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The $\exp(-\Phi(\eta))$-expansion method with application in the (1+1)-dimensional classical Boussinesq equations

Harun-Or- Roshid *, Md. Azizur Rahman

Department of Mathematics, Pabna University of Science and Technology, Bangladesh

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ABSTRACT

Periodic and soliton solutions are presented for the (1+1)-dimensional classical Boussinesq equation which governs the evolution of nonlinear dispersive long gravity wave traveling in two horizontal directions on shallow water of uniform depth. The equation is handled via the $\exp(-\Phi(\eta))$-expansion method. It is worth declaring that the method is more effective and useful for solving the nonlinear evolution equations. In particular, mathematical analysis and numerical graph are provided for those solitons, periodic, singular kink and bell type solitary wave solutions to visualize the dynamics of the equation.

Introduction

Recently nonlinear phenomena have become an interesting and important matter of study to many engineers, researcher and scientists. Nonlinear phenomena consist in plasma physics, fluid dynamics, mechanics and optical fibers, etc. The traveling wave solutions of nonlinear partial differential equations (NPDEs) play an important role in study of nonlinear physical phenomena.

In recent year many effective methods have been achieved such as the Adomian decomposition method [1], Ansatz method [2–4], Semi-inverse variation principle [5], the tanh method [6], the auxiliary equation method [7], the Darboux transformation method [8], the Backlund transformation method [9], the homogeneous balance method [10], the F-expansion method [11], the Jacobi elliptic function expansion [12], the (G'/G)-expansion method [13–18]. Many authors have studied the Boussinesq equation [19–23] and classical Boussinesq equations [18,19,24] in recent years because of their importance of applications in several areas of interest. Solitary wave solutions of Boussinesq equation in a power law media are investigated by Biswas et al. [19]. Ebadi et al. [20] examined the solitons and other nonlinear waves for the perturbed Boussinesq equation with power law nonlinearity. In the last year, Jawad et al. [21] examined the dynamics of shallow water waves with Boussinesq equation. Solitary wave and shock wave solutions of the variants Boussinesq equation are investigated by Triki et al.[22]. Very recently, Biswas et al. [23] obtained the soliton solutions to the Boussinesq equation with the effect of surface tension and the power law nonlinearity is considered for this equations. They also analyzed bifurcation of the Boussinesq equation with power law nonlinearity and dual-dispersion. Gepreel [17] has found some traveling wave solutions by using generalized (G'/G)-expansion method for some nonlinear evolution equations including the (1+1)-dimensional classical Boussinesq equations. Zayed and Joudi [18] has found traveling wave solutions more than [17] by using an extended (G'/G)-expansion method for the (1+1)-dimensional classical Boussinesq equations. The (1+1)-dimensional classical Boussinesq system has been derived by Wu and Zhang [24] which modeling nonlinear and dispersive long gravity wave traveling in two horizontal directions on shallow water of uniform depth. Recently, using the $\exp(-\Phi(\eta))$-expansion method [25] have obtained the exact traveling wave solutions of some nonlinear evolution equations. This method is very easy to implement and calculate and also gives new exact travelling solutions.

In this article, we use the $\exp(-\Phi(\eta))$-expansion method to find some exact new traveling wave solutions of the (1+1)-dimensional classical Boussinesq equations. The outline of this paper is as follows: Section 'Description of the $\exp(-\Phi(\eta))$-expansion method' contains the brief description of the $\exp(-\Phi(\eta))$-expansion method. In Section 'Application of the method', we find the solutions of the (1+1)-dimensional classical Boussinesq equations via the $\exp(-\Phi(\eta))$-expansion method. Section 'Physical explanation' contains the results and discussion. In Section 'Comparison' we compare our results with results of other's existing in the literature. Finally, Conclusions are given in the end at Section 'Conclusion'.

* Corresponding author.
E-mail address: harunorroshidmd@gmail.com (H.-O. Roshid).

