F}$, so $\varphi$ must be an inclusion. The other assertion follows immediately from this. $\square$

Recall [2] that a geometric morphism $\mathcal{S} \xrightarrow{\gamma} \mathcal{S}'$ is said to be locally connected if $\gamma^*$ has a left adjoint over $\mathcal{S}'$. In this case, we denote the left adjoint by $\gamma_!$, and the unit of $\gamma_! \dashv \gamma^*$ by $\eta$. “Over $\mathcal{S}'$” means that $\gamma_!(\gamma^*I \times_{\gamma^*K} B) \simeq I \times_K \gamma^*B$, i.e., if the left-hand square below is a pullback, then so is its transpose.

$$
\begin{array}{ccc}
P & \xrightarrow{b} & B \\
\downarrow & & \downarrow \\
\gamma^*P & \xrightarrow{\gamma^*b} & \gamma^*B \\
\downarrow \gamma^*a & & \downarrow \gamma^*a \\
\gamma^*I & \xrightarrow{\gamma^*a} & \gamma^*K \\
\end{array}
$$

Also, $\gamma_!(Y \rightarrow \gamma^*I)$ is canonically isomorphic to the transpose $\gamma_!Y \rightarrow I$. If a morphism $A \xrightarrow{b} B$ in $\mathcal{S}$ is $\mathcal{S}'$-definable, i.e., if there is a pullback as shown below (left), then it follows that the right-hand square below is a pullback.

$$
\begin{array}{ccc}
A & \xrightarrow{b} & B \\
\downarrow \eta_A & & \downarrow \eta_A \\
\gamma^*A & \xrightarrow{\gamma^*b} & \gamma^*B \\
\end{array}
$$

In other words, in the locally connected case, definable morphisms are definable by their components. From this, one immediately deduces that in the locally connected case,
(i) the composite of two definable morphisms is again definable. In addition, (ii) if \( \gamma \) is locally connected, then the morphism \( \varepsilon : \gamma^*\Omega_{\mathcal{F}} \to \Omega_{\mathcal{F}} \) is a monomorphism (locally connected geometric morphisms are subopen – see [2]). Thus, there is a bijection between definable subobjects of an object \( X \in \mathcal{F} \) and morphisms \( X \to \gamma^*\Omega_{\mathcal{F}} \). Finally, from the natural bijections (writing \( \Omega \) for \( \Omega_{\mathcal{F}} \))

\[
\begin{align*}
I & \to \gamma_*(\gamma^*\Omega^X) \\
\gamma^*I & \to \gamma^*\Omega^X \\
\gamma_!(X \times \gamma^*I) & \to \Omega \\
\gamma_!X \times I & \to \Omega \\
I & \to \Omega^{\pi X}
\end{align*}
\]

we see that (iii) for every \( X \in \mathcal{F} \), there is an object \( \gamma_!X \) and a lattice isomorphism \( \gamma_!(\gamma^*\Omega^X) \cong \Omega^\pi X \). Intuitively, this isomorphism says that the definable subobjects of \( X \) are exactly (the unions of) sets of components of \( X \). A result [2] which will be important to us (see Proposition 2.7 below) is that not only are conditions (i), (ii) and (iii) necessary for local connectedness, but that they are also sufficient.

Before proceeding with the factorization theorems we present the following results. They will not be further used in this paper.

**Proposition 2.5.** Let \( \mathcal{E} \xrightarrow{\varphi} \mathcal{F} \) be a geometric morphism over \( \mathcal{F} \) which is locally connected.

1. If \( \varphi \) is a spread, then \( \varphi \) is a local homeomorphism.

2. Assume that \( \mathcal{F} \xrightarrow{f} \mathcal{F} \) is subopen. If \( \varphi \) is localic and zero-dimensional, then \( \varphi \) is a local homeomorphism.

**Proof.**

1. Factor \( \varphi \) as \( \mathcal{E} \xrightarrow{\psi} \mathcal{F}/\varphi_1 \to \mathcal{F} \), where \( \psi \) is connected and locally connected. Connected geometric morphisms are pure, so \( \psi \) is a pure surjection. But if \( \varphi \) is a spread, then \( \psi \) is one as well (Proposition 1.2(3)). Thus, the spread \( \psi \) is a pure surjection, hence an equivalence (Proposition 2.4(3)).

2. Since locally connected geometric morphisms are open, the statement follows by Proposition 1.17 and 1. \( \square \)

**Definition 1.9** declares, in the special case that the codomain topos \( \mathcal{F} \) is taken to be the base topos, that a locale \( X \) in a topos is zero-dimensional if the object

\[
Clp(X) = \{U \in \mathcal{O}(X) \mid T = \bigvee\{U \leftrightarrow \omega \mid \omega \in \Omega\}\}
\]

sup-generates the frame \( \mathcal{O}(X) \).

**Theorem 2.6.** A locally connected zero-dimensional locale in a topos is discrete.
Proof. If $X$ denotes a locally connected zero-dimensional locale in a topos $\mathcal{S}$, then we can apply Proposition 2.5(2) to the diagram

\[
\begin{array}{ccc}
\text{Sh}(X) & \rightarrow & \mathcal{S} \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
\mathcal{S}. & \rightarrow & \square
\end{array}
\]

Returning to the factorization theorems, we will consider only the case when $e$ in

\[
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{\eta} & \mathcal{F} \\
\downarrow & & \downarrow \\
e & \rightarrow & f \\
\downarrow & & \downarrow \\
\mathcal{S} & \rightarrow & \mathcal{S}
\end{array}
\]

is locally connected. We will refer to a geometric morphism $\mathcal{E} \rightarrow \mathcal{F}$ over $\mathcal{S}$ for which $e$ is locally connected as having locally connected domain.

If $\phi$ has locally connected domain, then it follows from Proposition 1.5 and condition (i) above that $\phi$ is a spread iff for any generating family $D \rightarrow f^*K$ of $\mathcal{F}$ over $\mathcal{S}$, there is for every $X \in \mathcal{E}$ a diagram

\[
\begin{array}{ccc}
C & \rightarrow & X \\
\downarrow & & \downarrow \\
(\eta_C,e) & \rightarrow & e^*C \times \phi^*D
\end{array}
\]

in which $(\eta_C,e)$ is a definable subobject.

The following proposition is important for the factorization theorems. It will also be used in Section 3 (Proposition 3.12). We remind the reader that in a topos $\mathcal{F} \rightarrow \mathcal{S}$, the union of an $\mathcal{S}$-family of subobjects $S \rightarrow \gamma^*I \times T$ is its image under the projection $\gamma^*I \times T \rightarrow T$.

**Proposition 2.7.** Assume that $\mathcal{E} \rightarrow \mathcal{F}$ has locally connected domain. Then $\phi_*$ preserves $\mathcal{S}$-coproducts (as an $\mathcal{S}$-indexed functor) iff $f$ is locally connected and the canonical morphism $e_1 \cdot \phi^* \rightarrow f_1$ is an isomorphism. In this case, $\phi$ is pure and dense over $\mathcal{S}$. Conversely, if $\phi$ is pure and dense over $\mathcal{S}$ and if in $\mathcal{F}$ the union of an $\mathcal{S}$-family of definable subobjects is again one, then $\phi_*$ preserves $\mathcal{S}$-coproducts.

**Proof.** The direct image functor $\phi_*$ preserves $\mathcal{S}$-coproducts iff the unit $f^* \rightarrow \phi_\ast \phi^* f^*$ is an isomorphism. We have $e_1 \cdot \phi^* \rightarrow f_1 \cdot \phi^* f^*$, so $f_1$ exists and $e_1 \phi^* \simeq f_1$ iff $f^* \simeq \phi_\ast \phi^* f^*$. Clearly, such $\phi$ are pure and dense over $\mathcal{S}$.

Assume that $\phi$ is pure and dense over $\mathcal{S}$, i.e., that $f^*\Omega \simeq \phi_\ast e^*\Omega$, and that the given condition on $\mathcal{F}$ holds (and also that $e$ is locally connected). We will show that $f$ satisfies conditions (ii) and (iii) above, and that definable subobjects compose in $\mathcal{F}$.
Then together with the hypothesis that the union of an \( \mathcal{F} \)-family of definable subobjects is again one (see Example 2.8 below) we obtain that \( f \) is locally connected and that \( e; \varphi^* \simeq f_1 \) [2, Proposition 13]. For (iii), we have, using that \( e \) is locally connected, that for any \( I \in \mathcal{F}, X \in \mathcal{F}, \)

\[
\begin{align*}
I & \to \Omega^e; \varphi^*X \\
\varphi^*I \times X & \to \varphi^*\Omega \\
f^*I \times X & \to f^*\Omega \\
I & \to f_*(f^*\Omega^X)
\end{align*}
\]

For (ii), we have the commutative diagram

\[
\begin{array}{ccc}
f^*\Omega & \to & \varphi; e^*\Omega \\
\tau_{\mathcal{F}} & \downarrow & \varphi; \tau_{\mathcal{F}} \\
\Omega_{\mathcal{F}} & \to & \varphi; \Omega_{\mathcal{F}}
\end{array}
\]

where the bottom arrow is the unique frame morphism. The top arrow is an isomorphism, so \( \tau_{\mathcal{F}} \) is a monomorphism. Finally, let \( S \to F \) be a definable subobject in \( \mathcal{F} \). There is a morphism \( F \to f^*\Omega \) making the back face in the following diagram a pullback.

