A class of totally positive \( P \)-matrices whose inverses are \( M \)-matrices✩

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Abstract

In this work we introduce some technical conditions to prove that a \( P \)-matrix has an inverse \( M \)-matrix. We study a class of totally positive \( P \)-matrices whose inverses are \( M \)-matrices.

Keywords: \( P \)-matrix; \( M \)-matrix; Totally positive matrix; Inverse \( M \)-matrix

1. Introduction

A real matrix \( A = (a_{ij}) \) is nonnegative if \( a_{ij} \geq 0 \) for each pair \((i, j)\). \( A \) is called totally positive if all its minors are nonnegative. These matrices have become increasingly important in approximation theory, computer aided geometric design and other fields. Characterizations of totally positive matrices can be found in [1,3,4].

An \( n \times n \) real matrix is a \( P \)-matrix if all its principal minors are positive and it is an \( R \)-matrix if all its principal submatrices are nonsingular. Obviously, if \( A \) is a \( P \)-matrix, then \( A \) is an \( R \)-matrix. The converse does not hold.

Nonsingular \( M \)-matrices have many equivalent definitions. In fact, Berman and Plemmons [2] list fifty equivalent definitions. We shall use the following equivalent definitions.

Definition 1.1. Let \( A \) be an \( n \times n \) real matrix with nonpositive off-diagonal elements. Then the following concepts are equivalent:

(a) \( A \) is an \( M \)-matrix.  
(b) \( A^{-1} \) is nonnegative.  
(c) \( A \) is a \( P \)-matrix.  
(d) \( A^{-1} \) is a \( P \)-matrix.

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Nonsingular $M$-matrices have important applications, for instance, in iterative methods in numerical analysis, in the analysis of dynamical systems, in economics, and in mathematical programming.

For a nonsingular totally positive matrix $A$, Markham [5] and Peña [6], give necessary and sufficient conditions for $A^{-1}$ to be an $M$-matrix. In this work we introduce some technical conditions in order to prove that a $P$-matrix has an inverse $M$-matrix. We also analyze when a $P$-matrix has a totally positive inverse. Finally, we study a special class of totally positive $P$-matrices whose inverses are tridiagonal $M$-matrices.

2. Results

In general, we shall use similar notation to that of [1]. Given $k, n \in N$, $k \leq n$, $Q_{k,n}$ will denote the totality of strictly increasing sequences of $k$ natural numbers less than or equal to $n$:

$$\alpha = (\alpha_i)_{i=1}^k \in Q_{k,n}, \quad \text{if } 1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_k \leq n.$$  

Given $\alpha \in Q_{k,n}$, the complement $\alpha'$ is the strictly increasing sequence whose entries are $\{1, 2, \ldots, n\} \setminus \alpha$, so that $\alpha'$ is an element of $Q_{n-k,n}$. If $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ and $\beta = \{\beta_1, \beta_2, \ldots, \beta_k\}$ are elements of $Q_{k,n}$, we define $|\alpha - \beta| = \sum_{i=1}^k |\alpha_i - \beta_i|$.

The dispersion $d(\alpha)$ of $\alpha \in Q_{k,n}$ is defined by

$$d(\alpha) = \sum_{i=1}^{k-1} (\alpha_{i+1} - \alpha_i - 1) = \alpha_k - \alpha_1 - (k-1),$$

with the convention $d(\alpha) = 0$ for $\alpha \in Q_{1,n}$.

For $\alpha \in Q_{k,n}$, $\text{sgn}(\alpha)$ is defined as $\text{sgn}(\prod)$ for the permutation $\prod \in S_n$ that assigns $\alpha_i$ to $i$ for $i = 1, 2, \ldots, k$ and $\alpha'_j$ to $k+j$ for $j = 1, 2, \ldots, n-k$, so that

$$\text{sgn}(\alpha) = (-1)^{\sum_{i=1}^k \alpha_i - (k+1)/2}.$$  

Let $A$ be an $n \times n$ real matrix, and let $\alpha, \beta \in Q_{k,n}$. $A[\alpha|\beta]$ denotes the $k \times k$ submatrix of $A$ containing rows numbered by $\alpha$ and columns numbered by $\beta$. The principal submatrix $A[\alpha|\alpha]$ is abbreviated to $A[\alpha]$.

