Available online at www.sciencedirect.com
ScienceDirect
J. Math. Anal. Appl. 329 (2007) 677–689
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Two periodic solutions of second-order neutral functional differential equations

Jun Wu [a,*], Zhicheng Wang [b]

[a] College of Mathematics and Computer Science, Changsha University of Science Technology, Changsha 410076, China

[b] College of Mathematics and Econometrics, Hunan University, Changsha 410082, China

Received 1 January 2006

Available online 2 August 2006

Submitted by S.R. Grace

Abstract

In this paper, we consider a type of second-order neutral functional differential equations. We obtain some existence results of multiplicity and nonexistence of positive periodic solutions. Our approach is based on a fixed point theorem in cones.

Keywords: Fixed point theorem; Positive periodic solution; Multiplicity; Neutral functional differential equations

1. Introduction

In this paper, we consider existence, multiplicity and nonexistence of positive ω-periodic solutions for the following second-order neutral functional differential equation:

$$\big(x(t) - cx(t-\delta)\big)'' + a(t)x(t) = \lambda b(t) f\big(x\big(t-\tau(t)\big)\big), \tag{1.1}$$

where λ is a positive parameter, c and δ are constants and $|c| \neq 1$.

The existence of periodic solutions for functional differential equations has been derived from many fields such as physics, biology and mechanics [5,6]. Many results were obtained by Kuang [6], Freedman and Wu [4], Wang [7] and many others by applying fixed point index

* Corresponding author.
E-mail address: junwmath@hotmail.com (J. Wu).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2006.06.084

theory, theory of Fourier series, fixed point theorems in cones, Leray–Schauder continuation theorem, coincidence degree theory and so on. We refer to [8–14] for some recent results in this field.

Among the previous results on this problem, many of them concern neutral systems (Lu and Ge [10], Lu, Ge and Zheng [11] and Chen [13]). But to our best knowledge, papers on multiplicity of periodic solutions of neutral systems are few.

In this paper, we aim to establish existence, multiplicity and nonexistence of positive ω-periodic solutions for second-order neutral functional differential equation (1.1). Our approach is based on a fixed point theorem in cones as well as some analysis techniques used in [7,15].

Let

$$f_0 = \lim_{u\to 0^+} \frac{f(u)}{u}, \qquad f_\infty = \lim_{u\to\infty} \frac{f(u)}{u},$$
$$i_0 = \text{number of zeros in the set } \{f_0, f_\infty\},$$
$$i_\infty = \text{number of infinities in the set } \{f_0, f_\infty\}.$$

It is clear that $i_0, i_\infty = 0, 1$ or 2. We will show that (1.1) has i_0 or i_∞ positive ω-periodic solution(s) for certain λ, respectively.

Let $\bar{f} = \frac{1}{\omega}\int_0^\omega f(s)\,ds$, where a is a continuous ω-periodic function. In what follows, we set

$$X = \bigl\{x \mid x \in C(R, R),\ x(t+\omega) \equiv x(t)\bigr\}$$

with the norm defined by $\|x\|_X = \max\{|x(t)|:\ t \in [0, \omega]\}$. Then $(X, \|\cdot\|_X)$ is a Banach space. Let $A: X \to X$ defined by $(Ax)(t) = x(t) - cx(t-\delta)$.

Lemma 1.1. *If $|c| \neq 1$, then A has continuous bounded inverse A^{-1} on X and for all $x \in X$,*

$$\bigl(A^{-1}x\bigr)(t) = \begin{cases} \sum_{j\geqslant 0} c^j x(t - j\delta), & \text{if } |c| < 1, \\ -\sum_{j\geqslant 1} c^{-j} x(t + j\delta), & \text{if } |c| > 1, \end{cases} \tag{1.2}$$

and

$$\bigl\|A^{-1}x\bigr\|_X \leqslant \frac{\|x\|_X}{|1-|c||}.$$

Proof. According to [9], we can get the equality (1.2) and then verify Lemma 1.1.

We consider the following assumptions:

(A_1) $a, b \in C(R, (0, +\infty))$ are ω-periodic functions, $\max\{a(t):\ t \in [0, \omega]\} < (\frac{\pi}{\omega})^2$, and $\tau \in C(R, R)$ is a positive ω-periodic function.
(A_2) $f \in C([0, \infty), [0, \infty))$ and $f(u) > 0$ for $u > 0$.