The unit arrows in the bottom face of this diagram are isomorphisms, so the bottom face is a pullback. From this we conclude that \( S \to F \) is definable iff \( \varphi^*S \to \varphi^*F \) is definable and the top face in the above diagram is a pullback. It follows that definable subobjects compose.

**Example 2.8.** The inclusion of \( X \to Y \) of Example 1.7 is pure dense, and \( X \) is locally connected. However, \( Y \) is not locally connected. Clearly, the condition in Proposition 2.7 that the collection of definable, i.e., clopen, subsets of every open of \( Y \) be closed under arbitrary union is not satisfied (consider the open disk centered at \((0,0)\) of radius \( \frac{1}{2} \)). It was noted in Example 1.7 that this map is also a complete spread.
Thus, in general, it is not true that a pure dense complete spread is an equivalence. (This is so, however, if in the codomain topos the union of an \( S \)-family of definable subobjects is again one – see Theorem 2.15.)

\textbf{Proposition 2.9.} \textit{Let } \( \mathcal{E} \hookrightarrow \mathcal{F} \text{ be an inclusion with } \mathcal{F} \rightarrow \mathcal{S} \text{ locally connected. If } i_* \text{ preserves } \mathcal{S} \text{-coproducts, then } \mathcal{E} \rightarrow \mathcal{F} \text{ is locally connected with } e_1 \simeq f_1 \cdot i_* \text{. We also have } e_1 \cdot i^* \simeq f_1. \text{}}

\textit{Proof.} If the unit \( f^* \rightarrow i_* i^* f^* \) is an isomorphism, then we have an indexed adjointness \( f_1 \cdot i_* \vdash e^* \). The second statement follows from Proposition 2.7. \( \square \)

We will obtain the factorization theorems by first establishing a characterization of Lawvere distributions in terms of complete spreads (Theorem 2.13 below). For toposes \( \mathcal{F}, \mathcal{G} \) over \( \mathcal{S} \), a functor \( \mathcal{F} \rightarrow \mathcal{G} \) over \( \mathcal{S} \) is said to be \textit{cocontinuous} if it preserves all \( S \)-colimits. Let \( \text{Cocts}_{\mathcal{S}}(\mathcal{F}, \mathcal{G}) \) denote the category of cocontinuous functors with natural transformations over \( \mathcal{S} \) as morphisms.

Fix a presentation \( (D, J_f) \) of \( \mathcal{F} \rightarrow \mathcal{S} \). We associate with a \( \chi \in \text{Cocts}_{\mathcal{S}}(\mathcal{F}, \mathcal{S}) \) (i.e., to a distribution) the pullback

\[
\begin{array}{ccc}
\mathcal{D}_\chi & \xrightarrow{\pi} & \mathcal{S}^{\chi^*} \\
\downarrow i & & \downarrow u \\
\mathcal{F} & \xrightarrow{i} & \mathcal{S}^{D_{\chi^*}}
\end{array}
\]

in \( \text{Top}_\mathcal{S} \). The geometric morphism \( u \) is that induced by the discrete opfibration \( C \xrightarrow{U} D \), where \( C \) is the total category of the functor

\( D \rightarrow \mathcal{F} \xrightarrow{\chi} \mathcal{S} \).

The topos \( \mathcal{D}_\chi \xrightarrow{d} \mathcal{S} \) can be described as \( \text{Sh}(C, J_d) \), where \( J_d \) denotes the Grothendieck topology on \( C \) generated by the following covers \( R^{-1}c \). For any \( J_f \)-cover \( R = \{X_i \xrightarrow{x_i} X\} \) and any \( c \in \chi(X) \), we define

\( R^{-1}c = \{(X_i, d) \xrightarrow{x_i} (X, c) | x_i \in R \text{ and } \chi(x_i)(d) \simeq c\}. \)

\textbf{Proposition 2.10.} \textit{For any cocontinuous functor } \( \chi \), \( \mathcal{D}_\chi \xrightarrow{d} \mathcal{S} \text{ is locally connected, we have } d_1 \cdot \chi^* \simeq \chi, \text{ and } \mathcal{D}_\chi \xrightarrow{i} \mathcal{F} \text{ is a spread which is complete for } (D, J_f) \text{ (Definition 1.6).} \text{}}

\textit{Proof.} For the first assertion, we begin by showing that \( \pi_* \) (see diagram (1)) preserves \( \mathcal{S} \)-coproducts, or in other words, that the constant presheaves are sheaves. For any \( c \in \chi(X) \), \( R \in J_f \), the cover \( R^{-1}c \) is connected and inhabited as it is the preimage of \( c \) under the diagram \( \{\chi(x) | x \in R\} \) and since the induced morphism \( \varinjlim \mathcal{X} \rightarrow \chi(X) \)
is an isomorphism. This morphism is an isomorphism since $\chi$ is cocontinuous. Then

\[
C(R^{-1}c) \cong 1,
\]
and for any constant presheaf $C^*I$, we have

\[
\frac{R^{-1}c \to C^*I}{C((R^{-1}c)) \to I} \frac{1 \to I}{C(X, c) \to I} \frac{(X, c) \to C^*I}.
\]

This shows that $C^*I$ is a sheaf. By Proposition 2.9, since presheaf toposes are locally connected, $d$ must be locally connected and $d! \cdot \pi^* \cong C!$.

Next, for any $X \in D$, we have

\[
d!\chi^*(X) \cong d!\chi^*i_*i^!(X) \cong d!\pi^*u^*i_!(X) \cong C!u^*i_!(X)
\]

\[
\cong C! \bigsqcup_{c \in \chi(X)} (X, c) \cong \bigsqcup_{x \in \chi(X)} C(X, c) \cong \bigsqcup_{x \in \chi(X)} 1 \cong \chi(X).
\]

Therefore, $d! \cdot \chi^* \cong \chi$.

For the last assertion, it is not hard to see that, regarding $\mathcal{D}_\chi \xrightarrow{\chi} \mathcal{F}$, the toposes of sheaves for, respectively, the spread site for $(D, J_f)$, the complete spread site for $(D, J_f)$ (see Definition 1.6), and the site $(C, J_d)$ (i.e., $\mathcal{D}_\chi$) are equivalent. We omit the details of a precise argument. $\square$

Consider the functor $\Lambda$ which associates with a diagram

\[
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{\varphi} & \mathcal{F} \\
e & \swarrow & \searrow f \\
\mathcal{G} & & \mathcal{F}
\end{array}
\]

with $e$ locally connected, the cocontinuous functor $\Lambda_\varphi = e_1 \cdot \varphi^* : \mathcal{F} \to \mathcal{G}$.

**Proposition 2.11.** The passage $\chi \mapsto \mathcal{D}_\chi$ is right adjoint to $\Lambda$, i.e., there is a bijection

\[
\begin{array}{ccc}
geometric morphisms \mathcal{E} \longrightarrow \mathcal{D}_\chi & \text{over } \mathcal{F} \\
natural transformations \Lambda_\varphi \rightarrow \chi
\end{array}
\]

which is natural in $\mathcal{E} \xrightarrow{\varphi} \mathcal{F}$ with locally connected domain, and in $\chi \in \text{Cocets}_\mathcal{F}(\mathcal{F}, \mathcal{F})$. A spread $\mathcal{E} \xrightarrow{\varphi} \mathcal{F}$ is complete for $(D, J_f)$ iff $\mathcal{E} \cong \mathcal{D}_\Lambda_\varphi$.

**Proof.** The bijection is as follows. A natural transformation $t : \Lambda_\varphi \to \chi$ gives the geometric morphism $\mathcal{D}t \cdot \eta_\varphi$, where the unit $\eta_\varphi : \mathcal{E} \longrightarrow \mathcal{D}_\Lambda_\varphi$ is obtained as the
universal morphism arising from the commutative square

\[ \begin{array}{ccc}
\mathcal{E} & \xrightarrow{v} & \mathcal{E}^{e^\cdot} \\
\varphi \downarrow & & \downarrow u \\
\mathcal{F} & \xrightarrow{i} & \mathcal{F}^{d^\cdot}. \\
\end{array} \tag{2} \]

The geometric morphism \( v \) is that induced by the flat functor \( V : C \rightarrow \mathcal{E} \) such that for \( c \in e!(\varphi^\cdot X) \), \( V(X, c) \) is the pullback

\[ \begin{array}{ccc}
V(X, c) & \xrightarrow{} & \varphi^\cdot X \\
\downarrow & & \downarrow \\
1 & \xrightarrow{e^\cdot c} & e!e!(\varphi^\cdot X) \\
\end{array} \]

where the right-hand vertical arrow is the unit of \( e! \dashv e^\cdot \). The return passage associates with a diagram

\[ \begin{array}{ccc}
\mathcal{E} & \xrightarrow{\psi} & \mathcal{D} \chi \\
\varphi \downarrow & & \downarrow k \\
\mathcal{F} & \xrightarrow{\Lambda} & \mathcal{F} \\
\end{array} \]

the natural transformation \( \Lambda_\psi : e! \cdot \varphi^\cdot \rightarrow d! \cdot \chi^\cdot \simeq \chi \).

For the other assertion, if \( \mathcal{E} \xrightarrow{\varphi} \mathcal{F} \) is a spread which is complete for \((D, J_f)\), then it follows that diagram (2) is a pullback, i.e., that \( \mathcal{E} \cong D\Lambda_\varphi \).

If \((D', J_{f'})\) is any other presentation of \( \mathcal{F} \), then the corresponding functor \( \chi \mapsto D'\chi \) is also right adjoint to \( \Lambda \). Therefore, \( D \cong D' \). In particular, a spread with locally connected domain is complete for some site for \( \mathcal{F} \) iff it is complete for all sites for \( \mathcal{F} \). We make the following definition.