**Theorem 2.1.** Suppose $A = (a_{ij})$ is a $P$-matrix of size $n \times n$. Then the following properties are equivalent:

(i) Given $\alpha \in Q_{n-2,n}$, $A[\{i\}|\alpha]A[\alpha^{-1}\{j\}] \leq A[\{i\}|\{j\}]$, for all $i, j \notin \alpha, i \neq j$.

(ii) $A^{-1}$ is an $M$-matrix.

(iii) $\det A[\{i\}'|\{j\}'] \geq 0$, for $i+j = 2k+1$, and $\det A[\{i\}'|\{j\}'] \leq 0$, for $i+j = 2k$, where $k$ is a positive integer and $i \neq j$.

(iv) Given $\alpha, \beta \in Q_{n-1,n}$, $A[\alpha|\beta] \geq 0$, for $|\alpha - \beta| = 2k+1$, and $\det A[\alpha|\beta] \leq 0$, for $|\alpha - \beta| = 2k$, where $k$ is a positive integer and $\alpha \neq \beta$.

**Proof.** Let $B = A^{-1}$, $B = (b_{ij})_{1 \leq i, j \leq n}$ with

$$b_{ij} = \frac{1}{\det A}(-1)^{i+j}\det A[\{j\}'|\{i\}'].$$  

(1)

(i) $\Rightarrow$ (ii) Since $A$ is a $P$-matrix, we have to see that $b_{ij} \leq 0$, for all $i, j, i \neq j$. So consider $b_{ij}, i \neq j$, and assume, without loss of generality, that $i < j$. If we define $\alpha_1 = \{1, \ldots, i-1, i+1, \ldots, j-1, j+1, \ldots, n, i\}$ and $\beta_1 = \{1, \ldots, i-1, i+1, \ldots, j-1, j+1, \ldots, n, j\}$, then

$$\det A[\{j\}'|\{i\}'] = (-1)^{n-i-1}(-1)^{n-j-1}\det A[\alpha_1|\beta_1]$$

$$= (-1)^{n-i-1}(-1)^{n-j-1}\det \begin{bmatrix} A[\alpha] & A[\alpha|\{j\}] \\ A[\{i\}|\alpha] & a_{ij} \end{bmatrix}$$

$$= (-1)^{n-i-1}(-1)^{n-j-1}\det \begin{bmatrix} A[\alpha] & A[\alpha|\{j\}] \\ 0 & a_{ij} - A[\{i\}|\alpha]A[\alpha^{-1}\{j\}] \end{bmatrix}$$

$$= (-1)^{n-i-1}(-1)^{n-j}(a_{ij} - A[\{i\}|\alpha]A[\alpha^{-1}A[\alpha|\{j\}]] \det A[\alpha].$$
where \( \alpha = \{1, \ldots, i - 1, i + 1, \ldots, j - 1, j + 1, \ldots, n\} \in \mathcal{Q}_{n-2,n} \).

From (1) we have

\[
\begin{align*}
b_{ij} &= \frac{1}{\det A} (-1)^{i+j} (-1)^{n-i-1} (-1)^{n-j} (a_{ij} - A[i][\alpha]A[\alpha]^{-1}A[\alpha][j])) \det A[\alpha] \\
&= \frac{1}{\det A} (-1)^{2n-1} (a_{ij} - A[i][\alpha]A[\alpha]^{-1}A[\alpha][j])) \det A[\alpha].
\end{align*}
\]

(2)

Since \( A \) is a \( P \)-matrix and \( A[i][\alpha]A[\alpha]^{-1}A[\alpha][j] \leq a_{ij} \) we conclude that \( b_{ij} \leq 0 \).

(ii) \( \Rightarrow \) (iii) Since \( A^{-1} \) is \( M \)-matrix, \( b_{ij} \leq 0 \), for all \( i, j, i \neq j \). From (1) it is easy to see that the inequalities of (iii) hold since \( \det A > 0 \).

(iii) \( \Rightarrow \) (iv) Given \( \alpha, \beta \in \mathcal{Q}_{n-1,n} \), with \( \alpha \neq \beta \), there exist \( i, j \in \{1, 2, \ldots, n\} \), such that \( i \notin \alpha \) and \( j \notin \beta \), that is, \( \alpha = \{i\}' \) and \( \beta = \{j\}'. \) Since \( \det A[\alpha][\beta] = \det A[i]'[j]' \) and \( |\alpha - \beta| = |i - j| \), (iv) follows.