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Description of the exp($-\Phi(\eta)$)-expansion method

Let us consider a general nonlinear PDE in the form

$$F(v, v_t, v_x, v_{xx}, v_{tt}, v_{tx}, \ldots) = 0, \tag{1}$$

where, $v = v(x,t)$ is an unknown function, F is a polynomial in $v(x,t)$ and its derivatives in which highest order derivatives and nonlinear terms are involved and the subscripts stand for the partial derivatives. In the following, we give the main steps of this method:

Step 1: We combine the real variables x and t by a complex variable η,

$$v(x,t) = v(\eta), \quad \eta = x \pm Vt, \tag{2}$$

where V is the speed of the traveling wave. The traveling wave transformation (2) converts Eq. (1) into an ordinary differential equation for $v = v(\eta)$:

$$\Re(v, v', v'', v''', \cdots) = 0, \tag{3}$$

where, $\Re$ is a polynomial of v and its derivatives and the superscripts indicate the ordinary derivatives with respect to η.

Step 2: Suppose the traveling wave solution of Eq. (3) can be expressed as follows:

$$v(\eta) = \sum_{i=0}^{N} A_i(\exp(-\Phi(\eta)))^i, \tag{4}$$

where, $A_i (0 \leqslant i \leqslant N)$ are constants to be determined, such that $A_N \neq 0$ and $\Phi = \Phi(\eta)$ satisfies the following ordinary differential equation:

$$\Phi'(\eta) = \exp(-\Phi(\eta)) + \mu\exp(\Phi(\eta)) + \lambda, \tag{5}$$

Eq. (5) gives the following solutions:

Family 1: When $\mu \neq 0$, $\lambda^2 - 4\mu > 0$,

$$\Phi(\eta) = \ln\left(\frac{-\sqrt{(\lambda^2 - 4\mu)}\tanh\left(\frac{\sqrt{(\lambda^2-4\mu)}}{2}(\eta + E)\right) - \lambda}{2\mu}\right) \tag{6}$$

Family 2: When $\mu \neq 0$, $\lambda^2 - 4\mu < 0$,

$$\Phi(\eta) = \ln\left(\frac{\sqrt{(4\mu - \lambda^2)}\tan\left(\frac{\sqrt{(4\mu-\lambda^2)}}{2}(\eta + E)\right) - \lambda}{2\mu}\right) \tag{7}$$

Family 3: When $\mu = 0$, $\lambda \neq 0$, and $\lambda^2 - 4\mu > 0$,

$$\Phi(\eta) = -\ln\left(\frac{\lambda}{\exp(\lambda(\eta + E)) - 1}\right) \tag{8}$$

Family 4: When $\mu \neq 0$, $\lambda \neq 0$, and $\lambda^2 - 4\mu = 0$,

$$\Phi(\eta) = \ln\left(-\frac{2(\lambda(\eta + E) + 2)}{\lambda^2(\eta + E)}\right) \tag{9}$$

Family 5: When $\mu = 0$, $\lambda = 0$, and $\lambda^2 - 4\mu = 0$,

$$\Phi(\eta) = \ln(\eta + E) \tag{10}$$

$A_N, \cdots\cdots, V, \lambda, \mu$ are constants to be determined latter, $A_N \neq 0$, the positive integer N can be determined by considering the homogeneous balance between the highest order derivatives and the nonlinear terms appearing in Eq. (3).

Step 3: We substitute Eq. (4) into Eq. (3) and then we account the function $\exp(-\Phi(\eta))$. As a result of this substitution, we get a polynomial of $\exp(-\Phi(\eta))$. We equate all the coefficients of same power of $\exp(-\Phi(\eta))$ to zero. This procedure yields a system of algebraic equations whichever can be solved to find $A_N, \cdots\cdots, V, \lambda, \mu$. Substituting the values of $A_N, \cdots\cdots, V, \lambda, \mu$ into Eq. (4) along with general solutions of Eq. (5) completes the determination of the solution of Eq. (1).

Application of the method

In this section, we study the (1+1)-dimensional classical Boussinesq equations [17,18,24]:

$$\begin{aligned} &v_t + [(1+v)u]_x = -\frac{1}{3}u_{xxx} \\ &u_t + uu_x + v_x = 0 \end{aligned} \tag{11}$$

This system has been derived by Wu and Zhang [24] for modeling nonlinear and dispersive long gravity wave traveling in two horizontal directions on shallow water of uniform depth. Gepreel [17] has found some traveling wave solutions by using generalized (G'/G)-expansion method of the same equation while Zayed and Joudi [18] found more traveling wave solutions by using an extended (G'/G)-expansion method. We will solve (11) by the exp($-\Phi(\eta)$)-expansion method.