**Definition 2.12.** A spread \( \mathcal{E} \xrightarrow{\varphi} \mathcal{F} \) with locally connected domain will be said to be **complete** if it is complete for some (and hence, any) choice of a site for \( \mathcal{F} \).

We summarize the above results.

**Theorem 2.13.** Let \( \mathcal{F} \xrightarrow{f} \mathcal{G} \) denote an arbitrary bounded topos. There is an equivalence between the category of complete spreads with codomain \( \mathcal{F} \) and with locally connected domain, and the category \( \text{Cocts}_\mathcal{G}(\mathcal{F}, \mathcal{G}) \). This equivalence associates with a complete spread \( \mathcal{E} \xrightarrow{\varphi} \mathcal{F} \) with locally connected domain the cocontinuous functor \( e! \cdot \varphi^\cdot \). It associates with a cocontinuous functor \( \mathcal{F} \xrightarrow{k} \mathcal{G} \) the pullback in \( \mathcal{Top}_\mathcal{G} \) as in diagram (1).
Remark 2.14. If $\mathcal{F}$ is the topos of presheaves of a small category $\mathcal{D}$ in $\mathcal{S}$, then the complete spread over $\mathcal{F} = \mathcal{F}^{\mathcal{D}^\text{op}}$ corresponding to a cocontinuous functor $\mathcal{S}^{\mathcal{D}^\text{op}} \to \mathcal{S}$ is the geometric morphism $\mathcal{S}^{\mathcal{C}^\text{op}} \to \mathcal{S}^{\mathcal{D}^\text{op}}$ induced by the discrete opfibration corresponding to $\chi$.

We state the following theorems for localic geometric morphisms. Corresponding results for arbitrary (bounded) geometric morphisms can be obtained by applying the hyperconnected/localic factorization, and then recognizing that (hyper)connected morphisms are pure surjections.

Theorem 2.15 (Pure dense/complete spread). Any localic geometric morphism $\mathcal{S} \to \mathcal{F}$ with locally connected domain has a factorization $\mathcal{S} \to \mathcal{D} \to \mathcal{F}$, where $\mathcal{D}$ is locally connected, $\eta$ is pure dense, and $\chi$ is a complete spread.

Let $\mathcal{S} \to \mathcal{D} \to \mathcal{F}$ be an arbitrary factorization of $\gamma$ in which $\rho$ is pure dense, $\mathcal{D}$ has the property that the union of an $\mathcal{S}$-family of definable subobjects is again definable, and $\varphi$ is a spread which is complete for some choice of site for $\mathcal{F}$. Then $\mathcal{S}$ is locally connected, $\varphi$ is complete, and the given factorization is equivalent to the one specified in the existence statement.

Proof. Consider the factorization $\mathcal{S} \to \mathcal{D} \to \mathcal{F}$ constructed in the proof of Proposition 2.11. What remains to be shown is that the unit $\eta$ is pure dense. We have $\pi \cdot \eta \simeq \nu$ and $d_1 \cong C_1 \cdot \pi_*$ (see the proof of Proposition 2.10). We also have $e_1 \cdot \nu^* \simeq C_1$ since these two functors are cocontinuous and they agree on the generating category $C$. Therefore,

$$e_1 \cdot \eta^* \simeq e_1 \cdot (\nu^* \cdot \pi_*) \simeq C_1 \cdot \pi_* \simeq d_1,$$

and so by Proposition 2.7, $\eta$ is pure dense.

Turning to the uniqueness statement, by Proposition 2.7, $\mathcal{S}$ is locally connected with $g_1 \simeq e_1 \cdot \rho^*$, and $\varphi$ is a complete spread (Definition 2.12). But $e_1 \cdot \eta^* \simeq d_1$, and $g_1 \cdot \varphi^* \simeq d_1 \cdot \chi^*$ follows. Then, by Theorem 2.13 we have $\mathcal{D} \chi \simeq \mathcal{S}$ over $\mathcal{F}$. □

Example 2.16. 1. The pure dense/complete spread factorization of an inclusion is not necessarily an inclusion. A cookie cutter works by factoring into a pure dense map followed by a complete spread the inclusion of the plane minus a circle into the plane. The resulting middle space consists of the cookie and the remaining dough. The complete spread maps the cookie back to the space it left, and it is not an inclusion. (This is an example of Michael’s [22] concept of a cut.)

2. The comprehensive factorization [29] $E \to C \to D$ of a functor $E \to D$ into an initial functor followed by a discrete opfibration induces the pure dense/complete spread factorization $\mathcal{S}^{E^\text{op}} \to \mathcal{S}^{C^\text{op}} \to \mathcal{S}^{D^\text{op}}$ of the induced geometric morphism $f : \mathcal{S}^{E^\text{op}} \to \mathcal{S}^{D^\text{op}}$. The objects of $C$ are all pairs $(d, \alpha)$, where $d \in D$ and $\alpha$ is a connected component of the comma category $f/d$. A morphism $(d, \alpha) \to (d', \beta)$ is a morphism $d \to d'$ in $D$ such that $\beta$ is the connected component of $f(e) \to d \to d'$ in $f/d'$, where $\alpha$ is the connected component of $f(e) \to d$ in $f/d$. (The dual factorization
of \( f \) into an final functor followed by a discrete fibration induces the factorization
\[
G^{fp} \longrightarrow G^{dp} \rightarrow G^{fp},
\]
where \( f^* \).

**Corollary 2.17.** Let \( \mathcal{E} \xrightarrow{\varphi} \mathcal{F} \) be a localic pure surjection over \( \mathcal{G} \) with locally connected domain. Then the codomain \( \mathcal{F} \) is locally connected.

**Proof.** Consider the pure dense/complete spread factorization \( \mathcal{E} \xrightarrow{n} \mathcal{D} \xrightarrow{\chi} \mathcal{F} \) of \( \varphi \), in which \( \mathcal{D} \) is locally connected (Theorem 2.15). The geometric morphism \( \chi \) is pure dense (Proposition 2.4(2)). Since \( \varphi \) is a surjection, \( \chi \) is a pure surjection and also a spread. Therefore, \( \chi \) is an equivalence (Proposition 2.4(3)). \( \square \)

In general, the pure dense/complete spread factorization is not unique. Were that so, then a pure dense complete spread would be an equivalence and that is not generally the case (see Example 2.8). On the other hand, the pure surjection/spread factorization is unique.

**Theorem 2.18** (Pure surjection/spread). Any localic geometric morphism over \( \mathcal{G} \) with locally connected domain has a unique factorization into a pure surjection followed by a spread. The middle topos in this factorization is locally connected.

**Proof.** In any factorization \( \mathcal{E} \xrightarrow{p} \mathcal{G} \xrightarrow{i} \mathcal{D} \) of a pure geometric morphism in which \( i \) is an inclusion, \( p \) is pure (as follows easily from the definition of pure). Now factor the pure dense part of the previous factorization into a surjection followed by an inclusion. For \( \mathcal{E} \xrightarrow{\psi} \mathcal{F} \), this gives a factorization

\[
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{p} & \mathcal{G} \\
\downarrow \varphi & & \downarrow i \\
\mathcal{F} & \xleftarrow{\psi} & \mathcal{D}.
\end{array}
\]

where \( p \) is a pure surjection and \( \psi \cdot i \) is a spread (Proposition 1.2(2)). By Corollary 2.17, the topos \( \mathcal{G} \) is locally connected (as is \( \mathcal{D} \)).

For the uniqueness of the factorization, let \( \mathcal{E} \xrightarrow{p'} \mathcal{G}' \xrightarrow{i'} \mathcal{D}' \) be any pure surjection/spread factorization of \( \varphi \). Then \( \mathcal{G}' \) must be locally connected, and so we can apply the pure dense/complete spread factorization to \( \rho \) to give

\[
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{p'} & \mathcal{G}' \\
\downarrow \varphi & & \downarrow i' \\
\mathcal{F} & \xleftarrow{\psi'} & \mathcal{D}'.
\end{array}
\]
where $\mathcal{D}'$ is locally connected. Then $i' \cdot p'$ is pure dense as it is the composite of two pure dense morphisms, and so $\mathcal{D}' \cong \mathcal{D}$ over $\mathcal{F}$. Since $p$ is a spread, so is $i'$, and therefore, $i'$ is an inclusion. This allows us to conclude that $\mathcal{D}' \cong \mathcal{D}$ over $\mathcal{F}$. □

3. The symmetric monad on $\text{Top}_{\mathcal{F}}$

Recall that the symmetric topos construction [5] is part of an op-KZ-doctrine [17, 28] on the 2-category $\mathcal{A}$ of locally presentable $\mathcal{I}$-categories, $\mathcal{I}$-cocontinuous functors, and natural transformations. The biadjoint pair

$$
\eta, \varepsilon : \Sigma \dashv U : \text{Top}_{\mathcal{F}}^{\text{op}} \longrightarrow \mathcal{A}
$$

is 2-monadic. This says that a locally presentable $\mathcal{I}$-category $\mathcal{A}$ is the "frame" $U\mathcal{B}$ of an $\mathcal{I}$-topos $\mathcal{B}$ if and only if its structure $\theta_{\mathcal{A}} : U\Sigma \mathcal{A} \longrightarrow \mathcal{A}$ is an $\mathcal{I}$-cocontinuous coreflection of the unit $\eta_{\mathcal{A}} : \mathcal{A} \longrightarrow U\Sigma \mathcal{A}$, and that an $\mathcal{I}$-cocontinuous functor $F : U\mathcal{B} \longrightarrow U\mathcal{F}$ is a $\Sigma$-homomorphism if and only if $F$ is lex.

