2-Connected spanning subgraphs with low maximum degree in locally planar graphs

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Abstract

In this paper, we prove that there exists a function \( a : \mathbb{N}_0 \times \mathbb{R}_+ \to \mathbb{N} \) such that for each \( \varepsilon > 0 \), if \( G \) is a 4-connected graph embedded on a surface of Euler genus \( k \) such that the face-width of \( G \) is at least \( a(k, \varepsilon) \), then \( G \) has a 2-connected spanning subgraph with maximum degree at most 3 in which the number of vertices of degree 3 is at most \( \varepsilon |V(G)| \). This improves results due to Kawarabayashi, Nakamoto and Ota [K. Kawarabayashi, A. Nakamoto, K. Ota, Subgraphs of graphs on surfaces with high representativity, J. Combin. Theory Ser. B 89 (2003) 207–229], and Böhme, Mohar and Thomassen [T. Böhme, B. Mohar, C. Thomassen, Long cycles in graphs on a fixed surface, J. Combin. Theory Ser. B 85 (2002) 338–347].

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1. Introduction

All graphs in this paper are simple, with no loops or multiple edges. A closed surface means a connected compact 2-dimensional manifold without boundary. We denote the orientable and
nonorientable closed surfaces of genus $g$ by $S_g$ and $N_g$, respectively. For a closed surface $F^2$, let $\chi(F^2)$ denote the Euler characteristic of $F^2$. The number $k = 2 - \chi(F^2)$ is called the Euler genus of $F^2$. Let $F_k^2$ denote a closed surface of Euler genus $k$. It is well known that for every even $k \geq 0$, either $F_k^2 = S_{k/2}$ or $F_k^2 = N_k$, and for every odd $k$, $F_k^2 = N_k$. If a graph $G$ is embedded on a surface so that every noncontractible closed curve intersects $G$ at least $k$ times, we say the embedding is $k$-representative. The face-width or representativity is the smallest nonnegative integer $k$ for which the embedding is $k$-representative.

In 1931 Whitney [22] showed that 4-connected planar triangulations are hamiltonian, and in 1956, Tutte [21] proved that every 4-connected planar graph is hamiltonian. Almost thirty years later, Thomassen [19] (see also [5]) gave a short proof of Tutte’s theorem and extended it to show that every 4-connected planar graph is hamiltonian-connected, i.e., for any two distinct vertices $u, v$, there is a hamiltonian path from $u$ to $v$. There are many results inspired by these theorems of Whitney, Tutte and Thomassen. While we cannot survey all such results, we mention some that motivate the present paper.

Thomas and Yu [18] extended Tutte’s theorem to projective-planar graphs and proved that every 4-connected projective-planar graph is hamiltonian. However, Archdeacon, Hartsfield, and Little [1] proved that for each $k$ there exists a $k$-connected triangulation of some orientable surface having face-width $k$ in which every spanning tree has a vertex of degree at least $k$. In particular, such graphs are far from having hamiltonian cycles. So a fixed connectivity or face-width or both, independent of the surface, will not suffice for hamiltonicity on arbitrary surfaces.

If the surface is fixed and the face-width is large enough, then the situation is different. The first results in this direction were by Thomassen [20], who examined a generalization of hamiltonicity. A $k$-tree is a spanning tree of maximum degree at most $k$; this generalizes the idea of a Hamilton path, which is a 2-tree. Barnette [2] showed that every 3-connected planar graph has a 3-tree. Thomassen [20] showed that local planarity provides a similar result. He proved that a triangulation of a fixed orientable surface with large face-width has a 4-tree. Ellingham and Gao [6] modified the method of [20] to prove that a 4-connected triangulation of a fixed orientable surface with large face-width has a 3-tree.

These results were improved by examining another generalization of hamiltonicity. A $k$-walk is a spanning closed walk that uses every vertex at most $k$ times; this generalizes the idea of a Hamilton cycle, which is a 1-walk. Jackson and Wormald [10] noted that if a $k$-walk exists, then a $(k + 1)$-tree exists. Gao and Richter [8] improved Barnette’s result by showing that every 3-connected planar graph has a 2-walk. Yu [23] improved the results of Thomassen and Ellingham and Gao by showing that on a fixed surface, a 3-connected graph of large face-width has a 3-walk, and a 4-connected graph of large face-width has a 2-walk: the surface can be orientable or nonorientable, and the graph need not be a triangulation. Yu [23] also verified a conjecture of Thomassen [20] that every 5-connected triangulation of large face-width on a fixed surface is hamiltonian. Kawarabayashi [11] improved the conclusion here to hamiltonian-connected. Yu [23] posed the question of whether every 5-connected graph (not just triangulation) of large face-width on a fixed surface is hamiltonian, which is still unresolved. Thomassen [20] showed that for every surface of Euler genus greater than 2 there are 4-connected triangulations of arbitrarily large face-width that are not hamiltonian, so this would be best possible.

One way to tighten results on the existence of $k$-trees or $k$-walks is to bound the number of vertices of high degree, or visited more than once. Kawarabayashi, Nakamoto and Ota improved Thomassen’s result on 4-trees and Yu’s result on 3-walks as follows (the bounds are best possible).
Theorem 1.1. [12] For every nonspherical closed surface $F^2$ of Euler genus $k$, there exists a positive integer $N(F^2)$ such that every 3-connected $N(F^2)$-representative graph on $F^2$ has a 4-tree with at most $\max\{2k - 5, 0\}$ vertices of degree 4, and a 3-walk in which at most $\max\{2k - 4, 0\}$ vertices are visited 3 times.

A further way to generalize hamiltonicity is as follows. A $k$-covering (sometimes called a $k$-trestle) of a graph $G$ is a spanning 2-connected subgraph of $G$ with maximum degree at most $k$. Hence a 2-covering is exactly a hamiltonian cycle. The first result in this area was by Barnette [3], who showed that every 3-connected planar graph has a 15-covering; this was improved by Gao [7], who showed that every 3-connected graph on a surface with nonnegative Euler characteristic has a 6-covering. Barnette showed this would be best possible. For arbitrary surfaces, Sanders and Zhao [17] showed that 3-connected graphs on a fixed surface $F^2$ have a $K(F^2)$-covering, where $K$ is bounded by a linear function of the genus.

