MATHEMATICS

ON THE STRUCTURE OF WELL DISTRIBUTED SEQUENCES. (II)

BY

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Let (s_n) be a sequence of real numbers satisfying $0 \leqq s_n \leqq 1$ for every n, $(n=1, 2, \ldots)$. We take $0 \leqq a < b \leqq 1$ and let $I_{[a,b]}(x)$ denote the characteristic function of the interval $[a, b]$, so that

$$I_{[a,b]}(x) = \begin{cases} 1 & \text{if } x \in [a, b] \\ 0 & \text{otherwise.} \end{cases}$$

The sequence (s_n) is said to be *well-distributed* [1]) if

(1) $$\lim_{p \to \infty} \frac{1}{p} \sum_{k=n+1}^{n+p} I_{[a,b]}(s_k) = b - a$$

holds uniformly in n for every interval $[a, b]$. This may be regarded as a more stringent test of the regularity of distribution of a sequence (s_n) than the classical uniform distribution condition, where

(2) $$\lim_{p \to \infty} \frac{1}{p} \sum_{k=1}^{p} I_{[a,b]}(s_k) = b - a$$

for every interval $[a, b]$. By a well known theorem due to H. WEYL [2], the condition (2) may be expressed alternatively as

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} e(hs_k) = 0, \qquad (h = 1, 2, \ldots)$$

where $e(t)$ denotes the function $e^{2\pi i t}$. A similar condition for well distributed sequences has been given by G. M. PETERSEN, [1]. Thus, (s_n) is well distributed if, and only if

(3) $$\lim_{p \to \infty} \frac{1}{p} \sum_{k=n+1}^{n+p} e(hs_k) = 0, \qquad (h = 1, 2, \ldots)$$

uniformly in n. Throughout, we shall use $\{\theta\}$ to denote the *fractional part* of θ, i.e. $\theta - [\theta]$, where $[\theta]$ is the largest integer less than or equal to θ.

In [3] the following theorem is stated:

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1) See [1], § 2 page 189.

Theorem 5. *If p, q are positive integers, the sequence*

$$\left\{\left(\frac{p}{q}\right)^k \alpha\right\}, \qquad (k=1, 2, 3, \ldots)$$

is not well distributed for any α, $0<\alpha\leqq 1$.

The proof of this result as given in [3] is incorrect, though, as indicated by DOWIDAR and PETERSEN [4] in Theorem 6′ of their paper, the above theorem is true if p/q is an integer. We shall now show that the theorem is true if "*any* α" is replaced by "*almost all* α". The proof is a modification of that given in [3] and is of interest mainly because the non-well distribution of a sequence is deduced from the uniform distribution of each of a countable set of sequences.

Theorem A. *If p, q are positive integers, the sequence*

$$\left\{\left(\frac{p}{q}\right)^k \alpha\right\}, \qquad (k=1, 2, 3, \ldots)$$

is not well distributed for almost all α, $0<\alpha\leqq 1$.

Proof. In the first instance, we may suppose that the sequence is uniformly distributed; otherwise there is nothing to prove, since a sequence which is not uniformly distributed is not well distributed.

We denote by E_N $(N=0, 1, 2, \ldots)$ the set of α for which $\left\{\frac{p^k}{q^{k+N}}\alpha\right\}$ is uniformly distributed. Then, by a result due to H. WEYL [2],

$$\mu(E_N) = 1,$$

for all $(N=0, 1, 2, \ldots)$. Also, if

$$E = \bigcap_{N=0}^{\infty} E_N,$$

we have

$$\mu(E) = 1.$$

Sequences which are uniformly distributed are also everywhere dense in $[0, 1]$. Hence, if $\alpha \in E$, for every N we can find an $m=m(N)$ such that

$$\left\{\frac{p^m}{q^{m+N}}\alpha\right\} < \frac{1}{8p^N q^N}. \tag{4}$$

Consider,

$$\sum_{k=m+1}^{m+N} e(s_k) = \sum_{k=m+1}^{m+N} e\left(\left\{\left(\frac{p}{q}\right)^k \alpha\right\}\right) = \sum_{k=m+1}^{m+N} e\left(\frac{p^k}{q^k}\alpha\right) = \sum_{k=m+1}^{m+N} e\left(q^N \frac{p^k}{q^{k+N}}\alpha\right)$$
$$= \sum_{k=1}^{N} e\left(q^N \frac{p^k}{q^k}\left(\frac{p^m}{q^{m+N}}\alpha\right)\right) = \sum_{k=1}^{N} e\left(q^{N-k}p^k\left\{\frac{p^m}{q^{m+N}}\alpha\right\}\right).$$