Consider now the $\Sigma$ construction, but restricted to (frames of) $\mathcal{I}$-toposes and their morphisms. Taking opposites, (1) induces a 2-monad on $\text{Top}_{\mathcal{F}}$. A change of notation is called for, not only to avoid confusion, but to indicate a switch from an algebraic to a geometric point of view. Denote by $M : \text{Top}_{\mathcal{F}} \longrightarrow \text{Top}_{\mathcal{F}}$ the composite $\Sigma_{\text{op}} \mathcal{U}_{\text{op}}$, and by $\delta : \text{Id} \rightarrow M$ the natural transformation $\eta_{\text{op}}$. For an $\mathcal{I}$-topos $\mathcal{E}$, $\delta = \delta_{\mathcal{E}} : \mathcal{E} \longrightarrow M\mathcal{E}$ is the essential geometric morphism with $U(\delta_!) = \eta_{U\mathcal{E}}$, $U(\delta^*) = \theta_{U\mathcal{E}}$, and $U(\delta^*)$ its right adjoint. Since $\theta_{U\mathcal{E}} \cdot \eta_{U\mathcal{E}} \simeq 1$, $\delta^* \cdot \delta_! \simeq 1$, hence also $\delta^* \cdot \delta_* \simeq 1$, and $\delta$ is an essential inclusion in $\text{Top}_{\mathcal{F}}$. The notation $\delta : \text{Id} \rightarrow M$ intends to suggest that $M\mathcal{E}$ is the topos classifying generalized measures (distributions) on $\mathcal{E}$ [20, 21], and that $\delta$ induces the inclusion of the points of $\mathcal{E}$ into the measures on $\mathcal{E}$, where a point $p$ on $\mathcal{E}$ is regarded as the Dirac measure supported on $p$.

**Theorem 3.1.** The data given by $\delta : \text{Id} \rightarrow M$ is part of a KZ-doctrine on $\text{Top}_{\mathcal{F}}$. A topos $\mathcal{E}$ over $\mathcal{I}$ is an $M$-algebra iff $\delta_{\mathcal{E}}$ has a reflection left adjoint $\sigma_{\mathcal{E}}$ in $\text{Top}_{\mathcal{F}}$. Equivalently, $\mathcal{E}$ is an $M$-algebra iff $(\delta_{\mathcal{E}})_* : \mathcal{E} \longrightarrow M\mathcal{E}$ is $\mathcal{I}$-cocontinuous. A geometric morphism $\mathcal{E} \overset{f}{\longrightarrow} \mathcal{F}$ over $\mathcal{I}$, where $\mathcal{E}$ and $\mathcal{F}$ are $M$-algebras, is an $M$-homomorphism iff the canonical 2-cell $\zeta_f$ in

$$
\begin{array}{ccc}
M\mathcal{E} & \xrightarrow{\sigma_{\mathcal{E}}} & \mathcal{E} \\
\downarrow Mf & & \downarrow f \\
M\mathcal{F} & \xrightarrow{\sigma_{\mathcal{F}}} & \mathcal{F}
\end{array}
$$

arising from the identity 2-cell $\delta_{\mathcal{F}} \cdot f = Mf \cdot \delta_{\mathcal{E}}$ and the adjointnesses $\sigma_{\mathcal{E}} \dashv \delta_{\mathcal{E}}$ and $\sigma_{\mathcal{F}} \dashv \delta_{\mathcal{F}}$, is an isomorphism. Also $f$ is an $M$-homomorphism if $f$ is $\mathcal{I}$-essential with $f_!$ left exact, where $f_! \dashv f^*$. 
Proof. A KZ-doctrine \[17\] on a 2-category \(\mathcal{C}\) consists of an endofunctor \(T : \mathcal{C} \to \mathcal{C}\), two natural transformations \(\eta : \text{Id} \to T\), \(\mu : T \cdot T \to T\), and, for each \(C \in \mathcal{C}\), a 2-cell \(\gamma_C : \eta_{TC} \Rightarrow \eta_{TC}\) satisfying certain axioms \(T_0\)–\(T_3\). Such KZ-doctrines are exemplified by cocompletion processes. An op-KZ-doctrine is a 2-monad typified by a completion process, so that the additional structure on the monad \((T, \eta, \mu)\) is given by specifying, for each \(C \in \mathcal{C}\), a 2-cell \(\kappa_C : \eta_{TC} \Rightarrow \eta_{TC}\), satisfying axioms similar to \(T_0\)–\(T_3\).

The symmetric 2-monad \((\Sigma, \eta, \Sigma\eta, \mu)\) on \(A\) is an op-KZ-doctrine, so that, for each \(A \in A\), there is given a 2-cell \(\kappa : \eta_{\Sigma \eta} \Rightarrow \eta_{\Sigma \eta}\) which satisfies axioms \(T_0\)–\(T_3\). In particular, for an \(\mathcal{S}\)-topos \(\mathcal{E}\), the 2-cell \(\kappa_{\mathcal{U}}\mathcal{E}\) induces a 2-cell in the opposite direction between the right adjoints. This gives, by taking opposites, a 2-cell \(\lambda_\mathcal{E} : M(\delta_\mathcal{E}) \Rightarrow \delta_{\mathcal{M}}\). The verification of the axioms for a KZ-doctrine \((M, \delta, \mu, \lambda)\), where \(\mu = \Sigma_{\text{op}} \Sigma \mathcal{U}_{\text{op}}\), is an immediate consequence of the validity of these axioms for the op-KZ-doctrine structure on the symmetric monad on \(A\).

It follows from the general theory of KZ-doctrines \[17\] that, for \((M, \delta, \mu, \lambda)\), a topos \(\mathcal{E} \in \text{Top}_{\mathcal{S}}\) is an \(M\)-algebra iff the unit \(\delta_\mathcal{E} : \mathcal{E} \to M\mathcal{E}\) has a reflection left adjoint \(\sigma_\mathcal{E} : M\mathcal{E} \to \mathcal{E}\). A brief analysis shows this condition to be equivalent with the existence of a further right adjoint \((\delta_\mathcal{E})_* : (\delta_\mathcal{E})^*\). Since \((\delta_\mathcal{E})_*\) is a functor between \(\mathcal{S}\)-bounded toposes, it has a right adjoint iff it is \(\mathcal{S}\)-cocontinuous.

It also follows from the general theory that, in the present setting, a geometric morphism \(\mathcal{E} \xrightarrow{f} \mathcal{F}\) between \(M\)-algebras is an \(M\)-homomorphism iff the canonical 2-cell \(\xi_f\) (see (2)) is an isomorphism. As shown in \[17\] (Proposition 2.5) and applied here, a sufficient condition for a geometric morphism \(f\) to be an \(M\)-homomorphism is the existence of a right adjoint \(f \dashv g\) in \(\text{Top}_{\mathcal{S}}\). Analyzing this situation shows that we must have, equivalently, \(g^* \dashv f^*\) and that therefore, \(f\) is \(\mathcal{S}\)-essential with \(f^! \cong g^*\), therefore lex. \(\square\)

Example 3.2. 1. Consider the line \(\mathcal{R} = M\mathcal{S} = \mathcal{S}[U]\), the object classifier in \(\text{Top}_{\mathcal{S}}\), and the unit \(\delta_\mathcal{S} : \mathcal{S} \to M\mathcal{S}\), which is easily seen to be “supported on 1”, i.e., corresponds to the object 1 in \(\mathcal{S}\). Since \(\delta_\mathcal{S}\) is \(\mathcal{S}\)-continuous, \(\mathcal{S}\) is an \(M\)-algebra, and since \(\delta_\mathcal{S}\) is lex, \(\delta_\mathcal{S}\) is an \(M\)-homomorphism.

2. The line \(\mathcal{R}\) is also an \(M\)-algebra, in fact it is a free one. Denote its structure map by \(f : M\mathcal{R} \to \mathcal{R}\). The latter gives the action of distributions on \(\mathcal{S}\) by objects of \(\mathcal{S}\), as follows. If \(\mathcal{G} \xrightarrow{\phi} M\mathcal{E}\) is the geometric morphism corresponding to a distribution \(\mathcal{G} \xrightarrow{\mathcal{E}} \mathcal{G}\), and if \(\phi\) is the geometric morphism corresponding to an object \(X \in \mathcal{S}\), then the evaluation \(F(X)\) is the object of \(\mathcal{G}\) whose corresponding geometric morphism is the composite

\[
\mathcal{G} \xrightarrow{f} M\mathcal{E} \xrightarrow{M\phi} M\mathcal{R} \xrightarrow{f} \mathcal{R}.
\]

Corollary 3.3. Let \(\mathcal{E}\) be an \(M\)-algebra. Then for each \(\mathcal{G} \in \text{Top}_{\mathcal{S}}, \text{Top}_{\mathcal{S}}(\mathcal{G}, \mathcal{E})\) is \(\mathcal{S}\)-cocomplete, naturally in \(\mathcal{G}\). If \(\mathcal{G} \xrightarrow{\phi} \mathcal{F}\) is an \(M\)-homomorphism between \(M\)-algebras, then the induced functor \(\text{Top}_{\mathcal{S}}(\mathcal{G}, \mathcal{E}) \to \text{Top}_{\mathcal{S}}(\mathcal{G}, \mathcal{F})\) is \(\mathcal{S}\)-cocontinuous.
Proof. If $\mathcal{E}$ is an $M$-algebra, $\mathcal{E}$ is a retract of $M\mathcal{E}$, hence $\text{Top}_\mathcal{E}(\mathcal{E}, \mathcal{E})$ is a retract of $\text{Top}_\mathcal{E}(\mathcal{E}, M\mathcal{E}) \cong \text{Coc}t_\mathcal{E}(\mathcal{E}, \mathcal{E})$, which is $\mathcal{E}$-cocomplete, hence also $\text{Top}_\mathcal{E}(\mathcal{E}, \mathcal{E})$ is $\mathcal{E}$-cocomplete.