It is possible to obtain a result for graphs of large face-width on a fixed surface, and at the same time bound the number of vertices of high degree. Kawarabayashi, Nakamoto and Ota proved the following (the bounds “$4k - 8$” and “$2k - 4$” are best possible).

Theorem 1.2. [12] For every nonspherical closed surface $F^2$ of Euler genus $k$, there exists a positive integer $N(F^2)$ such that every 3-connected $N(F^2)$-representative graph on $F^2$ has an 8-covering with at most $\max\{4k - 8, 0\}$ vertices of degree 7 or 8, among which at most $\max\{2k - 4, 0\}$ have degree 8.

The bound “8” in Theorem 1.2 is not best possible. Kawarabayashi, Nakamoto and Ota improved this to 7, at the cost of increasing the number of vertices of large degree, as follows (the bound “$6k - 12$” is best possible).

Theorem 1.3. [13] For every nonspherical closed surface $F^2_k$ of Euler genus $k \geq 2$, there exists a positive integer $M(F^2)$ such that every 3-connected $M(F^2)$-representative graph on $F^2$ has a 7-covering with at most $6k - 12$ vertices of degree 7.

However, for each closed surface $F^2_k$ with $k > 2$, there exists a triangulation with arbitrarily large face-width having no 6-covering.

Now let us focus on the 4-connected case. Recently, Böhme, Mohar and Thomassen proved the following.

Theorem 1.4. [4] There exists a function $a : \mathbb{N}_0 \times \mathbb{R}_+ \rightarrow \mathbb{N}$ such that for each $\varepsilon > 0$, if $G$ is a 4-connected graph embedded on a closed surface of Euler genus $k$ such that the face-width of $G$ is at least $a(k, \varepsilon)$, then $G$ has a 4-covering such that the number of vertices of degree 3 or 4 is at most $\varepsilon|V(G)|$.

Kawarabayashi, Nakamoto and Ota were able to provide a linear bound on the number of vertices of degree 4.

Theorem 1.5. [12] For every nonspherical closed surface $F^2$ of Euler genus $k$, there exists a positive integer $N(F^2)$ such that every 4-connected $N(F^2)$-representative graph on $F^2$ has a 4-covering with at most $\max\{4k - 6, 0\}$ vertices of degree 4.
But the bound “4” in the above theorem is not best possible. The purpose of this paper is to prove that the bound “4” can be improved to 3.

**Theorem 1.6.** There exists a function \( a : \mathbb{N}_0 \times \mathbb{R}_+ \to \mathbb{N} \) such that for each \( \varepsilon > 0 \), if \( G \) is a 4-connected graph embedded on a closed surface of Euler genus \( k \) such that the face-width of \( G \) is at least \( a(k, \varepsilon) \), then \( G \) has a 3-covering (2-connected spanning subgraph with maximum degree at most 3) in which the number of vertices of degree 3 is at most \( \varepsilon |V(G)| \).

But perhaps the bound on the number of vertices of degree 3 in the above theorem is not best possible. The natural conjecture is the following.

**Conjecture 1.1.** [12] For every nonspherical closed surface \( F^2_k \) of Euler genus \( k \), there exists a positive integer \( M(F^2) \) such that every 4-connected \( M(F^2) \)-representative graph on \( F^2 \) has a 3-covering with at most \( ck \) vertices of degree 3, where \( c \) is a constant which does not depend on \( k \).

The bound “3” here would be best possible, as shown by Thomassen’s nonhamiltonian 4-connected triangulations of large face-width, mentioned earlier. If true, Conjecture 1.1 implies a conjecture of Mohar [14] which says for every nonspherical closed surface \( F^2_k \) of Euler genus \( k \), there exists a positive integer \( M(F^2) \) such that every 4-connected \( M(F^2) \)-representative graph on \( F^2 \) has a 3-tree with at most \( ck \) vertices of degree 3, where \( c \) is a constant which does not depend on \( k \).

However, Conjecture 1.1 seems to be difficult because it is closely related to the conjecture of Grünbaum [9] and Nash-Williams [15] that every 4-connected graph in the torus is hamiltonian. So far, we know from Sanders and Zhao [17] that every 4-connected graph in the torus or in the Klein bottle has a 3-covering.

### 2. Definitions and preliminary results

If \( P \) is a path containing vertices \( u \) and \( v \), let \( P[u, v] \) denote the subpath of \( P \) between \( u \) and \( v \). If \( C \) is a cycle with a particular assumed direction, let \( C[u, v] \) denote the subpath of \( C \) from \( u \) to \( v \) in the given direction.

A **disk graph** is a graph \( H \) embedded in a closed disk, such that a cycle \( Z \) of \( H \) bounds the disk. We write \( \partial H = Z \). An **internally 4-connected disk graph** or **I4CD graph** is a disk graph \( H \) such that from every internal vertex \( v \) (\( v \in V(H) - V(\partial H) \)) there are four paths, pairwise disjoint except at \( v \), from \( v \) to \( \partial H \).

A **cylinder graph** is a graph \( H \) embedded in a closed cylinder, such that two disjoint cycles \( Z_0, Z_1 \) of \( H \) bound the cylinder. We write \( \partial H = Z_0 \cup Z_1 \). An **internally 4-connected cylinder graph** or **I4CC graph** is a cylinder graph \( H \) such that from every internal vertex \( v \) there are four paths, pairwise disjoint except at \( v \), from \( v \) to \( \partial H \). Note that an I4CC graph is not necessarily connected: \( Z_0 \) and \( Z_1 \) may lie in different components.

If \( G \) is an embedded graph and \( Z \) is a contractible cycle of \( G \) bounding a closed disk, then the embedded subgraph consisting of all vertices, edges and faces in that closed disk is a **disk subgraph** of \( G \). Similarly, if \( Z_0 \) and \( Z_1 \) are disjoint homotopic cycles bounding a closed cylinder, then the embedded subgraph \( H \) consisting of all vertices, edges and faces in that closed cylinder is a **cylinder subgraph** of \( G \). We write \( H = Cyl_G[Z_0, Z_1] \) or just \( H = Cyl[Z_0, Z_1] \). If the surface is a torus or Klein bottle and \( Z_0, Z_1 \) are nonseparating, then this notation is ambiguous,
but it should be clear from context which one of the two possible cylinders we mean. We define \( \text{Cyl}(Z_0, Z_1) \) to be the graph \( \text{Cyl}(Z_0, Z_1) - V(Z_0) \), and define \( \text{Cyl}(Z_0, Z_1) \) and \( \text{Cyl}(Z_0, Z_1) \) similarly.