If $\alpha \in E$ and $m = m(N)$ then it follows from (4) that for all $k \leqq N$,

$$0 \leqq q^{N-k}p^k \left\{\frac{p^m}{q^{m+N}}\,\alpha\right\} < \frac{1}{8}\,\frac{q^{N-k}p^k}{p^N q^N} \leqq \frac{1}{8}.$$

This implies

$$\begin{aligned}\left|\sum_{k=m+1}^{m+N} e(s_k)\right| &= \left|\sum_{k=1}^{N} e\left(q^{N-k}p^k\left\{\frac{p^m}{q^{m+N}}\,\alpha\right\}\right)\right| \\ &\geqq \mathscr{R}\left(\sum_{k=1}^{N} e\left(q^{N-k}p^k\left\{\frac{p^m}{q^{m+N}}\,\alpha\right\}\right)\right) \\ &= \sum_{k=1}^{N} \mathscr{R}\left(e\left(q^{N-k}p^k\left\{\frac{p^m}{q^{m+N}}\,\alpha\right\}\right)\right) > \sum_{k=1}^{N} \cos\frac{\pi}{4} = \frac{N}{\sqrt{2}},\end{aligned}$$

and so, by (3), $\{(p/q)^k\,\alpha\}$ is not well distributed if $\alpha \in E$, which proves the theorem since $\mu(E) = 1$.

Remark: The technique used in the proof of Theorem A can clearly be used to establish the following stronger result:

Theorem B. *Let p_i and q_i $(i = 1, 2, \ldots, K)$ be positive integers with $p_i/q_i > 1$ for all $i = 1, 2, \ldots, K$, where K is fixed. Then the sequence $\{f(n)\,\alpha\}$, where*

$$f(n) = \left(\frac{p_1}{q_1}\right)^{n_1}\left(\frac{p_2}{q_2}\right)^{n_2}\cdots\left(\frac{p_K}{q_K}\right)^{n_K},$$

with $n = \sum_{i=1}^{K} n_i$, each n_i being an integer $\geqq 0$, is not well distributed for almost all α, $0 < \alpha \leqq 1$.

However, the case that arises when K is not fixed and which is not covered by Theorem 4 of [4], remains open.

We now prove a result on the uniform distribution of subsequences of a given sequence. To do so, it is necessary to introduce the idea of "*almost all subsequences of the given sequence* $\{s_k\}$", or "*a set of subsequences of measure one*".

Suppose that $\{s_k\}$ is a given sequence of real numbers and t is a real number in the interval $0 \leqq t \leqq 1$. Representing t by a non-terminating binary decimal expansion we can define a $1-1$ mapping of the infinite subsequences $\{s_{k_i}\}$ of $\{s_k\}$ onto the interval $[0, 1]$. For, let $\{s_{k_i}\}$ be any infinite subsequences of $\{s_k\}$, we define $t = 0\cdot\beta_1\beta_2\ldots$ (radix 2) by means of the equations

$$\beta_k = \begin{cases} 1 & \text{if } k = k_i \\ 0 & \text{otherwise.}\end{cases}$$

The inverse mapping is evident if we agree to use only the infinite decimal representation of t. With this mapping it is now possible to speak of "almost all subsequences of $\{s_k\}$" or "a set of subsequences of measure one" when the corresponding subset of the interval [0, 1] has measure one.