If $\mathcal{E} \xrightarrow{f} \mathcal{F}$ is an $M$-homomorphism, then $\text{Top}_\mathcal{E}(\mathcal{E}, \mathcal{E}) \rightarrow \text{Top}_\mathcal{E}(\mathcal{E}, \mathcal{F})$ is obtained by restriction from the corresponding $\text{Top}_\mathcal{E}(\mathcal{E}, M\mathcal{E}) \rightarrow \text{Top}_\mathcal{E}(\mathcal{E}, M\mathcal{F})$, equivalently given by composition with $f^*$, as a functor $\text{Coc}t_\mathcal{E}(\mathcal{E}, \mathcal{E}) \rightarrow \text{Coc}t_\mathcal{E}(\mathcal{F}, \mathcal{E})$. Since the latter is $\mathcal{E}$-cocontinuous, so is the former. □

Remark 3.4. The reader wishing to investigate the validity of the converse of Corollary 3.3 (which we ignore) should be aware that $\text{Top}_\mathcal{E}(\mathcal{E}, M\mathcal{E})$ cannot be the free $\mathcal{E}$-cocompletion of the $\mathcal{E}$-category $\text{Top}_\mathcal{E}(\mathcal{E}, \mathcal{E})$ since there exist locally connected Grothendieck toposes without points. If $\mathcal{E}$ is such a topos, then $\text{Top}_{\text{Set}}(\text{Set}, \mathcal{E})$ is vacuous, but $\text{Top}_{\text{Set}}(\text{Set}, M\mathcal{E})$ contains a copy of $\text{Set}$, by the local connectivity of $\mathcal{E}$.

We now turn to a comparison between the symmetric (or distributions) monad $M$ on $\text{Top}_\mathcal{E}$, and the lower power locale 2-monad on $\text{Loc}_\mathcal{E}$. In the process, we encounter the lower bag-domain 2-monad on $\text{Top}_\mathcal{E}$ and related constructions.

Power locales are so named because the points of any such locale $P(X)$ correspond to sets of points of the original locale $X$. The idea of replacing sets by bags (indexed families) of points led to toposes [31], and subsequently to the lower bag-domain $\mathcal{B}_L(\mathcal{E})$ as the classifying topos of bags of points of the topos $\mathcal{E}$. From this point of view, the bagdomain topos is a generalized lower power locale construction.

The symmetric topos $M\mathcal{E}$ can also be regarded as a generalized lower power locale, for the following reasons. First, there are similar constructions for $M\mathcal{E}$ and $P_L(X)$ as coinverters [5, 6], in which both involve lex completions of the sites of definition, and second, for $X$ a locale, the localic reflection of $M(\text{Sh}(X))$ is $P_L(X)$ [6] (this is also true of $\mathcal{B}_L(\text{Sh}(X))$ [16]). In addition, the pure dense/complete spread factorization of Section 2 bears an analogy to the strongly dense/weakly closed factorization for sublocales [14], and whereas $P_L(X)$ classifies weakly closed sublocales with open domain, $M\mathcal{E}$ classifies complete spreads with locally connected domain. We will examine this fact more closely, and supply further evidence supporting the claim that the symmetric topos is a generalized lower power locale.

Pitts [25] has shown that if in a bicomma square

$$
g \quad \Rightarrow \quad f$$


of locales, the "lower" leg \( f \) is essential, then \( g \) is open. Using this, Vermeulen [30] proved that the bijection between points of the lower power locale \( P_L(X) \) and weakly closed sublocales of \( X \) with open domain can be given by forming the comma square of a point of \( P_L(X) \) and (as the lower leg) the canonical inclusion \( X \to P_L(X) \). For an essential geometric morphism, the morphism opposite it in a bicomma square is locally connected (and the BCC condition is satisfied), also shown by Pitts [26]. We give below a new proof of this result using the characterization of the points of \( M^\mathcal{E} \) in terms of complete spreads with locally connected domain. Throughout, by a comma square we really mean a bicomma square.

**Proposition 3.5.** Assume that \( \varphi \) in the diagram

\[
\begin{array}{ccc}
\mathcal{G} & \rightarrow & \mathbb{F} \\
\downarrow \psi & & \\
\mathcal{E} & \xrightarrow{\varphi} & \mathcal{F}
\end{array}
\]

is an essential geometric morphism. Let

\[
\begin{array}{ccc}
\mathcal{L} & \rightarrow & \mathcal{G} \\
\downarrow \gamma & & \\
\mathcal{G} \times_{\mathcal{F}} \mathcal{E}
\end{array}
\]

be the complete spread with locally connected \( \mathcal{G} \)-domain corresponding (by Theorem 2.13) to the cocontinuous functor

\[
\mathcal{G} \times_{\mathcal{F}} \mathcal{E} \xrightarrow{(1 \times \varphi)} \mathcal{G} \times_{\mathcal{F}} \mathcal{F} \xrightarrow{(1, \psi)^*} \mathcal{G}.
\]

Then there is a canonical 2-cell \( \lambda^* \varphi^* \Rightarrow \gamma^* \psi^* \) for which the diagram

\[
\begin{array}{ccc}
\mathcal{L} & \rightarrow & \mathcal{G} \\
\downarrow \gamma & & \\
\mathcal{E} & \xrightarrow{\varphi} & \mathcal{F} \\
\downarrow \psi & & \\
\mathcal{G} & \xrightarrow{u} & \mathcal{F}
\end{array}
\]

is a comma square in \( \text{Top}_{\mathcal{F}} \).

**Proof.** By construction (see Theorem 2.13), we have \( \gamma_! (\lambda, \gamma)^* \simeq (1, \psi)^*(1 \times \varphi)_! \), and also \( \gamma_! \lambda^* \simeq \psi^* \varphi_! \). The latter isomorphism we denote by \( i \). Then let \( u : \lambda^* \varphi^* \Rightarrow \gamma^* \psi^* \) correspond under \( \gamma_! \downarrow \gamma^* \) to the 2-cell

\[
\gamma_! \lambda^* \varphi^* \Rightarrow \psi^* \varphi_! \varphi^* \Rightarrow \psi^*.
\]
where \( e \) is the counit of \( \varphi_1 \rightarrow \varphi^* \). Suppose we have

\[
\begin{array}{ccc}
\mathcal{H} & \xrightarrow{\beta} & \mathcal{G} \\
\downarrow \varepsilon & & \downarrow \psi \\
\mathcal{E} & \xrightarrow{\varphi} & \mathcal{F}
\end{array}
\]

in \( \text{Top}_{\mathcal{G}} \). This gives rise to a diagram

\[
\begin{array}{ccc}
\mathcal{H} & \xrightarrow{\beta} & \mathcal{G} \\
\downarrow \langle \beta, \xi \rangle & & \downarrow \langle 1, \psi \rangle \\
\mathcal{G} \times_{\mathcal{G}} \mathcal{E} & \xrightarrow{1 \times \varphi} & \mathcal{G} \times_{\mathcal{G}} \mathcal{F}
\end{array}
\]

in \( \text{Top}_{\mathcal{G}} \). Let \((E, J_e)\) be a presentation of \( \mathcal{E} \rightarrow \mathcal{G} \). Then we have the \( \mathcal{G} \)-site \( g^*E \) for \( \mathcal{G} \times_{\mathcal{G}} \mathcal{E} \) over \( \mathcal{G} \) as in the pullback

\[
\begin{array}{ccc}
\mathcal{G} \times_{\mathcal{G}} \mathcal{E} & \xrightarrow{p_0} & \mathcal{G} \\
\downarrow p_1 & & \downarrow g \\
\mathcal{E} & \xrightarrow{e} & \mathcal{F}
\end{array}
\]

By construction, the \( \mathcal{G} \)-site \( C \) giving \( \mathcal{L} \rightarrow \mathcal{G} \) in terms of \( g^*E \) has objects all pairs \((E, c), E \in \mathcal{E} \) and \( c \in \gamma_1 \lambda^*E \). Morphisms \((E, c) \rightarrow (E', c')\) are morphisms \( E \rightarrow E' \) such that \( \gamma_1 \lambda^*(x)(c) = c' \). Define \( II : C \rightarrow \mathcal{H} \) such that for \((E, c) \in C\), \( II(E, c) \) is the pullback

\[
\begin{array}{ccc}
H(E, c) & \xrightarrow{\beta e} & 1 \\
\downarrow y^*E & & \downarrow \beta \langle 1, \psi \rangle^*(1 \times \varphi), E \\
y^*(1 \times \eta) & & \lambda \delta \simeq \xi
\end{array}
\]

in \( \mathcal{G} \). Here, \( y \) denotes \( \langle \beta, \xi \rangle \), and \( \eta \) denotes the unit of \( \varphi_1 \rightarrow \varphi^* \). It follows that \( H \) is flat and cover preserving, and so determines a geometric morphism \( \mathcal{H} \rightarrow \mathcal{L} \). It is immediate that \( (\beta, \xi) \simeq (\gamma, \lambda) \delta \), and hence there is an isomorphism \( j : \lambda \delta \simeq \xi \). Then there is an isomorphism \( k : y \delta \simeq p_0(\gamma, \lambda) \delta \simeq p_0(\beta, \xi) \simeq \beta \), and we have \( t \cdot \varphi j = \psi k \cdot \eta \delta \).
Theorem 3.6. 1. (Pitts [26]) If the lower leg in a comma square in \( \text{Top}_{\mathcal{S}} \) is essential, then the geometric morphism opposite it is locally connected. Moreover, the Beck–Chevalley condition is satisfied.