The following is easy to prove.

**Lemma 2.1.** Suppose \( G \) is a 4-connected embedded graph. Any disk subgraph of \( G \) bounded by a cycle of length at least 4 is I4CD, and any cylinder subgraph of \( G \) is I4CC.

Suppose \( G \) is an embedded graph. If \( R = \{R_0, R_1, \ldots, R_m\} \) is a collection of pairwise disjoint homotopic cycles with \( R_i \subseteq \text{Cyl}(R_0, R_m) \) for each \( i \), and \( S = \{S_0, S_1, \ldots, S_{n-1}\} \) is a collection of disjoint paths with \( S_j \subseteq \text{Cyl}(R_0, R_m) \) for each \( j \), such that \( R_i \cap S_j \) is a nonempty path (possibly a single vertex) for each \( i \) and \( j \), then we say that \((R, S)\) is a cylindrical mesh in \( G \).

In two places in the proof of Theorem 1.6 (Steps 3 and 6) we will need to move two consecutive cycles in a cylindrical mesh closer together, so that there are no vertices between them. An arbitrary homotopic shifting of a cycle may not preserve the existence of a mesh, so we need the following technical lemma.

**Lemma 2.2.** Suppose \( N \) is an I4CC graph with \( \partial N = R_0 \cup R_1 \) that has a cylindrical mesh \((\{R_0, R_1\}, \{S_0, S_1, \ldots, S_{n-1}\})\).

(i) In \( N \) there are disjoint cycles \( R'_0 \) and \( R'_1 \) homotopic to \( R_0 \) (with \( R'_0 \) closer to \( R_0 \)) and pairwise disjoint paths \( S'_0, S'_1, \ldots, S'_{n-1} \), such that \( \text{Cyl}(R'_0, R'_1) \) is empty, each \( S'_j \) has the same ends as \( S_j \) (which may be outside \( \text{Cyl}(R'_0, R'_1) \)), and \( R'_i \cap S'_j \) is a nonempty path for each \( i \) and \( j \).

(ii) Moreover, if every component of \( \text{Cyl}(R_0, R_1) \) has at most two neighbors on \( R_0 \), we may take \( R'_0 = R_0 \).

**Proof.** (i) Embed \( N \) in the plane with \( R_1 \) as the outer face and \( R_0 \) as an inner face, with \( S_0, S_1, \ldots, S_{n-1} \) directed outwards from \( R_0 \) to \( R_1 \), and with all cycles directed clockwise. The proof is by induction on the number of vertices of \( \text{Cyl}(R_0, R_1) \). If there are none we are finished. Otherwise, let \( T \) be a component of \( \text{Cyl}(R_0, R_1) \). Since \( N \) is I4CC, \( T \) has at least two neighbors on one of \( R_0 \) or \( R_1 \).

Assume first that \( T \) has two neighbors on \( R_0 \). The graph \( A \) consisting of \( R_0, T, \) and all edges joining \( T \) to \( R_0 \) has a block \( B \) containing \( R_0 \) and at least one vertex of \( T \).

Suppose that some \( S_i \) has a subpath with both ends in \( B \) but containing an edge not in \( B \). This path has a subpath \( P \) whose ends are in \( B \) and all of whose edges and internal vertices are not in \( B \). If an internal vertex of \( P \) belongs to \( R_1 \), then \( R_1 \cap S_i \) is not a path, a contradiction, so \( V(P) \cap V(R_1) = \emptyset \). If both ends of \( P \) are in \( R_0 \), then \( R_0 \cap S_i \) is not a path, a contradiction, so at least one end of \( P \) is in \( T \). It follows that all internal vertices of \( P \) belong to \( V(T) - V(B) \), and all edges of \( P \) belong to \( E(A) - E(B) \). Thus, \( B \cup P \) is a 2-connected subgraph of \( A \) larger than \( B \), contradicting the fact that \( B \) is a block of \( A \). Hence, every subpath of every \( S_i \) with both ends in \( B \) lies completely in \( B \).

Let \( R'_0 \) be the outer cycle of \( B \). (The subgraph of \( N \) between \( R_0 \) and \( R'_0 * \) may contain vertices not in \( A \) or \( B \), from components of \( \text{Cyl}(R_0, R_1) \) other than \( T \), but this does not affect our argument.) For each \( i \), let \( r_i \) be the first vertex of \( S_i \), let \( s_i \) be the first vertex of \( S_i \) that belongs to \( R'_0 * \), let \( t_i \) be the last vertex of \( S_i \) that belongs to \( B \) (\( t_i \) is necessarily also the last vertex of \( S_i \) on \( R'_0 \)), and let \( u_i \) be the last vertex of \( S_i \). From above, each \( S_i[r_i, t_i] \) lies entirely in \( B \).
If $s_i \neq t_i$, then in the closed disk bounded by $R_0^*$ one of $R_0^*[s_i, t_i]$ or $R_0^*[t_i, s_i]$ lies on the same side of $S_i[s_i, t_i]$ as the interior of $R_0$, and the other lies on the opposite side. Let $Z_i$ denote the one on the opposite side, or let $Z_i = s_i = t_i$ if $s_i = t_i$. By planarity $S_j[r_j, t_j]$ does not intersect $Z_i$ for any $j \neq i$. If $Z_i$ intersects $Z_j$ then at least one of $s_i \in V(Z_j)$, $t_i \in V(Z_j)$, $s_j \in V(Z_i)$ or $t_j \in V(Z_i)$ must hold, which contradicts the fact that $S_j[r_j, t_j] \cap Z_i$ and $S_i[r_i, t_i] \cap Z_j$ are empty. Therefore, the paths $S_i^* = Z_i \cup S_i[t_i, u_i]$ for $0 \leq i \leq n - 1$ are pairwise disjoint, with $R_0^* \cap S_i^* = Z_i$ and $R_1 \cap S_i^* = R_1 \cap S_i$ both being paths for each $i$.