We utilize the traveling wave variables $v(\eta) = v(x,t)$, $\eta = x - Vt$, Eq. (11) is carried into following ODEs:

$$\begin{aligned} &-Vv' + [(1+v)u]' + \frac{1}{3}u''' = 0 \\ &-Vu' + uu' + v' = 0 \end{aligned} \tag{12}$$

Integrating (12) with respect to η once yields

$$K_1 - Vv + (1+v)u + \frac{1}{3}u'' = 0 \tag{13}$$

$$K_2 - Vu + \frac{1}{2}u^2 + v = 0 \tag{14}$$

where, K_1 and K_2 are constants of integration. Considering the homogeneous balance between highest order derivatives and nonlinear terms in Eqs. (13) and (14) we deduce that

$$u(\eta) = A_0 + A_1(\exp(-\Phi(\eta))) \tag{15}$$

$$v(\eta) = B_0 + B_1(\exp(-\Phi(\eta))) + B_2(\exp(-\Phi(\eta)))^2 \tag{16}$$

Substituting Eqs. (15) and (16) into Eq. (13) and then equating the coefficients of exp($-\Phi(\eta)$) to zero, we get

$$\begin{aligned} &-VB_0 + B_0A_0 + K_1 + \frac{1}{3}A_1\mu\lambda + A_0 = 0 \\ &A_1 - VB_1 + B_1A_0 + B_0A_1 + \frac{1}{3}A_1\lambda^2 + \frac{2}{3}A_1\mu = 0 \\ &B_1A_1 + B_2A_0 + A_1\lambda - VB_2 = 0 \\ &\frac{2}{3}A_1 + B_2A_1 = 0 \end{aligned} \tag{17}$$

Substituting Eqs. (15) and (16) into Eq. (14) and then equating the coefficients of exp($-\Phi(\eta)$) to zero, we get

$$\begin{aligned} &-VA_0 + B_0 + \frac{1}{2}A_0^2 + K_2 = 0 \\ &-WA_1 + B_1 + A_0A_1 = 0 \\ &B_2 + \frac{1}{2}A_1^2 = 0 \end{aligned} \tag{18}$$

Solving the Eqs. (17) and (18) yields

$$A_0 = A_0,\ A_1 = \pm\frac{2}{\sqrt{3}},\ B_0 = -1 - \frac{2}{3}\mu,\ B_1 = -\frac{2}{3}\lambda, B_2 = -\frac{2}{3},$$

$$K_1 = -A_0 \pm \frac{\lambda}{\sqrt{3}},\ K_2 = \frac{1}{2}A_0^2 \pm A_0\frac{\lambda}{\sqrt{3}} + 1 + \frac{2}{3}\mu \text{ and } V = A_0 \pm \frac{\lambda}{\sqrt{3}}$$

where λ, μ are arbitrary constants.

Now substituting the values of V, A_0, A_1 into Eq. (15) yields

$$u(\eta) = A_0 \pm \frac{2}{\sqrt{3}}(\exp(-\Phi(\eta))) \tag{19}$$

Again substituting the values of V, B_0, B_1, B_2 into Eq. (16) yields

$$v(\eta) = -\frac{2}{3}\left(\frac{3}{2} + \mu + \lambda(\exp(-\Phi(\eta))) + (\exp(-\Phi(\eta)))^2\right) \tag{20}$$

where, $\eta = x - \left(A_0 \pm \frac{\lambda}{\sqrt{3}}\right)t$

Now substituting Eqs. (6)–(10) into Eqs. (19) and (20) respectively, we get the following ten traveling wave solutions of the (1+1)-dimensional classical Boussinesq equations.

When $\mu \neq 0$, $\lambda^2 - 4\mu > 0$,

One pair:

$$u_1(\eta) = A_0 + \frac{1}{\sqrt{3}} \frac{4\mu}{\sqrt{(\lambda^2 - 4\mu)}\tanh\left(\frac{\sqrt{(\lambda^2-4\mu)}}{2}(\eta + E)\right) + \lambda}$$

$$v_1(\eta) = -\frac{2}{3}\left(\begin{array}{l} \frac{3}{2} + \mu + \frac{2\mu\lambda}{\sqrt{(\lambda^2-4\mu)}\tanh\left(\frac{\sqrt{(\lambda^2-4\mu)}}{2}(\eta+E)\right)+\lambda} \\ + \left(\frac{2\mu}{\sqrt{(\lambda^2-4\mu)}\tanh\left(\frac{\sqrt{(\lambda^2-4\mu)}}{2}(\eta+E)\right)+\lambda}\right)^2 \end{array}\right)$$

where, $\eta = x - \left(A_0 + \frac{\lambda}{\sqrt{3}}\right)t$.