2. For toposes \( \mathcal{E}, \mathcal{F} \) over \( \mathcal{S} \), let \( c\mathcal{S}_{\mathcal{F}}^{\mathcal{E}} \) (see below) denote the category of complete spreads over \( \mathcal{F} \times_{\mathcal{S}} \mathcal{E} = g^{*}\mathcal{E} \) with locally connected \( \mathcal{S} \)-domain. Then the functor which associates with a geometric morphism \( \mathcal{F} \to M\mathcal{E} \) the complete spread \((\gamma, \lambda): \mathcal{L} \to g^{*}\mathcal{E}, \) where

\[
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{\delta} & M\mathcal{E} \\
\downarrow{\gamma} & & \downarrow{\rho} \\
\mathcal{L} & \xrightarrow{\lambda} & \mathcal{F}
\end{array}
\]

is a comma square, is an equivalence \( \text{Top}_{\mathcal{S}}(\mathcal{F}, M\mathcal{E}) \cong c\mathcal{S}_{\mathcal{E}}^{\mathcal{F}}. \)

Proof. 1 is clear from the proof of Proposition 3.5. The passage described in statement 2 is isomorphic to the series of equivalences

\[
\text{Top}_{\mathcal{S}}(\mathcal{F}, M\mathcal{E}) \cong \text{Cocts}_{\mathcal{S}}(\mathcal{E}, \mathcal{F}) \cong \text{Cocts}_{\mathcal{S}}(g^{*}\mathcal{E}, \mathcal{F}) \cong c\mathcal{S}_{\mathcal{E}}^{\mathcal{F}},
\]

where the last equivalence is by Theorem 2.13. \( \square \)

The lower power locale \( P_{L}(X) \) enters also into the characterization of open locales [32]. This result says that a locale \( X \) is open iff the locale-based fibration induced by \( P_{L}(X) \) has a terminal object. We will establish a corresponding result for toposes, replacing \( P_{L}(X) \) by \( M\mathcal{E} \) and “open” by “locally connected”. To do this, we first set down some terminology.

Let \( \mathcal{X} \) denote an arbitrary 2-category. By a fibration on \( \mathcal{X} \) we mean a homomorphic bifunctor \( \mathcal{A}: \mathcal{X}^{\text{op}} \to \text{CAT.} \) For every 0-cell \( K \in \mathcal{X}, \) we denote the category \( \mathcal{A}(K) \) by \( \mathcal{A}_{K}, \) for every 1-cell \( f: J \to K, \) the functor \( \mathcal{A}(f) \) by \( f^{*}, \) and for every 2-cell \( f \Rightarrow g, \) the natural transformation \( \mathcal{A}(t) \) by \( t^{*}. \) For any 0-cell \( K \in \mathcal{X}, \) the representable fibration \( \mathcal{A}(-, K) \) will be denoted by \( \mathcal{K}. \)

For any topos \( \mathcal{E} \to \mathcal{S}, \) we now define a fibration on \( \text{Top}_{\mathcal{S}} \) which we denote by \( c\mathcal{S}_{\mathcal{E}}. \) For \( \mathcal{F} \to \mathcal{S}, \) the fiber \( c\mathcal{S}_{\mathcal{E}}^{\mathcal{F}} \) is the category of complete spreads over \( \mathcal{F} \times_{\mathcal{S}} \mathcal{E} = g^{*}\mathcal{E} \) with locally connected \( \mathcal{S} \)-domain. For any geometric morphism \( \mathcal{F} \to \mathcal{G}, \) the functor \( p^{*} \) (not to be confused with the inverse image functor \( p^{*} \)) associates with a complete spread \( \mathcal{L} \to g^{*}\mathcal{E} \) the pullback

\[
\begin{array}{ccc}
\mathcal{M} & \to & \mathcal{L} \\
p^{*}\varphi & \downarrow{\varphi} & \downarrow{g^{*}\varphi} \\
f^{*}\mathcal{E} & \xrightarrow{p^{*}\mathcal{E}} & \mathcal{G}^{*}\mathcal{E}.
\end{array}
\]
In view of Theorem 3.6(2), the left side of this pullback is a complete spread (with locally connected \( \mathcal{F} \)-domain) since the composite of a pullback on the left of a comma square is again a comma square. For any natural transformation \( p \Rightarrow q \) between geometric morphisms over \( \mathcal{F} \), there is a natural transformation \( t^* \) such that for any complete spread \( \mathcal{L} \xrightarrow{\varphi} \mathcal{G} \), \( t^* \) is a geometric morphism \( t^*_\varphi : p^*(\varphi) \Rightarrow q^*(\varphi) \) over \( f^* \).

**Corollary 3.7.** \( cS \mathcal{F} \) is naturally equivalent to the representable fibration \( \overline{M \mathcal{F}} \). This equivalence is given at “stage” \( \mathcal{G} \xrightarrow{\varphi} \mathcal{F} \) by the comma square construction described in Theorem 3.6(2).

The comma pair \( (I, A) \downarrow (J, B) \) of two objects \( (I, A) \) and \( (J, B) \) in a fibration \( \mathcal{A} \) on \( \mathcal{K} \) consists of data \( (P, p, q, u) \), where

\[
P \xrightarrow{q} J \\
p \downarrow \\
I
\]

are 0-cells and 1-cells in \( \mathcal{K} \), and \( p^*A \xrightarrow{i} q^*B \) is a morphism in \( \mathcal{A}^P \), which is universal in the sense that if \( (H, m, n, t) \) are any other such data, then there is an essentially unique 1-cell \( H \xrightarrow{f} P \) and isomorphisms \( m \cong pf \), \( n \cong qf \) such that

\[
m^*A \xrightarrow{i} n^*B \\
\downarrow_{i^*} \downarrow_{j^*} \\
f^*p^*A \xrightarrow{f^*u} f^*q^*B
\]

commutes. The next two lemmas follow immediately from the definitions.

**Lemma 3.8.** A diagram

\[
P \xrightarrow{q} J \\
p \downarrow \downarrow \\
I \xrightarrow{u} K \\
\uparrow_{a} \uparrow_{b}
\]

is a comma square in \( \mathcal{K} \) iff \( (P, p, q, u) \) is the comma pair \( (I, a) \downarrow (J, b) \) in the fibration \( \mathcal{K} \).

A fibration \( \mathcal{A} \) will be said to have a terminal object if for every 0-cell \( H \in \mathcal{K} \), the category \( \mathcal{A}^H \) has a terminal object \( T_H \), and furthermore, if these terminal objects are stable under the transition functors, i.e., if for every 1-cell \( H \xrightarrow{f} I \) in \( \mathcal{K} \), the unique
morphism \( f^*(T_1) \to T_H \) is an isomorphism. (If \( \mathcal{K} \) has a pseudo-terminal object \( 1_\mathcal{K} \), then for any \( K \in \mathcal{K} \), the representable \( \hat{K} \) has a terminal object iff the essentially unique 1-cell \( K \to 1_\mathcal{K} \) has a right adjoint.)

**Lemma 3.9.** Assume that the 2-category \( \mathcal{K} \) has pseudo-products. Let \( \mathcal{A} \) be a fibration on \( \mathcal{K} \), and assume that \( \mathcal{A} \) has a terminal object. Then for any objects \( I, J \in \mathcal{K} \), and any object \( A \in \mathcal{A} \), the representable \( K \to \mathcal{X} \) has a terminal object iff the essentially unique 1-cell \( K \to 1_\mathcal{K} \) has a right adjoint.

**Theorem 3.10.** Let \( \mathcal{E} \xrightarrow{\gamma} \mathcal{P} \) be an object of \( \text{Top}_{\mathcal{A}} \). Then \( \gamma \) is locally connected iff the representable fibration \( M\mathcal{E} \) on \( \text{Top}_{\mathcal{A}} \) has a terminal object.

**Proof.** Assume that \( M\mathcal{E} \) has a terminal object. In particular, this says that there is a terminal object \( T \in \text{Top}_{\mathcal{A}}(\mathcal{P}, M\mathcal{E}) \). Let \( u \) denote the unique natural transformation \( \delta \to T \cdot \gamma \), where \( \mathcal{E} \xrightarrow{\delta} M\mathcal{E} \) is the canonical essential inclusion. By Lemmas 3.8 and 3.9, the square

\[
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{\gamma} & \mathcal{P} \\
\downarrow{\delta} & & \downarrow{T} \\
\mathcal{E} & \xrightarrow{\delta} & M\mathcal{E}
\end{array}
\]

is a comma square in \( \text{Top}_{\mathcal{A}} \), and so by Theorem 3.6(1), \( \gamma \) is locally connected.