Since $Cyl(R_0^*, R_1)$ has fewer vertices than $Cyl(R_0, R_1)$, we may apply induction to $Cyl(R_0^*, R_1)$, $R_0^*$, $R_1$, $S_0^*, \ldots, S_n^*$, to obtain $R_0'$, $R_1'$, and paths $S_0'', \ldots, S_n''$. Let $S_i' = S_i[r_i, s_i] \cup S_i''$ for each $i$, then the required conclusion holds.

Similarly, if $T$ has two neighbors on $R_1$ then we may construct an $R_1^*$ and apply induction to $Cyl(R_0, R_1^*)$.

(ii) If every component of $N - V(R_0 \cup R_1)$ has at most two neighbors on $R_0$, then in the above $T$ always has at least two neighbors on $R_1$, and we can always construct $R_1^*$ rather than $R_0^*$. The components of $Cyl(R_0, R_1^*)$ are subgraphs of the components of $Cyl(R_0, R_1)$, and so also have at most two neighbors on $R_0$. Thus, by induction we may take $R_0' = R_0$. \hfill $\square$

3. Proof of Theorem 1.6

We divide the proof into ten steps. Since 4-connected graphs on the plane (and hence on the sphere) or projective plane are hamiltonian [18,21], we assume $F^2$ has Euler genus at least 2.

Step 1 (Cylindrical meshes on handles). Let $\Phi$ and $\Psi$ be graph embeddings on the same closed surface $F^2$. We say that $\Psi$ is a surface minor of $\Phi$ if $\Psi$ can be obtained (up to homeomorphism) from $\Phi$ by a sequence of contractions and deletions of edges. The following deep result by Robertson and Seymour will be used to guarantee that the embedding of $G$ contains certain cylindrical meshes.

Lemma 3.1. (Robertson and Seymour [16]) Let $\Omega$ be a fixed graph embedding on a closed surface $F^2$. Then, there exists a positive integer $R(\Omega)$ such that if $\Phi$ is an $R(\Omega)$-representative graph embedding on $F^2$, then $\Phi$ has $\Omega$ as a surface minor.

Suppose $F^2$ has Euler genus $2g$ or $2g + 1$, where $g \geq 1$. Let $q \geq 2$ be an integer so that $1/q \leq \varepsilon$. We can find a connected graph $M$ with an embedding $\Omega$ on $F^2$ that contains $g$ pairwise disjoint cylindrical embeddings of copies of $Q = P_{7q + 1} \times C_{40}$ (“$\times$” denotes Cartesian product), in such a way that deleting the vertices of one $C_{40}$ in each of the $g$ copies results in a planar or projective-planar graph. Take the representativity of the embedding of $G$ to be at least $\max\{4, R(\Omega)\}$, with $R(\Omega)$ from Lemma 3.1. Then the embedding of $G$ has $\Omega$ as a surface minor, with pairwise disjoint subgraphs $Q_1, Q_2, \ldots, Q_g$ of $G$ contracting to the copies of $Q$ in $M$. Each $Q_i$ has pairwise disjoint cycles $R_{0i}, R_{1i}, \ldots, R_{7q}$ (in that order) and paths $S_{i0}, S_{i1}, \ldots, S_{i39}$ (in that cyclic order) such that each $R_{ij}$ contracts to one of the $C_{40}$ in a copy of $Q$, each $S_{ik}$ contracts to one of the $P_{7q + 1}$ in a copy of $Q$, and $(\{R_{ij} : 0 \leq j \leq 7q\}, \{S_{ik} : 0 \leq k \leq 39\})$ is a cylindrical mesh in $G$. Deleting the vertices of one $R_{ij}$ for each $i$ from $G$ results in a planar or projective-planar graph.

Step 2 (Small cylinders). For each $i, 1 \leq i \leq g$, choose $m_i \in \{0, 1, \ldots, q - 1\}$ so as to minimize $|V(Cyl(R_{i,7m_i}, R_{i,7m_i+7}))|$. Then $\bigcup_{i=1}^g V(Cyl(R_{i,7m_i}, R_{i,7m_i+7})) < |V(G)|/q \leq \varepsilon|V(G)|$. We
will construct a 3-covering all of whose degree 3 vertices lie in this set. To simplify our notation, we assume without loss of generality that \( m_i = 0 \) for each \( i \), so we will be concerned with \( \text{Cyl}[R_{i0}, R_{i7}] \) for each \( i \).

**Step 3 (Empty spaces for cutting).** For each \( i \), \( 1 \leq i \leq g \), define \( X_{2i-1} = R_{i0} \), \( Y_{2i-1} = R_{i1} \), \( Z_{2i-1} = R_{i2} \), \( Z_{2i} = R_{i5} \), \( Y_{2i} = R_{i6} \), and \( X_{2i} = R_{i7} \). By Lemma 2.1 we may apply Lemma 2.2(i) to each cylinder \( \text{Cyl}(Y_j, Z_j) \), \( 1 \leq j \leq 2g \), modifying the paths \( S_{ij} \), \( 0 \leq k \leq 39 \), as specified by Lemma 2.2 to preserve the existence of a cylindrical mesh. Thus, we may assume that \( \text{Cyl}(Y_j, Z_j) \) is empty for each \( j \).

**Step 4 (Cut \( G \) into a planar or projective-planar subgraph and \( g \) cylinder subgraphs).** Define \( H = G - \bigcup_{i=1}^{g} V(\text{Cyl}(Z_{2i-1}, Z_{2i})) \), then \( H \) has \( g \) cylindrical faces, each bounded by \( Y_{2i-1} \) and \( Y_{2j} \) for some \( i \). By cutting around each such cylindrical face, and filling in the resulting pair of holes with two disks, we obtain an embedding of \( H \) in the plane or projective plane, in which each cycle \( Y_j \), \( 1 \leq j \leq 2g \), bounds a face. Now \( V(G) \) is partitioned by \( H \) and \( \text{Cyl}(Z_{2i-1}, Z_{2i}), i \leq i \leq g \). These are all 2-connected graphs, because if there were a cutvertex, either it would be a cutvertex in \( G \), or there would be a nonseparating simple closed curve intersecting \( G \) only at the cutvertex, contradicting the fact that \( G \) is 4-connected and embedded 4-representatively. For similar reasons, any 2-cut or 3-cut \( S \) in \( H \) must contain at least two vertices of some \( Y_j \). Moreover, \( H - S \) has exactly two components, one of which is a subgraph of \( \text{Cyl}(X_j, Y_j) \).