Another pair:

$$u_2(\eta) = A_0 - \frac{1}{\sqrt{3}} \frac{4\mu}{\sqrt{(\lambda^2 - 4\mu)}\tanh\left(\frac{\sqrt{(\lambda^2-4\mu)}}{2}(\eta + E)\right) + \lambda}$$

$$v_1(\eta) = -\frac{2}{3}\left(\begin{array}{l} \frac{3}{2} + \mu + \frac{2\mu\lambda}{\sqrt{(\lambda^2-4\mu)}\tanh\left(\frac{\sqrt{(\lambda^2-4\mu)}}{2}(\eta+E)\right)+\lambda} \\ + \left(\frac{2\mu}{\sqrt{(\lambda^2-4\mu)}\tanh\left(\frac{\sqrt{(\lambda^2-4\mu)}}{2}(\eta+E)\right)+\lambda}\right)^2 \end{array}\right)$$

where, $\eta = x - \left(A_0 - \frac{\lambda}{\sqrt{3}}\right)t$.

When $\mu \neq 0$, $\lambda^2 - 4\mu < 0$,

One pair:

$$u_3(\eta) = A_0 + \frac{1}{\sqrt{3}} \frac{4\mu}{\sqrt{(4\mu - \lambda^2)}\tan\left(\frac{\sqrt{(4\mu-\lambda^2)}}{2}(\eta + E)\right) - \lambda}$$

$$v_2(\eta) = -\frac{2}{3}\left(\begin{array}{l} \frac{3}{2} + \mu + \frac{2\mu\lambda}{\sqrt{(4\mu-\lambda^2)}\tan\left(\frac{\sqrt{(4\mu-\lambda^2)}}{2}(\eta+E)\right)-\lambda} \\ + \left(\frac{2\mu}{\sqrt{(4\mu-\lambda^2)}\tan\left(\frac{\sqrt{(4\mu-\lambda^2)}}{2}(\eta+E)\right)-\lambda}\right)^2 \end{array}\right)$$

where, $\eta = x - \left(A_0 + \frac{\lambda}{\sqrt{3}}\right)t$.

Another pair:

$$u_4(\eta) = A_0 - \frac{1}{\sqrt{3}} \frac{4\mu}{\sqrt{(4\mu - \lambda^2)}\tan\left(\frac{\sqrt{(4\mu-\lambda^2)}}{2}(\eta + E)\right) - \lambda}$$

$$v_2(\eta) = -\frac{2}{3}\left(\begin{array}{l} \frac{3}{2} + \mu + \frac{2\mu\lambda}{\sqrt{(4\mu-\lambda^2)}\tan\left(\frac{\sqrt{(4\mu-\lambda^2)}}{2}(\eta+E)\right)-\lambda} \\ +\left(\frac{2\mu}{\sqrt{(4\mu-\lambda^2)}\tan\left(\frac{\sqrt{(4\mu-\lambda^2)}}{2}(\eta+E)\right)-\lambda}\right)^2 \end{array}\right)$$

where, $\eta = x - \left(A_0 - \frac{\lambda}{\sqrt{3}}\right)t$.

When $\mu = 0$, $\lambda \neq 0$, and $\lambda^2 - 4\mu > 0$,

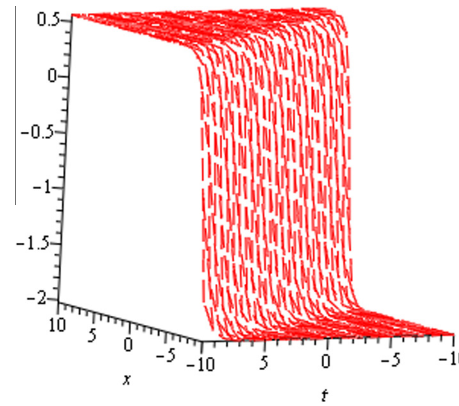

Fig. 1. Kink solution of $u_2(\eta)$, for $a_0 = 1$, $\mu = 1$, $E = 1$, $\lambda = 3$.