Conversely, assume that \( \gamma \) is locally connected. Then the fibration \( c\mathcal{E} \) has a terminal object. In fact, for any topos \( \mathcal{E} \xrightarrow{g} \mathcal{P} \), the terminal object \( T_{\mathcal{E}} \) in \( c\mathcal{E} \) is the complete spread \( g^*\mathcal{E} \xrightarrow{1} g^2\mathcal{E} \). Moreover, it is clear that for any geometric morphism \( \mathcal{F} \xrightarrow{p} \mathcal{E} \), we have \( p^*(T_{\mathcal{E}}) \cong T_{\mathcal{F}} \). By Corollary 3.7, the proof of Theorem 3.10 is complete. \( \square \)

The essential inclusion \( \delta \) factors through the bagdomain \( \mathcal{B}_L\mathcal{E} \) by essential inclusions as in the following diagram:

\[
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{\delta} & M\mathcal{E} \\
& \searrow \quad \nwarrow & \\
& \mathcal{B}_L\mathcal{E} &
\end{array}
\]
These morphisms are induced by corresponding site inclusions

$$(C,J) \rightarrow (C_{fp}, J_{fp}) \rightarrow (C^*, J^*),$$

where $(C,J)$ is a site definition of $\mathcal{E}$, $C_{fp}$ is the finite products completion of $C$ (this site for $\mathcal{B}_1\mathcal{E}$ is described in [16]), and $C^*$ is the lex completion of $C$ (the site for $M\mathcal{E}$ in terms of one for $\mathcal{E}$ [5, 6]).

From the point of view of the models of the theories which these toposes classify, the geometric morphism $\mathcal{E} \rightarrow \mathcal{B}_1\mathcal{E}$ corresponds to forgetting that a lex cocontinuous functor $\mathcal{E} \rightarrow \mathcal{B}$ preserves the terminal object, whereas $\mathcal{B}_1\mathcal{E} \rightarrow M\mathcal{E}$ corresponds to forgetting that a pullback preserving cocontinuous functor preserves pullbacks. For this, recall [31, 16] that $\mathcal{B}_1\mathcal{E}$ classifies bags of points of $\mathcal{E}$, or partial points (i.e., pullback-preserving cocontinuous functors $\mathcal{E} \rightarrow \mathcal{B}$ over $\mathcal{S}$), and that $M\mathcal{E}$ classifies distributions on $\mathcal{E}$ (i.e., cocontinuous functors $\mathcal{E} \rightarrow \mathcal{B}$).

The above comparison can also be made in terms of complete spreads. $M$ classifies complete spreads with locally connected domains, and $\mathcal{B}_1$ classifies complete spreads whose domains have totally connected components, in the following sense.

**Definition 3.11.** A topos $\mathcal{D} \xrightarrow{d} \mathcal{S}$ will be said to be totally connected (have totally connected components) if it is locally connected and the left adjoint $d_!$ is left exact (preserves pullbacks).

**Proposition 3.12.** A topos $\mathcal{D} \xrightarrow{d} \mathcal{S}$ has totally connected components iff there is an $I \in \mathcal{S}$ and a factorization $\mathcal{D} \rightarrow \mathcal{S}/I \rightarrow \mathcal{S}$ such that the $\text{Top}_{\mathcal{S}/I}$-fibration $\mathcal{D}$ has a terminal object (in which case $I$ is uniquely determined as $d_1$). Such a topos has a pure dense bag of points, i.e., there is an $I \in \mathcal{S}$, and a pure dense geometric morphism $\mathcal{S}/I \rightarrow \mathcal{D}$ over $\mathcal{S}$. If a topos $\mathcal{D}$ over $\mathcal{S}$ has a pure dense bag of points, and furthermore has the property that its definable subobjects are closed under arbitrary union (see Proposition 2.7), then $\mathcal{D}$ has totally connected components.

**Proof.** Assume that $\mathcal{D}$ has totally connected components. Let $I = d_1$, and factor $d$ as $\mathcal{D} \xrightarrow{\tilde{d}} \mathcal{S}/I \rightarrow \mathcal{S}$, where $\tilde{d}$ is locally connected such that $\tilde{d}_!$ is lex. Thus, there is a geometric morphism $\mathcal{S}/I \xrightarrow{p} \mathcal{D}$ such that $p^*$ is $\tilde{d}_!$ and $p_*$ is $\tilde{d}^*$. Then for any $\mathcal{D} \xrightarrow{g} \mathcal{S}/I$, $p \circ g$ is terminal in $\text{Top}_{\mathcal{S}/I}(\mathcal{D}, \mathcal{D})$ since its corresponding distribution $g^* \cdot p^* = g^* \cdot \tilde{d}_!$ is the terminal such (its corresponding $\mathcal{S}/I$-complete spread over $\mathcal{D}$ is the identity on $\mathcal{D} \times \mathcal{D}$). Now assume that there is an $I \in \mathcal{S}$ and a factorization $\tilde{d}$ through $\mathcal{S}/I$ such that $\mathcal{D}$ has a terminal object as a $\text{Top}_{\mathcal{S}/I}$-fibration. In particular, there is a terminal geometric morphism $\mathcal{S}/I \xrightarrow{p} \mathcal{D}$. We want to show that $p^* \dashv \tilde{d}^*$ so that $\tilde{d}$ is locally connected with $\tilde{d}_!$ lex, whence $\mathcal{D}$ has totally connected components. There is a (unique) natural transformation $1_{\mathcal{D}} \xrightarrow{t} \tilde{d}^* \cdot p^*$ since $p$ is stably terminal. Let $i$ denote the structure isomorphism $p^* \cdot \tilde{d}^* \simeq 1$. The triangle identity on the left below holds since $t$ is over $\mathcal{S}/I$. The identity on the right holds since there is only
one natural transformation $p^* \rightarrow p^*$.

\[
\begin{array}{c}
\begin{array}{c}
d^* \to d^*p^*d^* \quad p^* \to p^*d^*p^*
\end{array}
\end{array}
\]

If $\mathcal{D}$ has totally connected components, then the geometric morphism $p$ is pure dense over $\mathcal{S}$ since from $p^* = d!$ we obtain $\Sigma_1 \cdot p^* \simeq d_1$. Conversely, but under the stated proviso, a topos $\mathcal{D}$ with a pure dense bag of points is locally connected with $\Sigma_1 \cdot p^* \simeq d_1$ (Proposition 2.7). Thus, $d_!$ preserves pullbacks, i.e., $\mathcal{D}$ has totally connected components. \(\blacksquare\)

Note that a pure dense geometric morphism $\mathcal{S}/I \rightarrow \mathcal{D}$ over $\mathcal{S}$ is an inclusion. This is because any geometric morphism $\mathcal{S}/I \rightarrow \mathcal{D}$ is a spread over $\mathcal{S}$, so that Proposition 2.4(3), applies.

**Corollary 3.13.** An $\mathcal{S}$-topos $\mathcal{E}$ is locally connected iff $M\mathcal{E}$ is totally connected.

**Proof.** This follows from Proposition 3.12 and Theorem 3.10. \(\blacksquare\)

**Theorem 3.14.** $\mathcal{B}_L$ classifies complete spreads whose domains have totally connected components. Equivalently, a geometric morphism $\mathcal{S} \xrightarrow{p} M\mathcal{E}$ factors through $\mathcal{B}_L\mathcal{E}$ iff the $\mathcal{G}$-domain of the corresponding complete spread $\mathcal{L} \rightarrow g^*\mathcal{E}$ for the comma square

\[
\begin{array}{ccc}
\mathcal{S} & \xrightarrow{\delta} & M\mathcal{E} \\
\downarrow & & \downarrow \rho \\
\mathcal{E} & \xrightarrow{\mathcal{B}_L\mathcal{E}} & \mathcal{S}/I
\end{array}
\]

has totally connected components (in which case the resulting inside square is a comma square).

**Proof.** $\mathcal{B}_L\mathcal{E}$ classifies bags of points of $\mathcal{E}$. For example, points $\mathcal{S} \rightarrow \mathcal{B}_L\mathcal{E}$ correspond to diagrams

\[
\begin{array}{ccc}
S/I & \xrightarrow{\phi} & \mathcal{E} \\
\downarrow & & \downarrow \epsilon \\
\mathcal{S} & \xrightarrow{\iota} & \mathcal{S}/I
\end{array}
\]

(3)
φ in diagram (3). $L'$ is the locally connected middle topos in the pure dense/complete spread factorization of φ, so that $L' \rightarrow \mathcal{E}$ is a complete spread whose domain has totally connected components (by Proposition 3.12). On the other hand, if we start with such a complete spread $L \rightarrow \mathcal{E}$, then we know there is a (pure dense) geometric morphism $\mathcal{F}/\mathcal{L}_{1} \rightarrow \mathcal{E}$ which when composed with $L \rightarrow \mathcal{E}$ gives a bag of points of $\mathcal{E}$. These two passages are mutual inverses (up to isomorphism).

Theorem 3.17 below expresses another connection between $M$ and $\mathcal{B}_{L}$. Denote by $C\mathcal{E}$ the topos classifier of probability distributions on $\mathcal{E}$, i.e., of cocontinuous functors $\mathcal{E} \rightarrow \mathcal{G}$ which preserve 1. Probability distributions correspond to complete spreads with connected locally connected domain. This follows by Theorem 2.13 in view of the fact that a locally connected geometric morphism $\gamma$ is connected iff $\gamma_{!}1 \simeq 1$.