Now for \( 1 \leq j \leq 2g \), add a vertex \( v_j \) in each face of \( H \) bounded by \( Y_j \), joining \( v_j \) to each vertex of \( Y_j \) that is adjacent in \( G \) to a vertex of \( Z_j \). Let \( H' \) be the resulting graph embedded in the plane or projective plane. Since \( H \) is 2-connected, \( H' \) is 2-connected. Consider any minimal cutset \( S' \) of \( H' \) with \( |S'| \leq 3 \). If \( S' \) contains no \( v_j \), it is a cutset in \( H \), using two vertices of some \( Y_j \). Let \( T \) be the component of \( H - S' \) contained in \( \text{Cyl}(X_j, Y_j) \). Since \( G \) is 4-connected, \( v_j \) and \( T \) are part of the same component of \( H' - S' \). But then there is a nonseparating simple closed curve intersecting \( G \) only at \( S' \), contradicting the fact that the embedding of \( G \) is 4-representative. Therefore \( S' \) contains some \( v_j \). Then \( S = S' - \{v_j\} \) is a cutset in \( H \), so \( |S| = 2 \), and both vertices of \( S \) belong to some \( Y_k \). Since \( S' \) is minimal, \( v_j \) is adjacent to vertices in more than one component of \( H' - S' \), so \( k = j \). Thus, we have proved that \( H' \) is 3-connected, and any 3-cut \( S' \) in \( H' \) consists of some \( v_j \) and two vertices on \( Y_j \). Moreover, \( H' - S' \) has exactly two components, one of which is a subgraph of \( \text{Cyl}(X_j, Y_j) \).

**Step 5 (Tutte cycle).** A Tutte cycle \( C \) in a graph \( G \) is a cycle so that every component of \( G - V(C) \) has at most three neighbors on \( C \). If \( C' \) is a cycle in \( G \), then a Tutte cycle with respect to \( C' \) in \( G \) is a Tutte cycle \( C \) with the added property that any component of \( G - V(C) \) containing a vertex of \( C' \) has at most two neighbors on \( C \). We construct a Tutte cycle in \( H' \) to form the skeleton of our 3-covering of \( G \). Some care is required to avoid getting a 3-cycle, or a cycle restricted to the disk subgraph of \( H' \) bounded by \( X_j \) for some \( j \).

Since \( q \geq 2 \), there is \( w \in V(G) \) at distance at least two from \( \bigcup_{j=1}^{g} \text{Cyl}(X_{2i-1}, X_{2i}) \). Let \( ww_1, \ldots, \ldots, w_k \) be the edges around \( w \) in cyclic order, where \( k \geq 4 \) since \( G \) is 4-connected. Since the embedding of \( G \) is 3-representative, there is a cycle \( W \) in \( G \), and hence in \( H' \), containing \( w_1, w_2, \ldots, w_k \) in that order, bounding a closed disk containing all faces incident with \( w \). The cycle \( W' = ww_1 \cup W[w_1, w_3] \cup w_3w \) is a face of \( G - ww_2 \) and also of the planar or projective-planar embedding of \( H' - ww_2 \). Since \( H' - ww_2 \) is 2-connected, by [21] (if \( H' \) is planar) or [18] (if \( H' \) is projective-planar) we can find a Tutte cycle \( C \) with respect to \( W' \) in \( H' - ww_2 \).
through \( w_3 \). If \( w_2 \notin V(C) \), let \( A \) denote the component of \( H' - w_2 - V(C) \) containing \( w_2 \), which has at most two neighbors on \( C \).

Suppose \( C \) is a 3-cycle. Then \( C \) is a cycle in \( G \). Since \( G \) is embedded 4-representatively and is 4-connected, \( C \) is contractible and does not separate \( G \). In other words, \( C \) is a face of \( G \), so it must be \( w_2 w_3 w_4 \). But now \( A \) contains the path \( W = \{w_3, w_4\} \), and so has the three neighbors \( w, w_3, w_4 \) on \( C \), a contradiction. Therefore, \( C \) is not a 3-cycle.

If \( w_2 \notin V(C) \), restoring \( w_2 w_3 \) to \( H' - w_2 \) adds at most one neighbor on \( C \) to the component \( A \), which therefore has at most three neighbors on \( C \). Thus, \( C \) is a Tutte cycle in \( H' \).

Let \( T \) be a component of \( H' - V(C) \). Since \( C \) is a Tutte cycle in \( H' \) and \( H' \) is 3-connected, \( T \) has a set \( S' \) of exactly three neighbors on \( C \). Since \( C \) is not a 3-cycle, \( S' \) is a cutset. From above, \( S' \) consists of \( v_j \) and two vertices of \( Y_j \), for some \( j \), and \( H' - S' \) has exactly two components: \( T \), and another component \( T' \) that contains \( C - S' \). Moreover, one of \( T \) or \( T' \), call it \( T_1 \), is a subgraph of \( \text{Cyl}(X_j, Y_j) \). By choice of \( w \), \( w \) is not adjacent to a vertex of \( S' \), so \( w \in V(C - S') \). However, \( w \notin \bigcup_{j=1}^{2g} V(\text{Cyl}(X_{2i-1}, X_{2i})) \), so \( w \), and hence \( C - S' \), are not in \( T_1 \). Thus, \( T_1 = T \), so that \( T \) is a subgraph of \( \text{Cyl}(X_j, Y_j) \).

Such a \( T \) cannot contain any vertex \( v_k \), so \( C \) contains all vertices \( v_1, v_2, \ldots, v_{2g} \).

**Step 6** (Absorb vertices not used by \( C \) into the cylinders). Let \( T \) denote the set of components of \( H' - V(C) \). From above, every \( T \in \mathcal{T} \) is adjacent in \( H' \) to three vertices of \( C \), namely \( v_j \) for some \( j \), and two vertices \( y_T, y'_T \in V(Y_j) \), where we choose the order of \( y_T, y'_T \) relative to a fixed orientation of \( Y_j \). If \( T \cap Y_j = \emptyset \), then \( \text{Cyl}(X_j, Y_j) \) is a subgraph of \( G \) not used by \( C \).