We now state the following result, which has been proved in substance by BUCK and POLLARD [5]:

Lemma 1. *A bounded sequence* $\{s_k\}$ *is* (C, 1) *summable to s if, and only if, almost all subsequences of* $\{s_k\}$ *are* (C, 1) *summable to s.*

We use this to prove the following:

Theorem C. *The sequence* $\{s_k\}$ *is uniformly distributed if, and only if, almost all subsequences of* $\{s_k\}$ *are uniformly distributed.*

Proof. By the lemma, it follows that if $e(hs_k)$ is Cesàro summable to zero for every $(h=1, 2, \ldots)$, then the set of subsequences, E_h say, which are Cesàro summable to zero is of measure one. Hence,

$$\mu(\bigcap_{h=1}^{\infty} E_h) = 1,$$

and a set of subsequences of measure one is uniformly distributed.

If almost all subsequences of $\{s_k\}$ are uniformly distributed then, for each h, the sequence $(e(hs_k))$ has a set of subsequences of measure one which are Cesàro summable to zero. Hence, by the lemma, $(e(hs_k))$ is Cesàro summable to zero, $(h = 1, 2, 3 \ldots)$, and so $\{s_k\}$ is uniformly distributed, which completes the proof of Theorem C.

At this point we shall give two modifications of a theorem due to DOWIDAR and PETERSEN [4], which is stated as:

Theorem 4. *Let* $(n(k))$ *be a subsequence of the integers,*

$$\frac{n(k)}{n(k-1)} = r(k), \ r(k) \nearrow \infty,$$

then, for almost all α, $0<\alpha\leqq 1$, *the sequence* $\{n(k)\alpha\}$ *is not well distributed.*

Firstly, we show that if the condition on $r(k)$ is relaxed so that $r(k) \nearrow \infty$ is replaced by $\lim\limits_{k->\infty} r(k) = \infty$, then the theorem is still true in the new form given by:

Theorem D. *Let* $(n(k))$ *be a subsequence of the integers*

$$\frac{n(k)}{n(k-1)} = r(k), \ \lim_{k->\infty} r(k) = \infty,$$

then, for almost all α, $0<\alpha\leqq 1$, *the sequence* $\{n(k)\alpha\}$ *is not well distributed.*

Proof. This is simply a modification of the proof of Theorem 4 given in [4] which depends upon the fact that if the sequence $\{n(k)\alpha\}$ *is* well distributed, then we cannot have, for instance,

$$\{n(k)\alpha\} \leqq \tfrac{1}{2} \text{ for } k = k_\nu+1, k_\nu+2, \ldots, k_\nu+[\log_2 \nu]$$

for infinitely many ν, as this violated criterion (1).

For the most part we shall adhere to the notation of [4], i.e. we denote by E_k the set of α for which $\{n(k)\alpha\} \leqq \frac{1}{2}$. This set consists of the closed intervals:

$$\left(0, \frac{1}{2n(k)}\right), \left(\frac{1}{n(k)}, \frac{3}{2n(k)}\right), \ldots, \left(\frac{n(k)-1}{n(k)}, \frac{2n(k)-1}{2n(k)}\right),$$

which may be written as

$$E_k = \bigcup_{r=0}^{n(k)-1} J'(r, n(k)),$$

where $J'(r, n(k))$ is the closed interval $\left(\frac{r}{n(k)}, \frac{2r+1}{2n(k)}\right)$ of length $\frac{1}{2n(k)}$. We note that $\mu(E_k) = \frac{1}{2}$. In addition, $J(r, n(k))$ denotes the closed interval $\left(\frac{r}{n(k)}, \frac{r+1}{n(k)}\right)$.

As in [4], lower and upper bounds for $\mu(\bigcap_{i=k}^{k+p} E_i)$ are obtained as

$$(5) \qquad \begin{cases} \mu(\bigcap_{i=k}^{k+p} E_i) \geqq \frac{1}{2}\left(\frac{1}{2} - \frac{2}{r(k+1)}\right) \cdots \left(\frac{1}{2} - \frac{2}{r(k+p)}\right) \\ \qquad = (\frac{1}{2})^{p+1} - \lambda(k, p) = L(k, p), \end{cases}$$

and

$$(6) \qquad \begin{cases} \mu(\bigcap_{i=k}^{k+p} E_i) \leqq \frac{1}{2}\left(\frac{1}{2} + \frac{2}{r(k+1)}\right) \cdots \left(\frac{1}{2} + \frac{2}{r(k+p)}\right) \\ \qquad = (\frac{1}{2})^{p+1} + \zeta(k, p) = U(k, p) \end{cases}$$

where

$$\lim_{k->\infty} \lambda(k, p) = \lim_{k->\infty} \zeta(k, p) = 0, \qquad (p = 1, 2, \ldots).$$