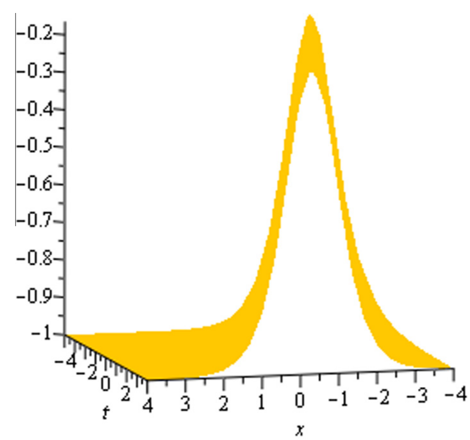

Fig. 2. Bell Shape Soliton solution of $v_1(\eta)$, for $a_0 = 2$, $\mu = 1$, $E = 0.5$, $\lambda = 3$.

One pair:

$$u_5(\eta) = A_0 + \frac{2}{\sqrt{3}} \frac{\lambda}{\exp(\lambda(\eta + E)) - 1}$$

$$v_3(\eta) = -\frac{2}{3}\left(\frac{3}{2} + \mu + \frac{\lambda^2}{\exp(\lambda(\eta+E)) - 1} + \left(\frac{\lambda}{\exp(\lambda(\eta+E)) - 1}\right)^2\right)$$

where, $\eta = x - \left(A_0 + \frac{\lambda}{\sqrt{3}}\right)t$.

Another pair:

$$u_6(\eta) = A_0 - \frac{2}{\sqrt{3}} \frac{\lambda}{\exp(\lambda(\eta + E)) - 1}$$

$$v_3(\eta) = -\frac{2}{3}\left(\frac{3}{2} + \mu + \frac{\lambda^2}{\exp(\lambda(\eta+E)) - 1} + \left(\frac{\lambda}{\exp(\lambda(\eta+E)) - 1}\right)^2\right)$$

where, $\eta = x - \left(A_0 - \frac{\lambda}{\sqrt{3}}\right)t$.

When $\mu \neq 0$, $\lambda \neq 0$, and $\lambda^2 - 4\mu = 0$,

One pair:

$$u_7(\eta) = A_0 + \frac{1}{\sqrt{3}} \frac{\lambda^2(\eta + E)}{(\lambda(\eta + E) + 2)}$$

$$v_4(\eta) = -\frac{2}{3}\left(\frac{3}{2} + \mu - \frac{\lambda^3(\eta+E)}{2(\lambda(\eta+E)+2)} + \left(\frac{\lambda^2(\eta+E)}{2(\lambda(\eta+E)+2)}\right)^2\right)$$

where, $\eta = x - \left(A_0 + \frac{\lambda}{\sqrt{3}}\right)t$.

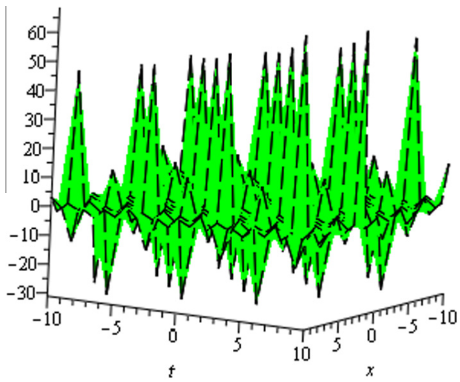

Fig. 3. Periodic solution of $u_3(\eta)$, for $a_0 = 1$, $\mu = 2$, $E = 1$, $\lambda = 2$.

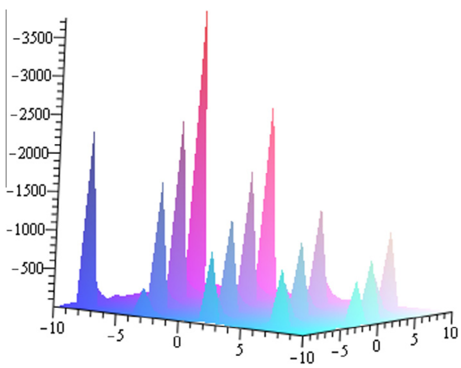

Fig. 4. Periodic solution of $v_2(\eta)$, for $a_0 = 1$, $\mu = 2$, $E = 1$, $\lambda = 2$.