For each \( j, 1 \leq j \leq 2g \), let \( T_j \) be the set of components of \( H' - V(C) \) that are adjacent in \( H' \) to \( v_j \), so that \( T = \bigcup_{j=1}^{2g} T_j \). Let \( Y'_j \) be the cycle in \( G' \) obtained from \( Y_j \) by replacing the segment \( Y_j[y_T, y'_T] \) by the edge \( y_T y'_T \). For each \( T \in T_j \); then \( V(Y'_j) = V(Y_j) \cap V(C) \). Modify each path \( S_{ik} \) to obtain \( S'_{ik} \) in \( G' \) by replacing any segment \( Y_j[y_T, y'_T] \) by the edge \( y_T y'_T \). Then \( \{X_{2i-1}, Y'_{2i-1}, Z_{2i-1}, R_{i3}, R_{i4}, Z_{2i}, Y'_{2i}, X_{2i}\}, \{S'_{ik} | 0 \leq k \leq 39\} \) forms a cylindrical mesh in \( G' \) for each \( i \).

For each \( j \), the components of each \( \text{Cyl}_C(Y'_j, Z_j) \) are precisely the elements of \( T_j \), each of which is adjacent to two vertices of \( Y'_j \). Thus, Lemma 2.1 allows us to apply Lemma 2.2(ii) for each \( j \) to find \( Z'_j \) (not changing \( Y'_j \)) such that \( \text{Cyl}_G(Y'_j, Z'_j) \) is empty, modifying the paths \( S'_{jj/2i,k}, 0 \leq k \leq 39 \), appropriately, so that for each \( i, 1 \leq i \leq g \), \( \{X_{2i-1}, Y'_{2i-1}, Z'_{2i-1}, R_{i3}, R_{i4}, Z_{2i}, Y'_{2i}, X_{2i}\}, \{S'_{ik} | 0 \leq k \leq 39\} \) forms a cylindrical mesh in \( G' \). Each \( Z'_j \) is a cycle in \( G \) as well as in \( G' \) (since it contains no edge \( y_T y'_T \)), and every vertex of \( G \) is either in \( C \) or belongs to a cylinder subgraph \( \text{Cyl}(Z'_{2i-1}, Z'_{2i}) \).

**Step 7** (Two large subgraphs in each cylinder). For each \( j \), let \( r_j, r'_j \in V(Y'_j) \) denote the neighbors of \( v_j \) in \( C \). Then in \( G \) or \( G' \), each \( r_j \) is adjacent to some \( s_j \in V(Z'_j) \) and each \( r'_j \) is adjacent to some \( s'_j \in V(Z'_j) \). If \( s_j \neq s'_j \), let \( W_j = \{r_j, r'_j\} \) and \( V_j = \{s_j, s'_j\} \). If \( s_j = s'_j \), then we let \( x_j \) and \( x'_j \) denote the vertices closest to \( s_j \) in either direction along \( Z'_j \) that have a neighbor in \( Y'_j \), and we let \( w_j \) and \( w'_j \), respectively, be those neighbors. In this case, let \( W_j = \{r_j, r'_j, w_j, w'_j\} \) and \( V_j = \{s_j = s'_j, x_j, x'_j\} \).

We now claim that for each \( i, 1 \leq i \leq g \), \( \text{Cyl}(Z'_{2i-1}, Z'_{2i}) \) has disjoint disk subgraphs \( L_{2i-1}, L_{2i} \) with the following properties.
(i) \( L_{2i-1} \cap Z'_{2i-1}, L_{2i-1} \cap Z'_2 \), \( L_{2i} \cap Z'_{2i-1}, L_{2i} \cap Z'_2 \) are all paths with at least one edge;
(ii) for \( j = 2i - 1 \) and \( 2i \), every neighbor of \( W_j \) on \( Z'_j \) (including every vertex of \( V_j \)) belongs to \( L_j \);
(iii) for \( j = 2i - 1 \) and \( 2i \), no vertex of \( Y'_j \) is adjacent to both components of \( Z'_j - V(L_{2i-1} \cup L_{2i}) \); and
(iv) subject to (i), (ii) and (iii), \( |V(L_{2i-1} \cup L_{2i})| \) is as large as possible.

We prove this for \( i = 1 \); the proof for general \( i \) is similar. We need only find \( L_1 \) and \( L_2 \) satisfying (i), (ii) and (iii).

Define \( R'_{11} = Y'_1, R'_{12} = Z'_1, R'_{13} = R_{13}, R'_{14} = R_{14}, R'_{15} = Z'_2 \) and \( R'_{16} = Y'_2 \). For each \( j, 1 \leq 1 \leq 5 \), and for each \( k \in \mathbb{Z}_{40} \), let \( U_{jk} \) denote the disk subgraph of \( G' \) bounded by subpaths of \( R'_{1j} \), \( R'_{1j+1} \), \( S'_{1k} \) and \( S'_{1,k+1} \) that does not contain vertices of any other paths of the cylindrical mesh. We call \( U_{jk} \) a cell of the mesh. Let \([i, j] \) denote the set \( \{i, i + 1, \ldots, j \} \) either as an interval in the integers, or as a cyclic interval in \( \mathbb{Z}_{40} = \{0, 1, \ldots, 39\} \)—it will be clear from context which is intended. Let \( U_{j,[k_1,k_2]} = \bigcup_{k \in [k_1,k_2]} U_{jk} \) and \( U_{j,[j_1,j_2],[k_1,k_2]} = \bigcup_{j \in [j_1,j_2]} U_{j,[k_1,k_2]} U_{jk} \).

Let \( U_{1,\{a,a+a+1\}} \) be a contiguous block of cells that contains \( V_1 \), such that \( a \) is as small as possible. Then \( \alpha \leq 20 \). The neighbors of \( V_1 \) on \( R'_{11} \), including \( W_1 \), lie in \( U_{1,\{a-1,a+a+1\}} \). Therefore, the neighbors of \( W_1 \) on \( R'_{12} = Z'_1 \) lie in \( U_{1,\{a-2,a+a+2\}} \cap R'_{12} \subseteq U_{2,\{a-3,a+a+3\}} \cap R'_{12} \). Similarly, there are \( b \) and \( \beta \leq 20 \) such that the neighbors of \( W_2 \) on \( R'_{15} = Z'_2 \) lie in \( U_{4,\{b-3,b+b+3\}} \cap R'_{15} \).