For $\nu = 1, 2$, let $\xi_\nu = E_{k_\nu}$, where $k_2 > k_1$ and $r(k) > 16$ for $k > k_1$. For $\nu \geqq 3$ let

$$\xi_\nu = \bigcap_{i=k_\nu}^{k_\nu + p(\nu)} E_i,$$

where $p(\nu)$ is given by

$$p(\nu) = \begin{cases} [\log_2 \nu] & \text{when } \nu \neq 2^k \\ \log_2 \nu - 1 & \text{when } \nu = 2^k. \end{cases}$$

Since $\lim_{k->\infty} \lambda(k, p) = 0$, we have

$$L(k_\nu, p(\nu)) \geqq \frac{1}{\nu}$$

for $k_\nu > K'$. Also, we can choose K'' so that for $k_\nu > K''$

$$U(k_\nu, p(\nu)) \leqq \frac{5}{2\nu}.$$

Consequently, we can choose $K=\max(K', K'')$ such that for $k_\nu > K$,

$$\frac{5}{2\nu} \geqq \mu(\xi_\nu) \geqq \frac{1}{\nu}.$$

So far our argument is similar to that used in [4] since there, up to this point, only the condition $\lim_{k\to\infty} r(k) = \infty$ is assumed. We now consider the set

$$\xi_\nu = \bigcap_{i=k_\nu}^{k_\nu+p(\nu)} E_i,$$

where k_ν must be chosen so that $k_\nu > \max(K, k_{\nu-1}+p(\nu-1))$. With this choice of k_ν we ensure that the sets ξ_ν and $\xi_{\nu-1}$ have no sets of intervals in common. The rest of the argument proceeds as in [4] provided that $r(k) > 16$ for all the k used in the construction, i.e. for $k_\nu, \ldots, k_\nu+p(\nu)$, $(\nu=1, 2, \ldots)$. We now give the second modification of Theorem 4, [4], namely:

Theorem E. *Let $(n(k))$ be a subsequence of the integers which possesses a rearrangement, $(n'(k))$ say, such that*

$$\lim_{k\to\infty} \frac{n'(k)}{n'(k-1)} = \infty.$$

Then, for almost all α, $0<\alpha\leqq 1$, the sequence $\{n(k)\alpha\}$ is not well distributed.

Proof. Again, this depends essentially upon the proof of Theorem 4 in [4]. However, we must introduce the following technique which enables us to estimate the measure of the sets ξ_ν.

At the outset we choose

$$p(\nu) = \begin{cases} [\log_2 \nu] & \text{when } \nu \neq 2^k \\ \log_2 \nu - 1 & \text{when } \nu = 2^k. \end{cases}$$

To obtain an estimate for

$$\mu\Big(\bigcap_{i=k}^{k+p(\nu)} E_i\Big)$$

we can regard the finite sequence $n(k), n(k+1), \ldots, n(k+p(\nu))$ to be arranged as an increasing sequence $n''(k), n''(k+1), n''(k+p(\nu))$. We note that, in almost all cases, the sequence $n''(k), n''(k+1), \ldots, n''(k+p(\nu))$ will *not* be identical with $n'(k), n'(k+1), \ldots, n'(k+p(\nu))$. However, there exists an integer K such that $n''(k)=n'(K)$ and, for sufficiently large k, i.e. $k>N$, we have

$$\frac{n''(k+1)}{n''(k)} \geqq \frac{n'(K+1)}{n'(K)}.$$

If

$$\frac{n''(k+1)}{n''(k)} = r''(k), \ldots, \frac{n''(k+p(\nu))}{n''(k+p(\nu)-1)} = r''(k+p(\nu)-1),$$