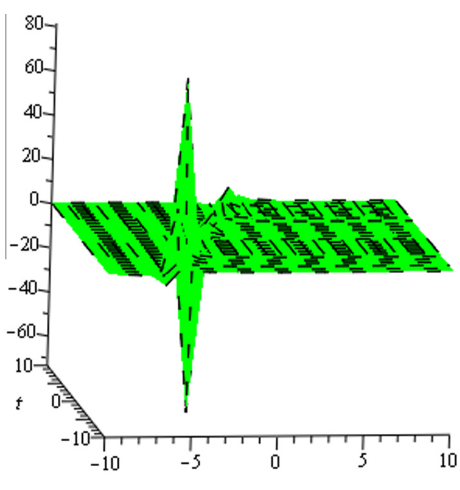

Fig. 5. Singular kink solution of $u_8(\eta)$, for $a_0 = 1.5$, $\mu = 1$, $E = 2$, $\lambda = 2$.

Another pair:

$$u_8(\eta) = A_0 - \frac{1}{\sqrt{3}} \frac{\lambda^2(\eta + E)}{(\lambda(\eta + E) + 2)}$$

$$v_4(\eta) = -\frac{2}{3}\left(\frac{3}{2} + \mu - \frac{\lambda^3(\eta + E)}{2(\lambda(\eta + E) + 2)} + \left(\frac{\lambda^2(\eta + E)}{2(\lambda(\eta + E) + 2)}\right)^2\right)$$

where, $\eta = x - \left(A_0 - \frac{\lambda}{\sqrt{3}}\right)t$.

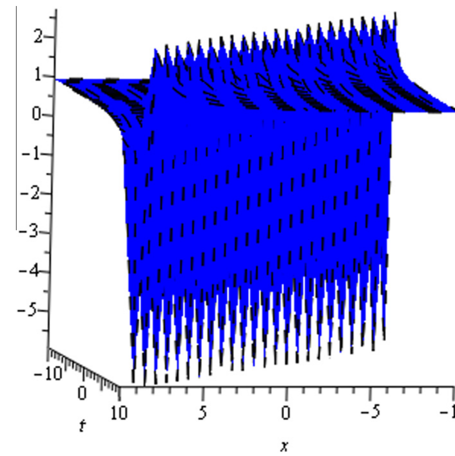

Fig. 6. Singular kink solution of $u_{10}(\eta)$, for $a_0 = 1$, $\mu = 0$, $E = 1$, $\lambda = 0$.

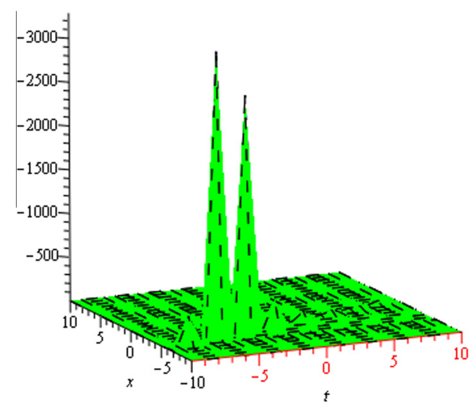

Fig. 7. Soliton solution of $v_4(\eta)$, for $a_0 = 1.5$, $\mu = 1$, $E = 2$, $\lambda = 2$.

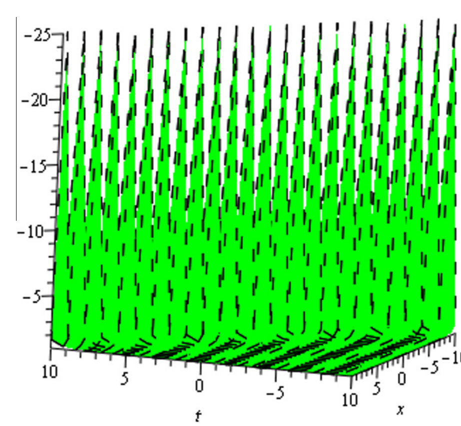

Fig. 8. Soliton solution of $v_5(\eta)$, for $a_0 = 1$, $\mu = 0$, $E = 1$, $\lambda = 0$.