Now \( L_1^2 = U_{4,\{b-3,b+b+3\}} \) and \( L_2^2 = U_{2,\{a-7,a-5\}} \). Together use up at most \( 27 + 3 = 30 \) of the 40 cells \( U_{4j} \), in one or two contiguous blocks. Therefore there is a block of at least 5 contiguous unused cells. Hence, we can choose \( c \) so that \( U_{4,\{c,c+2\}} \) is a block of 3 cells disjoint from \( L_1^2 \cup L_2^2 \).

If \( \{b-3,b+b+3\} \cup \{a-7,a-5\} \) is a cyclic interval in \( \mathbb{Z}_{40} \), define \( L_3^2 = \emptyset \); otherwise, define \( L_3^2 \) to be whichever of \( U_{4,\{b+b+4,a-8\}} \) or \( U_{4,\{a-4,b-4\}} \) does not intersect \( U_{4,\{c,c+2\}} \). Let \( L_1^1 = U_{2,\{a-3,a+a+3\}} \) and \( L_2^1 = U_{2,\{a-7,a-5\}} \). If \( \{a-3,a+a+3\} \cup \{c,c+2\} \) is a cyclic interval in \( \mathbb{Z}_{40} \), define \( L_3^1 = \emptyset \); otherwise, define \( L_3^1 = U_{2,\{a+a+c-1\}} \).

Then \( L_1 = L_1^1 \cup L_1^2 \cup L_3^1 \) and \( L_2 = L_2^1 \cup L_2^2 \cup L_3^2 \) are both unions of contiguous blocks of cells, using cyclic intervals of cells along \( R'_{12} = Z'_1 \) and \( R'_{15} = Z'_2 \), giving (i). Property (ii) is immediate from our construction. For (iii), consider any \( v \) on \( R'_{11} = Y'_1 \). Since \( v \) belongs to at most two cells \( U_{1j} \), the neighbors of \( v \) on \( R'_{12} = Z'_1 \) lie in \( U_{1,\{d,d+1\}} \) for some \( d \). Since both \( L_1^1 \cup L_3^1 \) and \( L_2^2 \) use at least three contiguous blocks \( U_{2j} \), it is not possible for \( U_{1,\{d,d+1\}} \) to intersect both components of \( Z'_1 - V(L_1 \cup L_2) \) except trivially. A similar argument applies to vertices of \( Y'_2 \).

**Step 8 (The remainder of each cylinder).** Now we show that for each \( i \), \( \text{Cyl}(Z'_{2i-1}, Z'_{2i}) \) contains four additional subgraphs \( M_{ji} \), \( j = 2i - 1 \) or \( 2i \) and \( l = 1 \) or \( 2 \), each of which intersects \( L_{2i-1} \cup L_{2i} \) at exactly two vertices \( u_{j,2i-1}, u_{j,2i} \) of \( Z'_j \). We begin with the case \( i = 1 \).

There are vertices \( u_{11}, u_{12}, u_{13}, u_{14} \) in order along \( Z'_1 \), and \( u_{21}, u_{22}, u_{23}, u_{24} \) in order along \( Z'_2 \), such that \( a_{L_1} = Z'_1[u_{14}, u_{11}] \cup Z'_2[u_{24}, u_{21}] \cup P_4 \cup P_1 \) and \( a_{L_2} = Z'_1[u_{12}, u_{13}] \cup Z'_2[u_{22}, u_{23}] \cup P_2 \cup P_3 \), where each \( P_k \) is a path from \( u_{1k} \) to \( u_{2k} \) internally disjoint from \( Z'_1 \cap Z'_2 \).

We first claim that \( u_{11} \) and \( u_{12} \) lie on a common face of \( G' \). Consider the boundaries of the faces containing \( u_{11} \). If they do not contain \( u_{12} \), then there must exist a path joining \( Z'_1[u_{11}, u_{12}] - \{u_{11}, u_{12}\} \) and \( (P_1 \cup Z'_2[u_{21}, u_{22}] \cup P_2) - \{u_{11}, u_{12}\} \). This contradicts the maximality of \( |V(L_1 \cup L_2)| \).

Thus, we can add an edge \( u_{11}u_{12} \) (if it does not already exist) through this face. In the same way, we can add an edge \( u_{21}u_{22} \). Consider the disk subgraph \( U_{1j} \) bounded by \( P_1, P_2 \) and \( u_{11}u_{12}, u_{21}u_{22} \). If \( U_1 \) contains an interior vertex \( v \), then since \( G \) is 3-connected, there exist three dis-
joint paths joining \( v \) to the boundary of \( U_1 \). However, this also contradicts the maximality of \( |V(L_1 \cup L_2)| \). Thus, \( U_1 \) has no interior vertices.

Similarly, \( U_2 \) has no interior vertices, where \( U_2 \) is bounded by \( P_3, P_4 \) and \( u_{13}u_{14}, u_{23}u_{24} \) (we add these edges as before).

If \( Z'[u_{11}, u_{12}] \) is the single edge \( u_{11}u_{12} \), define \( M_{11} = Z'[u_{11}, u_{12}] \). Otherwise, let \( M_{11} \) denote the disk subgraph bounded by \( Z'[u_{11}, u_{12}] \cup \{u_{11}u_{12}\} \). Let \( q_{11} \) denote the vertex of \( Z'[u_{11}, u_{12}] \) that is closest to \( u_{11} \) that has a neighbor \( p_{11} \) on \( Y'_1 \), and let \( q_{12} \) denote the vertex of \( Z'[u_{11}, u_{12}] \) that is closest to \( u_{12} \) that has a neighbor \( p_{12} \) on \( Y'_1 \). Since \( G \) is 4-connected, \( q_{11} \neq q_{12} \), and we may assume that \( p_{11} \neq p_{12} \).