then, since $\lim_{k\to\infty} \frac{n'(k)}{n'(k-1)} = \infty$, it follows that

$$r''(k+j) > M \qquad (j = 0, 1, \ldots, p(\nu)-1) \tag{7}$$

for $k > N$ and any fixed M. Lower and upper bounds for

$$\mu(\bigcap_{i=k}^{k+p(\nu)} E_i),$$

similar to (5) and (6) can now be obtained, and, for some k_ν, we have

$$\frac{5}{2\nu} \geqq \mu(\xi_\nu) \geqq \frac{1}{\nu}, \tag{8}$$

where

$$\xi_\nu = \bigcap_{i=k_\nu}^{k_\nu+p(\nu)} E_i.$$

We note that as k takes different integer values so the values of the ratios $r''(k+j)$, $(j=0, 1, \ldots, p(\nu)-1)$ may differ. However, inequality (7) indicates that k_ν may be chosen so that (8) is valid.

The next stage of the proof is similar to that for Theorem 4 in [4]. Although, ξ_ν is not now covered by intervals of the form $J'(r, n(k_\nu+p(\nu))$, but by intervals of the form $J'(r, h)$, where

$$h = \max_{0\leqq j\leqq p(\nu)} n(k_\nu+j) = n''(k_\nu+p(\nu)),$$

whose number does not exceed

$$n''(k_\nu)\, U(k_\nu, p(\nu)).$$

In addition, k_ν may be chosen so that, for $k \geqq k_\nu$,

$$n(k) > n''(k_{\nu-1}+p(\nu-1)),$$

since in the sequence $(n'(k))$, which is increasing in k, only finitely many terms precede the term which is now $n''(k_{\nu-1}+p(\nu-1))$. With this choice of k_ν, and with k_1 chosen so that

$$\frac{n'(K_1+j)}{n'(K_1+j-1)} > 16, \qquad (j = 1, 2, \ldots)$$

where $n'(K_1) = n''(k_1)$, the rest of the proof follows as in Theorem 4, [4], without difficulty.

In conclusion, we give two further results. In the first instance, from Theorem D we deduce that if

$$\lim_{k\to\infty} \frac{n(k)}{n(k-1)} = \infty,$$

then, for almost all α, $0 < \alpha \leqq 1$, we have

$$\{n(k)\alpha\} \leqq \tfrac{1}{2} \text{ for } k = k_\nu+1, k_\nu+2, \ldots, k_\nu+[\log_2 \nu],$$

and infinitely many ν. The number $\frac{1}{2}$ was chosen in a purely arbitrary way, and with minor modifications the proofs of Theorem 4, [4], and of Theorems D and E could have been carried out using any rational number in (0, 1), e.g. the number $1/R$, $0<1/R<1$. Hence, if

$$\lim_{k\to\infty} \frac{n(k)}{n(k-1)} = \infty,$$

then, for almost all α, $0<\alpha\leqq 1$, we have

$$\{n(k)\alpha\} \leqq \frac{1}{R} \text{ for } k = k_\nu+1,\, k_\nu+2,\, \ldots,\, k_\nu+[\log_R \nu]$$

for infinitely many ν and any fixed positive integer R. Moreover, the same arguments will show that if $(n(k))$ possesses a subsequence, $(n(k_i))$ say, such that

$$\lim_{i\to\infty} \frac{n(k_i)}{n(k_{i-1})} = \infty,$$

then, for almost all α, $0<\alpha\leqq 1$,

$$\{n(k_i)\alpha\} \leqq \frac{1}{R} \text{ for } i = i_\nu+1,\, i_\nu+2,\, \ldots,\, i_\nu+[\log_R \nu] \tag{9}$$

for infinitely many ν, and any fixed positive integer R.