(iv) \( \Rightarrow \) (i) Given \( \alpha \in \mathcal{Q}_{n-2,n} \) there exist \( i, j \in \{1, 2, \ldots, n\} \) such that \( \alpha = \{i, j\}'. \) Let \( \gamma = \{i\}' \) and \( \beta = \{j\}'. \) As we have seen,

\[
\det A[\beta][\gamma] = (-1)^{2n-i-j-1} \det A[\alpha](a_{ij} - A[i][\alpha]A[\alpha]^{-1}A[\alpha][j])).
\]

If \( |\beta - \gamma| = 2k + 1 \) then \( (-1)^{2n-i-j-1} = 1 \), and by (iv) \( a_{ij} - A[i][\alpha]A[\alpha]^{-1}A[\alpha][j]) \geq 0 \). Analogously, if \( |\beta - \gamma| = 2k \) then \( (-1)^{2n-i-j-1} = -1 \), and by hypothesis we obtain the desired inequality.

The condition (i) of the above theorem allows us to establish a relation between the sign of \( \det A \) and the sign of all its principal minors, when \( A \) is an \( R \)-matrix.

**Theorem 2.2.** Let \( A = (a_{ij}) \) be an \( R \)-matrix of size \( n \times n \). If for any \( i \in \{1, 2, \ldots, n\} \)

\[
A[i][\alpha]A[\alpha]^{-1}A[\alpha][i]) < a_{ii}, \quad \forall \alpha \in \mathcal{Q}_{k,n}, \quad k = 1, 2, \ldots, n - 1,
\]

then all principal minors have the sign of \( \det A \).

**Proof.** If \( \alpha \) is an element of \( \mathcal{Q}_{n-1,n} \) then there exists \( i \in \{1, 2, \ldots, n\} \) such that \( i \notin \alpha \). By permutation we put the \( i \)th row and column in the last position. Then

\[
\begin{align*}
\det A &= (-1)^{n-i} (-1)^{n-i} \det \begin{bmatrix}
A[\alpha] & A[\alpha][i] \\
A[i][\alpha] & a_{ii}
\end{bmatrix} \\
&= (-1)^{2(n-i)} \det \begin{bmatrix}
A[\alpha] & A[\alpha][i] \\
0 & a_{ii} - A[i][\alpha]A[\alpha]^{-1}A[\alpha][i])
\end{bmatrix} \\
&= \det A[\alpha](a_{ii} - A[i][\alpha]A[\alpha]^{-1}A[\alpha][i]).
\end{align*}
\]

Therefore \( \text{sign}(\det A[\alpha]) = \text{sign}(\det A) \).

Now, let \( \gamma \in \mathcal{Q}_{n-2,n} \) and \( i, j \in \{1, 2, \ldots, n\} \) such that \( i, j \notin \gamma \). By a similar reasoning \( \text{sign} \det A[\gamma] = \text{sign} \det A[\alpha] \), where \( \alpha = \gamma \cup \{i\}, \alpha \in \mathcal{Q}_{n-1,n} \). Therefore \( \text{sign}(\det A[\gamma]) = \text{sign}(\det A) \). And so on.

The next results deal with the following question: when does a \( P \)-matrix have a totally positive inverse? They can be obtained by applying Theorem 2.1 of [1] and Theorem 3.1 of [4], respectively.

**Proposition 2.1.** Let \( A \) be an \( n \times n \) nonsingular \( P \)-matrix such that for all \( k, 1 \leq k \leq n \)

\[
\text{sgn}(\alpha)\text{sgn}(\beta) \det A[\beta'][\alpha'] \geq 0, \quad \forall \alpha, \beta \in Q_{k,n}, \text{ with } d(\beta) = 0.
\]

Then \( A^{-1} \) is a totally positive matrix.