In a similar way we can construct \( M_{21} \) bounded by \( Z'[u_{21}, u_{22}] \cup \{u_{21}u_{22}\} \), \( M_{22} \) bounded by \( Z'[u_{13}, u_{14}] \cup \{u_{13}u_{14}\} \), and \( M_{22} \) bounded by \( Z'[u_{23}, u_{24}] \cup \{u_{23}u_{24}\} \). More generally, for every \( j \) and \( l \), \( 1 \leq j \leq 2g \) and \( 1 \leq l \leq 2 \), we can construct \( M_{jl} \) and, if appropriate, \( q_{j,2l-1}, p_{j,2l-1}, q_{j,2l}, p_{j,2l}. \)

By property (iii) of Step 7, \( p_{j1}, p_{j2}, p_{j3}, p_{j4} \) are all distinct, and by property (ii) of Step 7 none of these vertices belong to \( W_j \). Because the disk subgraphs \( U_{2l-1}, U_{2l} \) have no interior vertices, every vertex of \( Cyl[Z'_{2l-1}, Z'_{2l}] \) belongs to exactly one of \( L_{2i-1}, L_{2i}, M_{jl} - \{u_{j,2l-1}, u_{j,2l}\}, j = 2i - 1 \) or \( 2i \) and \( l = 1 \) or \( 2 \).

**Step 9 (Spanning each \( L_j \) and \( M_{jl} \)).** In [17], Sanders and Zhao proved the following theorem. They stated it for “2-connected graphs without any interior component 3-cuts” but these are exactly our I4CD graphs.

**Theorem 3.7.** (Sanders and Zhao [17, Lemma 6.2]) Let \( G \) be an I4CD graph and let \( x, y \) be two distinct vertices in \( \partial G \). Then \( G \) has a 3-covering \( K \) such that \( E(\partial G) \subseteq E(K) \) and \( \deg_K(x) = 2, \deg_K(y) = 2 \).

For each \( L_j \) we construct two subgraphs which together include all vertices of \( L_j \), and connect \( L_j \) to \( C \). First suppose that \( s_j \neq s'_j \). Let \( D'_j \) denote whichever of \( Z'_j[s_j, s'_j] \) and \( Z'_j[s'_j, s_j] \) lies in \( L_j \), and let \( D_j = D'_j \cup \{r_j s_j, r'_j s'_j\} \). By Lemma 2.1 we may apply Theorem 3.7 to \( L_j \) to obtain a 3-covering \( E_j \) in which \( s_j, s'_j \) have degree 2. Note that \( D_j \) and \( E_j \) share the path \( D'_j \). Now suppose that \( s_j = s'_j \). Let \( D_j \) be the path \( r_j s_j r'_j \). The graph \( L_j \cup \{x_j s_j, x'_j s_j\} \) (adding the edges if not already present) is a disk subgraph of the 4-connected embedded graph \( G \cup \{x_j s_j, x'_j s_j\} \) and so is I4CD by Lemma 2.1. Apply Theorem 3.7 to this graph to obtain a 3-covering \( E'_j \) in which \( s_j \) has degree 2, which contains \( x_j s_j \) and \( x'_j s_j \). Let \( E_j = (E'_j - \{s_j\}) \cup \{w_j x_j, w'_j x'_j\} \).

Now for each \( M_{jl} \) we construct a subgraph which includes all vertices of \( M_{jl} - \{u_{j,2l-1}, u_{j,2l}\} \), and which connects this subgraph to \( C \). First, if \( M_{jl} \) is just a single edge \( u_{j,2l-1}u_{j,2l} \), let \( F_{jl} = \emptyset \). Now suppose \( M_{jl} \) is not a single edge. The graph \( M_{jl} \cup \{u_{j,2l-1}q_j, u_{j,2l-1}q'_j, u_{j,2l}q_j, u_{j,2l}q'_j\} \) (adding the edges if not already present) is a disk subgraph of the 4-connected embedded graph \( G \cup \{u_{j,2l-1}u_{j,2l}, u_{j,2l-1}q_j, u_{j,2l-1}q'_j, u_{j,2l}q_j, u_{j,2l}q'_j\} \), so it is I4CD by Lemma 2.1. Apply Theorem 3.7 to this graph to obtain a 3-covering \( F'_{jl} \) in which \( u_{j,2l-1}, u_{j,2l} \) have degree 2, which contains \( u_{j,2l-1}u_{j,2l} \), \( u_{j,2l-1}q_j, u_{j,2l-1}q'_j, u_{j,2l}q_j, u_{j,2l}q'_j \). Let \( F_{jl} = (F'_{jl} - \{u_{j,2l-1}, u_{j,2l}\}) \cup \{p_j,2l-1q_j,2l-1, p_j,2lq_j,2l\} \).

**Step 10 (Join everything together and verify 2-connectedness).** The proof of the following lemma is straightforward.
Lemma 3.2. Let \( G_1 \) and \( G_2 \) be disjoint 2-connected graphs, and suppose we form \( G \) from \( G_1 \) and \( G_2 \) in one of the following ways.

(i) Identify a path on at least two vertices in \( G_1 \) with a path of the same length in \( G_2 \).
(ii) Take a path \( u_0u_1\ldots u_k \) in \( G_2 \), such that all of \( u_1, u_2, \ldots, u_{k-1} \) have degree 2, and let \( G = G_1 \cup (G_2 - \{u_1, u_2, \ldots, u_{k-1}\}) \cup \{u_0v, u_kv\} \) where \( v \) and \( w \) are distinct vertices of \( G_1 \).

Then \( G \) is 2-connected.

Let \( C' = C - \{v_1, v_2, \ldots, v_{2g}\} \). We claim that \( C' \cup \bigcup_{j=1}^{2g} (D_j \cup E_j \cup F_{jl} \cup F_{j2}) \) is the required 3-covering. By construction it spans all vertices of \( G \), and has at most \( \epsilon |V(G)| \) vertices of degree greater than 2. It does not use any of the edges we added to \( G \) in Step 6 or Step 8. By the last paragraph of Step 8, we do not create any vertices of degree greater than 3. We use Lemma 3.2 to verify that it is 2-connected. By our construction, \( C' \cup \bigcup_{j=1}^{2g} D_j \) is a cycle. For each \( j \), we may apply Lemma 3.2(i) with \( G_2 = E_j \) if \( \sigma_j \neq \sigma_j' \), or Lemma 3.2(ii) with \( G_2 = E_j' \) if \( \sigma_j = \sigma_j' \), to show that we retain 2-connectedness when we add \( E_j \). For each \( j \) and \( l \), we may also apply Lemma 3.2(ii) with \( G_2 = F_{jl} \) to show that we retain 2-connectedness when we add \( F_{jl} \). This completes the proof.

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References