For the subsequence $(n(k_i))$ we now define the *index sequence* (x_j) of $(n(k_i))$ by the equations

$$x_j = \begin{cases} 1 \text{ when } j = k_1, k_2, \ldots, k_i, \ldots \\ 0 \text{ otherwise.} \end{cases}$$

Let us suppose that, for some positive integer R,

$$\frac{1}{p} \sum_{j=m+1}^{m+p} x_j > \frac{2}{R} > 0$$

for all m and all $p>P$. If this is so we say that the *lower density* of $(n(k_i))$ exceeds $2/R$. From (9) it follows that, for almost all α, $0<\alpha\leqq 1$,

$$\frac{1}{\lambda(\nu)} \sum_{k=\eta}^{\theta} I_{[0,1/R]}\,(\{n(k)\alpha\}) > \frac{2}{R}$$

for infinitely many ν, where

$$\eta = k_{(i_\nu)+1},\ \theta = k_{(i_\nu+[\log_R \nu])} \text{ and } \lambda(\nu) = \theta - k_{(i_\nu)}.$$

This violates (1) thereby establishing the following:

Theorem F. *If $(n(k))$ possesses a subsequence $(n(k_i))$ satisfying*

$$\lim_{i\to\infty} \frac{n(k_i)}{n(k_{i-1})} = \infty,$$

and having a lower density that is positive, then $\{n(k)\alpha\}$ *is not well distributed for almost all* α, $0<\alpha\leqq 1$.

Finally, we prove a stronger result than the above, but first we make some preliminary remarks about arithmetic means. If (x_j) is a bounded sequence it is evident that

$$\lim_{p->\infty}\left|\frac{1}{p}\sum_{j=m_1+1}^{m_1+p} x_j-\frac{1}{p}\sum_{j=m_2+1}^{m_2+p} x_j\right|=0. \tag{10}$$

Next we give:

Lemma 2. *If the bounded sequence* (x_j) *is not almost convergent to zero, then there exists a positive real number* γ, *and sequences* (p_u), (m_v) *such that*

$$\left|\frac{1}{p_u}\sum_{j=m_v+1}^{m_v+p_u} x_j\right|>\gamma,$$

where

$$\lim_{v->\infty} m_v=\lim_{u->\infty} p_u=\infty.$$

Proof. If (x_j) is not (C, 1) summable to zero, and

$$\lim_{p->\infty}\sup\left|\frac{1}{p}\sum_{j=m_1+1}^{m_1+p} x_j\right|>2\,\gamma, \tag{11}$$

for some fixed positive integer m_1, then it follows from (10) that the conditions of the lemma are satisfied.

If the inequality (11) is not satisfied for some integer m_1, thereby indicating that (x_j) is (C, 1) summable to zero, then

$$\lim_{p->\infty}\left|\frac{1}{p}\sum_{j=m+1}^{m+p} x_j\right|=0$$

for all $m=1, 2, \ldots$, and if the conditions of the lemma are not satisfied, then

$$\lim_{p->\infty}\left|\frac{1}{p}\sum_{j=m+1}^{m+p} x_j\right|=0$$

uniformly in m, and (x_j) is almost convergent to zero.

In addition, we require:

Lemma 3. *If* (x_j) *is a sequence of* 1*'s and* 0*'s which is not almost convergent to zero then, for every* p $(p=1, 2, \ldots)$ *there exists an infinite sequence* $(m_v)=(m_v(p))$ *such that*:

$$\left|\frac{1}{p}\sum_{j=m_v+1}^{m_v+p} x_j\right|>\gamma.$$

Proof. Firstly, we choose γ to be an irrational number so that the conditions of Lemma 2 are satisfied. Then, if Lemma 3 is not valid we have for some p and all $m > M$,

$$\left| \frac{1}{p} \sum_{j=m+1}^{m+p} x_j \right| < \gamma < (1-\varepsilon)\gamma.$$

Consequently, for all $m > M$

$$\left| \frac{1}{\omega p} \sum_{j=m+1}^{m+\omega p} x_j \right| < (1-\varepsilon)\gamma,$$

ω a positive integer; and if $(\omega-1)p < h \leqq \omega p$, then

$$\left| \frac{1}{h} \sum_{j=m+1}^{m+h} x_j \right| \leqq \left| \frac{1}{h} \sum_{j=m+1}^{m+\omega p} x_j \right| = \left| \frac{\omega p}{h} \cdot \frac{1}{\omega p} \sum_{j=m+1}^{m+\omega p} x_j \right| < \frac{\omega p}{h} \gamma(1-\varepsilon).$$

For ω sufficiently large, i.e. $\omega > \omega_0$ say,

$$\frac{\omega p}{h} < \frac{1}{1-\varepsilon},$$

and so

$$\left| \frac{1}{h} \sum_{j=m+1}^{m+h} x_j \right| < \gamma,$$

for $h > \omega_0 p$ and $m > M$, which contradicts the conclusion of Lemma 2.