When $\mu = 0$, $\lambda = 0$, and $\lambda^2 - 4\mu = 0$,

One pair:

$$u_9(\eta) = A_0 + \frac{2}{\sqrt{3}} \frac{1}{\eta + E}$$

$$v_5(\eta) = -\frac{2}{3}\left(\frac{3}{2} + \mu + \frac{\lambda}{\eta + E} + \left(\frac{1}{\eta + E}\right)^2\right)$$

where, $\eta = x - A_0 t$.

Table 1
Comparison between Zayed and Joudi [18] and our solutions.

Zayed and Joudi [18]	Our solution
(i) If $A = 0$, $\mu < 0$, $\sigma = 1$ and $B \neq 0$ then from equation (3.40) we obtain $u(\xi) = a_0 + 2\sqrt{-\frac{\mu}{3}}\coth(\sqrt{-\mu}\xi)$ and $v(\xi) = -1 + \frac{2\mu}{3}\csc h^2(\sqrt{-\mu}\xi)$	(i) If $\lambda = 0$, $\eta = \xi$, $E = 0$ then $\sqrt{\lambda^2 - 4\mu} > 0$ becomes $\mu < 0$, and then our solutions $u_2(\eta)$, $v_1(\eta)$ reduced to$u_2(\xi) = A_0 + 2\sqrt{-\frac{\mu}{3}}\coth(\sqrt{-\mu}\xi)$ and $v_1(\xi) = -1 + \frac{2\mu}{3}\csc h^2(\sqrt{-\mu}\xi)$
(ii) If $A = 0$, $\mu > 0$, $\sigma = 1$ and $B \neq 0$ then from equation (3.44) we obtain $u(\xi) = a_0 + 2\sqrt{\frac{\mu}{3}}\cot(\sqrt{\mu}\xi)$ and $v(\xi) = -1 - \frac{2\mu}{3} - \frac{2\mu}{3}\cot^2(\sqrt{\mu}\xi)$	(ii) If $\lambda = 0$, $\eta = \xi$, $E = 0$ then $\sqrt{\lambda^2 - 4\mu} < 0$ becomes $\mu > 0$, and then our solutions $u_3(\eta)$, $v_2(\eta)$ reduced to $u_3(\xi) = A_0 + 2\sqrt{\frac{\mu}{3}}\coth(\sqrt{\mu}\xi)$ and $v_2(\xi) = -1 - \frac{2\mu}{3} - \frac{2\mu}{3}\cot^2(\sqrt{\mu}\xi)$

Another pair:

$$u_{10}(\eta) = A_0 - \frac{2}{\sqrt{3}}\frac{1}{\eta + E}$$

$$v_5(\eta) = -\frac{2}{3}\left(\frac{3}{2} + \mu + \frac{\lambda}{\eta + E} + \left(\frac{1}{\eta + E}\right)^2\right)$$

where, $\eta = x - A_0 t$.

Physical explanation

In this section we will put forth the physical significances and graphical representations of the obtained results of the classical Boussinesq equations.

Results and discussion

It is nice-looking to point out that the delicate balance between the nonlinearity effect of and the dissipative effect of and gives rise to solitons, that after a fully interaction with others, the solitons come back retaining their identities with the same speed and shape. The classical Boussinesq equation has solitary wave solutions with exponentially decaying wings. If two solitons of the classical Boussinesq equation collide, the solitons just overtake through each other and come into view unchanged.