**Theorem 3.15.** For any topos $\mathcal{E}$ over $\mathcal{F}$, there is a subtopos $C\mathcal{E}$ of $M\mathcal{E}$ which classifies probability distributions. Furthermore, there is a factorization

$$
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{\delta} & M\mathcal{E} \\
\downarrow \delta & & \downarrow 1 \\
C\mathcal{E} & \xrightarrow{\delta} & \mathcal{E}
\end{array}
$$

where $\delta$ is essential and satisfies $\delta_{!}1 \simeq 1$. A geometric morphism $\mathcal{G} \rightarrow M\mathcal{E}$ factors through $C\mathcal{E}$ iff in the comma square

$$
\begin{array}{ccc}
L & \xrightarrow{\gamma} & \mathcal{G} \\
\downarrow \gamma & & \downarrow \nu \\
\mathcal{E} & \xrightarrow{\delta} & M\mathcal{E}
\end{array}
$$

$\gamma$ is connected (in which case the resulting inside square is a comma square).

**Proof.** Let $C\mathcal{E}$ denote the subtopos of $M\mathcal{E}$ given by the least topology forcing the morphism $\delta_{!}1 \rightarrow 1$ to be an isomorphism [11, 3.59 (iv)]. Then a geometric morphism $\mathcal{G} \rightarrow M\mathcal{E}$ factors (uniquely) through $C\mathcal{E}$ iff $\rho_{!}\delta_{!}1 \simeq 1$ iff $\gamma_{!}1 \simeq \gamma_{!}\delta_{!}1 \simeq 1$ iff the locally connected $\gamma$ is connected. Note that $\delta$ factors through $C\mathcal{E}$ since $\delta_{!}\delta_{!}1 \simeq 1$. We have $\delta_{!} = \delta_{!}i_{*}$, and since $i_{!}\delta_{!}1 \simeq \delta_{!}i_{*}$, $\delta$ is essential with $\delta_{!}1 \simeq i_{!}\delta_{!}1 \simeq 1$.

We remark that in terms of models of the theories classified by these toposes, $\delta$ corresponds to forgetting that a lex cocontinuous functor $\mathcal{E} \rightarrow \mathcal{G}$ (i.e., a geometric morphism $\mathcal{G} \rightarrow \mathcal{E}$) preserves pullbacks, and the second factor corresponds to forgetting that an 1-preserving cocontinuous functor $\mathcal{E} \rightarrow \mathcal{G}$ (i.e., a geometric morphism $\mathcal{G} \rightarrow C\mathcal{E}$) preserves 1.

The proof of Theorem 3.17, given below, relies on the following.

**Lemma 3.16.** Let $\mathcal{E} \rightarrow \mathcal{F}$ have locally connected domain. Then in the canonical factorization

$$
\mathcal{E} \xrightarrow{\phi} \mathcal{F}
$$
\[ \varphi \text{ is a complete spread over } \mathcal{S} \text{ iff } \psi \text{ is a complete spread over } \mathcal{S}/e_1. \] (Here, \( \psi^* \) is given by pullback of \( \varphi^* \) along the unit \( \eta_1 : 1 \to e^*e_1.1. \))

**Proof.** Let \( I \) denote \( e_1 \), and let \((D,J_f)\) be a site for \( \mathcal{F} \) over \( \mathcal{S} \). We compare the complete spread site for \( \psi \) over \( \mathcal{S}/I \), but described from the point of view of \( \mathcal{S} \), with the complete spread site for \( \varphi \) over \( \mathcal{S} \). In order to do this, first observe that a morphism \( A \to B \) in \( \mathcal{E} \) is \( \mathcal{S}/I \)-definable iff it is \( \mathcal{S} \)-definable. In fact, this is clear from the following diagram in which the bottom two squares are pullbacks:

\[
\begin{array}{ccc}
A & \xrightarrow{e^*e,A} & B \\
\downarrow \phi_{e,A} & & \downarrow \phi_{e,B} \\
\phi_{e,A} & \xrightarrow{e^*e_1} & \phi_{e,B} \\
\end{array}
\]

Next, we analyze \( \psi \). Given \( X \xrightarrow{\phi} f^*I \), and a component \( 1 \xrightarrow{c} I \), consider the pullback

\[
\begin{array}{ccc}
X & \xrightarrow{x} & X \\
\downarrow \phi & & \downarrow \phi \\
1 & \xrightarrow{f^*} & f^*I \\
\end{array}
\]

Then the action of \( \psi \) in its relation to \( \varphi \) is summed up in the following diagram:

\[
\begin{array}{ccc}
\psi^*x & \xrightarrow{b} & \psi^*x \\
\downarrow a & & \downarrow a \\
\varphi^*X & \xrightarrow{c} & \varphi^*X \\
\downarrow \phi^* & & \downarrow \phi^* \\
1 & \xrightarrow{\eta_1} & 1 \\
\end{array}
\]

Every face in this diagram is a pullback, and the morphism \( a_c \) is a definable monomorphism since \( c \xrightarrow{1} 1 \) is. Here, \( x_c \) is the object \( X_c \to 1 \xrightarrow{f^*} f^*I \). An object of the complete
spread site for $\psi$ is an $I$-indexed family $X \xrightarrow{f} f^* I$ of objects of $D$ paired with an $\mathcal{F}/I$-definable (equivalently, $\mathcal{F}$-definable) subobject of $\psi^* x$, but viewed, by pullback along the $b_c$'s, as an $I$-indexed family of $\mathcal{F}$-definable subobjects $\{S_c \hookrightarrow \psi^* x_c \mid c \in I\}$. If $\psi$ is a spread over $\mathcal{F}/I$, then these families generate $\mathcal{E}$ over $\mathcal{F}/I$. Since the composite of a definable $S_c \hookrightarrow \psi^* x_c$ with the corresponding $a_c$ yields a definable subobject of $\phi^* X_c$, it follows that the definable subobjects of the $\phi^* X_c$, now considered individually, generate $\mathcal{E}$ over $\mathcal{F}$. This says that $\phi$ is a spread. Conversely, if $\phi$ is a spread, then by considering the constant families $X \times f^* I \to f^* I, X \in D$, we see that $\psi$ is a spread over $\mathcal{F}/I$.

A similar analysis can be given for completeness. □

Theorem 3.17. Let $\mathcal{E}$ denote an arbitrary topos over $\mathcal{F}$. Then $M\mathcal{E} \simeq \mathcal{B}_{L}(C(\mathcal{E}))$.

Proof. Let $c\mathcal{S}_\mathcal{E}$ denote the fibration of complete spreads with connected locally connected domain. It will be shown that the fibration $c\mathcal{S}_\mathcal{E}$ is the free $\mathcal{F}$-coproduct completion [16] of $c\mathcal{S}_\mathcal{F}$. I.e., it will be shown that there is an equivalence

\[
\text{Fam}_\mathcal{E}(c\mathcal{S}_\mathcal{F}) \simeq c\mathcal{S}_\mathcal{E}
\]  

of $\mathcal{F}$-indexed categories, which is natural in $\mathcal{E}$. Then the theorem follows from Theorems 3.15 and 3.6 (2). A typical object of $\text{Fam}_\mathcal{E}(c\mathcal{S}_\mathcal{F})$ (in the fiber over $1 \in \mathcal{E}$) is a pair $(J \in \mathcal{E}, \psi)$, where in

\[
\begin{array}{ccc}
\mathcal{L} & \xrightarrow{\psi} & g^* \mathcal{E} \\
\downarrow \gamma & & \downarrow \xi \\
\mathcal{F}/J & \xrightarrow{\rho} & \mathcal{E}
\end{array}
\]  

$\psi$ is complete spread over $\mathcal{F}/J$ and $\gamma$ is connected locally connected. By Lemma 3.16, the outer perimeter of this diagram describes an object of $c\mathcal{S}_\mathcal{E}$. Conversely, given a typical object of $c\mathcal{S}_\mathcal{E}$, as described by the perimeter of (5), we factor $\mathcal{L} \xrightarrow{\gamma} \mathcal{F}$ into a connected locally connected morphism followed by a local homeomorphism. The front half of this factorization produces, again by Lemma 3.16, an object of $\text{Fam}_\mathcal{E}(c\mathcal{S}_\mathcal{F})$ (with $J = \gamma_1 1$). The uniqueness of the clc/lh factorization gives the equivalence (4). □

The following diagram, depicting the two canonical factorizations of $\mathcal{E} \xrightarrow{\delta} M\mathcal{E}$, summarizes the situation:

\[
\begin{array}{ccc}
\mathcal{B}_{1}\mathcal{E} & \xrightarrow{M\mathcal{E}} & C\mathcal{E}
\end{array}
\]
In terms of the domains of the complete spreads classified, this diagram corresponds to the following:

We conclude with a comment concerning another topos $C'\mathcal{E}$ for which it is also true that

$$M(\mathcal{E}) \cong \mathcal{B}_L(C'(\mathcal{E})).$$

Given a site $(C,J_e)$ for $\mathcal{E}$, there is induced a site whose underlying category $T$ is taken to be the finite connected limit completion of $C$. (We use here only that this construction exists [27].) Let $C'\mathcal{E}$ be defined as the topos of sheaves on the site (with underlying category) $T$. By results of Carboni and Johnstone [7], the finite product completion of $T$ must be the lex completion $C^*$ of $C$. There are functors $C \to T \to C^*$ which induce geometric morphisms $\mathcal{E} \to C'\mathcal{E} \to M\mathcal{E}$, and it follows from the previous remark that (6) holds. It seems reasonable that $C'\mathcal{E}$ should also classify probability distributions, and thus be equivalent to $C\mathcal{E}$, but at present this is not known to the authors.

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**References**