Definition: If (x_j) is the index sequence of $(n(k_i))$, and (x_j) is almost convergent to zero, we say that the subsequence $(n(k_i))$ has *density* zero. We are now in a position to prove:

Theorem G. *If the sequence $(n(k)\alpha)$ is well distributed for almost all α, $0 < \alpha \leqq 1$, then any subsequence $(n(k_i))$ of $(n(k))$ satisfying*

$$\lim_{i \to \infty} \frac{n(k_i)}{n(k_{i-1})} = \infty$$

has density zero.

Proof. If the sequence $(n(k_i))$ has a positive lower densit, then, by Theorem F, we have the above result. Arguments similar to those used in the proof of Theorem F are applicable in this instance if, for each ν, $(\nu = 1, 2, \ldots)$ we have as is implied in the theorem

$$\xi_\nu = \bigcap_{i=i_\nu}^{i_\nu + [\log_R \nu]} E_{k_i},$$

where the basic intervals $J'(r, n(k))$, $(r = 0, \ldots, n(k)-1)$, are of length $1/Rn(k)$ where $1/R < \gamma$, γ being chosen so that Lemmas 2 and 3 are valid. In the induction process for the construction of the sets ξ_ν, the only restriction imposed on each i_ν is $i_\nu > N$ for some positive integer N. We now choose i_ν so that $i_\nu > N$ and also

$$\left| \frac{1}{p(\nu)} \sum_{j=m(\nu)+1}^{m(\nu)+p(\nu)} x_j \right| > \gamma,$$

where

$$k_{(i_\nu)} < m(\nu) < m(\nu) + p(\nu) < k_{(i_\nu + [\log_R \nu])}.$$

and

$$\lim_{\nu -> \infty} p(\nu) = \lim_{\nu -> \infty} m(\nu) = \infty.$$

With this choice of the i_ν the usual contradiction of (1) follows and the theorem is proved.

NOTE IN PROOF:

We have used the notation of [4] although some of the material on pages 483 and 484 there is not strictly correct. We would like to note the following changes.

1. On page 483, $p(\nu)$ should be defined as $p(\nu) = [\log_2 \nu] - 1$, $\nu \neq 2^k$; $p(\nu) = k-2$, $\nu = 2^k$.

2. Last line of page 483 to top line of page 484 should read: "it is clear that ξ_ν is covered by intervals of the form $J'(r, n(k_\nu + p(\nu)))$ whose number does not exceed $2n(k_\nu + p(\nu))\, U(k_\nu, p(\nu))$".

3. Page 484: "$J'(r, n(k_{\nu+h}))$ is intersected by no more than

$$\frac{2n(k_{\nu+h} + p(\nu+h))}{n(k_{\nu+h})}\, U(k_{\nu+h}, p(\nu+h))$$

intervals of the form $J'(r, n(k_{\nu+h} + p(\nu+h)))$ which belong to $\xi_{\nu+h}$".

4. The formula for $\mu(\xi_\nu, \xi_{\nu+h})$ becomes:

$$\mu(\xi_\nu, \xi_{\nu+h}) \leqq 2n(k_\nu + p(\nu)) \cdot U(k_\nu, p(\nu)) \cdot \frac{5n(k_{\nu+h})}{8n(k_\nu + p(\nu))} \cdot \frac{2n(k_{\nu+h} + p(\nu+h))}{n(k_{\nu+h})} \cdot$$

$$\cdot\, U(k_{\nu+h} + p(\nu+h)) \cdot \frac{1}{2n(k_{\nu+h} + p(\nu+h))} \leqq \frac{125}{16} \cdot \frac{1}{\nu} \cdot \frac{1}{\nu+h}.$$

5. In the calculations on the lower half of page 484, $\frac{125}{16}$ must replace $\frac{125}{64}$ throughout and a must be taken close to $\frac{1}{8}$.

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