Solutions $u_1(\eta)$ and $u_2(\eta)$ are the kink solution of the classical Boussinesq equation which rise or descent from one asymptotical state at $\eta \to -\infty$ to another asymptotical state at $\eta \to +\infty$. This soliton referred to as topological solitons. Fig. 1 shows the shape of the solitary kink-type solution of the classical Boussinesq equation (only shows the shape of $u_2(\eta)$ with $a_0 = 1$, $\mu = 1$, $E = 1$, $\lambda = 3$). Shape of the solution $u_1(\eta)$ is similar to the Fig. 1 and we omitted its shape. Solution $v_1(\eta)$ is the bell-shaped soliton solution of classical Boussinesq equation. It has infinite wings or infinite tails which referred to as non-topological solitons. This solution does not depend on the amplitude and high frequency. Fig. 2 shows the shape of the exact bell-shaped soliton solution i.e., non-topological soliton solution $v_1(\eta)$ of the classical Boussinesq equation (solution of $v_1(\eta)$, for $a_0 = 2$, $\mu = 1$, $E = 0.5$, $\lambda = 3$). Solutions $u_3(\eta)$, $u_4(\eta)$ and $v_2(\eta)$ are the exact periodic traveling wave solutions (only shows the shape of $u_3(\eta)$, for $a_0 = 1$, $\mu = 2$, $E = 1$, $\lambda = 2$ and $v_2(\eta)$, for $a_0 = 1$, $\mu = 2$, $E = 1$, $\lambda = 2$). Figs. 3 and 4 below shows the periodic solution of $u_3(\eta)$ and $v_2(\eta)$. Figure of solution $u_4(\eta)$ is similar to the figure of $u_3(\eta)$ and for convenience the figure is omitted. Solutions $u_5(\eta)$, $u_6(\eta)$, $u_7(\eta)$, $u_8(\eta)$, $u_9(\eta)$ and $u_{10}(\eta)$ are singular Kink type soliton solutions of classical Boussinesq equation. Figs. 5 and 6 show the shape of singular Kink type soliton solutions of $u_8(\eta)$ and $u_{10}(\eta)$ respectively (we only shows the shape of $u_8(\eta)$, for $a_0 = 1.5$, $\mu = 1$, $E = 2$, $\lambda = 2$ and $u_{10}(\eta)$, for $a_0 = 1$, $\mu = 0$, $E = 1$, $\lambda = 0$) wave speed within the interval. The figure of the solutions $u_{5,6}(\eta)$, $u_7(\eta)$ are similar to the figure of $u_8(\eta)$. The figure of the solutions $u_9(\eta)$ is similar to the figure of $u_{10}(\eta)$ and we omitted the similar figures for convenience. Solution $v_3(\eta)$, $v_4(\eta)$ and $v_5(\eta)$ are the multiple soliton solution. Fig. 7 shows the shape of the soliton solution of $v_4(\eta)$ (only shows the shape of solution of $v_4(\eta)$, for $a_0 = 1.5$, $\mu = 1$, $E = 2$, $\lambda = 2$. We omitted the similar figures for convenience.

Graphical representation

The graphical representations of the achieved solutions for particular values of the arbitrary constants are shown in Figs. 1–8 figures with the aid of commercial software Maple-13.

Comparison

Many author implemented different methods to the classical Boussinesq equations for obtaining traveling wave solutions, such as, Gepreel [17] used the generalized (G'/G)-expansion method for constructing traveling wave solutions. Zayed and Joudi [18] implemented the extended (G'/G)-expansion method for getting exact traveling wave solutions. To the best of our knowledge, the classical Boussinesq equations [18] have not been investigated by the $\exp(-\Phi(\eta))$ method to construct exact traveling wave solutions. Beyond Table 1 Zayed and Joudi [18] obtained other hyperbolic solutions (3.41), (3.42) and (3.43) and trigonometric solutions (3.45) and (3.46). It is worth mentioning that (G'/G)-expansion method is special case of the extended (G'/G)-expansion method. So, comparison our solution with the extended (G'/G)-expansion method is sufficient. Beside this, we achieved our solutions though the $\exp(-\Phi(\eta))$ method with different auxiliary equation while the extended (G'/G)-expansion method performed with others. Thus taking special values of parameters we see that the few solutions (in Table 1) reduce to our solutions i.e. similar to our obtained solutions. But, in this paper the obtained wave solutions u_1, v_1; u_4, v_2; $u_{5,6}$, v_3; $u_{7,8}$, v_4 and $u_{9,10}$, v_5 are completely new and have not been found in the previous literature.

Conclusion

In this article, we have seen that different types of traveling wave solutions of the (1+1)-dimensional classical Boussinesq equations are successfully found by using the $\exp(-\Phi(\eta))$-expansion method. The performance of this method is trustworthy, useful and giving new solutions of the given equations. We notice that, our new solution might have significant impact on future researchers. The $\exp(-\Phi(\eta))$-expansion method is also applicable for other NPDEs.

Appendix A. Supplementary data

Supplementary data associated with this article can be found, in the online version, at http://dx.doi.org/10.1016/j.rinp.2014.07.006.

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