**Proposition 2.2.** Let \( A \) be an \( n \times n \) \( P \)-matrix. \( A^{-1} \) is totally positive if and only if for all \( k, 1 \leq k \leq n \),

\[
\begin{align*}
\text{sgn}(\alpha) \det A[k+1, \ldots, n][\alpha'] &> 0, \quad \forall \alpha \in Q_{k,n} \\
\text{sgn}(\beta) \det A[\beta'][k+1, \ldots, n] &> 0, \quad \forall \beta \in Q_{k,n}.
\end{align*}
\]
3. A special class of \( P \)-matrices

In this section we are going to analyze a special class of oscillatory matrices introduced by Markham in [5]. Let \( v_n \) be the vector of positive numbers \( v_n = \{a_n, a_{n-1}, \ldots, a_2, a_1\} \) such that
\[
a_n > a_{n-1} > \cdots > a_2 > a_1 > 0.
\]

Consider the symmetric matrix \( A \), of size \( n \times n \), defined in the following way: the main diagonal is \( v_n \), the first superdiagonal is \( v_{n-1} = \{a_{n-1}, \ldots, a_2, a_1\} \), and so on. The matrix \( A \) has the form,
\[
A = \begin{bmatrix}
a_n & a_{n-1} & a_{n-2} & \cdots & a_2 & a_1 \\
a_{n-1} & a_n & a_{n-2} & \cdots & a_2 & a_1 \\
a_{n-2} & a_{n-1} & a_n & \cdots & a_2 & a_1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
a_2 & a_2 & a_2 & \cdots & a_2 & a_1 \\
a_1 & a_1 & a_1 & \cdots & a_1 & a_1
\end{bmatrix}.
\]

We are going to prove that \( A \) is a totally positive \( P \)-matrix. In addition, by using the above results we are going to provide an easy proof that its inverse is an \( M \)-matrix.

Let us see that \( A \) is a \( P \)-matrix. Since \( A \) is symmetric, it is sufficient to prove that \( \det A[\{1, 2, \ldots, k\}] > 0 \), for \( k = 1, 2, \ldots, n \). The proof is by induction on \( k \). The case \( k = 1 \) is trivial since \( a_1 > 0 \). For \( k = 2 \), \( \det A[\{1, 2\}] = a_1(a_n - a_{n-1}) > 0 \), since \( a_n > a_{n-1} \). By the induction hypothesis we suppose that \( \det A[\{1, 2, \ldots, k-1\}] > 0 \). Then
\[
\det A[\{1, 2, \ldots, k\}] = \det A[\{1, 2, \ldots, k-1\}] \left( a_{n-k+1} - a_{n-k+1}^2 \right) > 0,
\]
since \( a_{n-k+2} > a_{n-k+1} \).

**Proposition 3.1.** Let \( A = (a_{ij}) \) be the above \( n \times n \) matrix. Then \( A \) is a totally positive matrix.

**Proof.** Since \( A \) is a \( P \)-matrix, by using Theorem 3.1 of [4] it is sufficient to prove that for all \( k, 1 \leq k \leq n \),
\[
\det A[\{1, 2, \ldots, k\}] > 0, \quad \forall \beta \in Q_{k,n}.
\]
\[
\det A[\{\alpha\} | \{1, 2, \ldots, k\}] > 0, \quad \forall \alpha \in Q_{k,n}.
\]

Let us see the first inequality; the other is completely analogous. Let \( k, 1 \leq k \leq n \), and \( \beta \in Q_{k,n} \). If \( \{1, 2, \ldots, k-1\} \subseteq \beta \), it is easy to observe that \( \det A[\{1, 2, \ldots, k\}]|_{\beta} = 0 \). Suppose, now, \( \{1, 2, \ldots, k-1\} \nsubseteq \beta \). There exists an index \( j \), \( j \geq k \), such that \( j \in \beta \). Then
\[
\det A[\{1, 2, \ldots, k\}]|_{\beta} = \det A[\{1, 2, \ldots, k-1\}] a_{n-j+1} a_{n-k+2}^2 (a_{n-k+2} - a_{n-k+1}) > 0,
\]
since \( A \) is a \( P \)-matrix and \( a_{n-k+2} > a_{n-k+1} \). \( \Box \)

In the next result we are going to use the Neville elimination process. The essence of this process is producing zeros in a column of a matrix by adding to each row an appropriate multiple of the previous one.

**Theorem 3.1.** The inverse of matrix \( A \) is a tridiagonal \( M \)-matrix.

**Proof.** Since \( A \) is totally positive, by applying Theorem 2.1 of [6] we see that
\[
\det A[\{i\} | \{j\}] = 0, \quad \text{when } |i - j| = 2.
\]
(4)

Since \( A \) is symmetric, we shall assume \( j = i + 2 \). By applying the Neville elimination process we transform \( A[\{i\} | \{j\}] \) into an upper triangular matrix with a zero entry in the main diagonal. Therefore, we obtain easily (4). \( \Box \)
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