Let

$$M = \max\bigl\{a(t):\ t \in [0, \omega]\bigr\}, \qquad m = \min\bigl\{a(t):\ t \in [0, \omega]\bigr\},$$
$$\beta = \sqrt{M}, \qquad L = \frac{1}{2\beta \sin\frac{\beta\omega}{2}}, \qquad l = \frac{\cos\frac{\beta\omega}{2}}{2\beta\sin\frac{\beta\omega}{2}},$$
$$k = l(M+m) + LM, \qquad \alpha = \frac{l[m - |c|(M+m)]}{LM(1-|c|)}.$$

If the assumption (A_1) holds, then $M < (\frac{\pi}{\omega})^2$. Thus we can see that $L \geqslant l > 0$.
Additionally, define

$$M(r) = \max\left\{ f(t)\colon\ 0 \leqslant t \leqslant \frac{r}{1-|c|} \right\},$$

$$m(r) = \min\left\{ f(t)\colon\ \alpha r \leqslant t \leqslant \frac{r}{1-|c|} \right\}, \qquad k_1 = \frac{k - \sqrt{k^2 - 4LlMm}}{2LM}. \qquad \square$$

In this paper, we discuss existence of positive ω-periodic solutions of Eq. (1.1) when $c \in (-\min\{k_1, \frac{m}{M+m}\}, 0]$.

Theorem 1.1. *Suppose the assumptions* (A_1), (A_2) *hold and* $-\min\{\frac{m}{M+m}, k_1\} < c \leqslant 0$.

(a) *If* $i_0 = 1$ *or* 2, *then* (1.1) *has* i_0 *positive* ω*-periodic solution(s) for* $\lambda > \frac{1}{m(1)\bar{b}l\omega} > 0$;
(b) *If* $i_\infty = 1$ *or* 2, *then* (1.1) *has* i_∞ *positive* ω*-periodic solution(s) for* $0 < \lambda < \frac{m-(M+m)|c|}{L\bar{b}\omega(M-M|c|)M(1)}$;
(c) *If* $i_\infty = 0$ *or* $i_0 = 0$, *then* (1.1) *has no positive* ω*-periodic solution for sufficiently small or large* $\lambda > 0$, *respectively.*

Theorem 1.2. *Suppose the assumptions* (A_1), (A_2) *hold and* $-\min\{\frac{m}{M+m}, k_1\} < c \leqslant 0$.

(a) *If there exists a constant* $c_1 > 0$ *such that* $f(u) \geqslant c_1 u$ *for* $u \in [0, +\infty)$, *then* (1.1) *has no positive* ω*-periodic solution for* $\lambda > \frac{1-c^2}{\bar{b}l\omega c_1(\alpha-|c|)}$;
(b) *If there exists a constant* $c_2 > 0$ *such that* $f(u) \leqslant c_2 u$ *for* $u \in [0, +\infty)$, *then* (1.1) *has no positive* ω*-periodic solution for* $0 < \lambda < \frac{m-(M+m)|c|}{\bar{b}\omega c_2 LM}$.

Theorem 1.3. *Suppose the assumptions* (A_1), (A_2) *hold,* $c \in (-\min\{k_1, \frac{m}{M+m}\}, 0]$ *and* $i_0 = i_\infty = 0$. *If*

$$\frac{1-c^2}{\max\{f_0, f_\infty\}\bar{b}\omega l(\alpha - |c|)} < \lambda < \frac{m-(M+m)|c|}{\min\{f_0, f_\infty\}\bar{b}\omega LM},$$

then (1.1) *has one positive* ω*-periodic solution.*

The rest of this paper is organized as follows: Section 2 is about statement of the method (a fixed point theorem in cones) and some prior estimations in order to prove our main results; in Section 3, we give the proofs of our main results by using our lemmas and present an example.

2. Preliminaries

We first state the well-known fixed point theorem in cones [1–3]. For the proof, we refer to the classical works [1–3].

Lemma 2.1. (Deimling [2], Guo and Lakshmikantham [3] and Krasnoselskii [1]) *Let* E *be a Banach space and* K *a cone in* E. *For* $r > 0$, *define* $K_r = \{u \in K\colon\ \|u\| < r\}$. *Assume that* $T : \overline{K}_r \to K$ *is completely continuous such that* $Tx \neq x$ *for* $x \in \partial K_r = \{u \in K\colon\ \|u\| = r\}$.

(i) *If* $\|Tx\| \geqslant \|x\|$ *for* $x \in \partial K_r$, *then* $i(T, K_r, K) = 0$;
(ii) *If* $\|Tx\| \leqslant \|x\|$ *for* $x \in \partial K_r$, *then* $i(T, K_r, K) = 1$.

Next, we transfer existence of positive ω-periodic solutions of neutral equation (1.1) into existence of positive fixed points of some fixed point mapping.

In order to establish existence, multiplicity and nonexistence of positive ω-periodic solutions for (1.1), we consider the following equation:

$$y''(t) + a(t)\big(A^{-1}y\big)(t) = \lambda b(t) f\big(\big(A^{-1}y\big)\big(t-\tau(t)\big)\big), \tag{2.1}$$

where A^{-1} is defined by (1.2). By Lemma 1.1, we conclude that

Lemma 2.2. *$y(t)$ is an ω-periodic solution of* (2.1) *if and only if $(A^{-1}y)(t)$ is an ω-periodic solution of* (1.1).

Aiming to apply Lemma 2.1 to Eq. (2.1), we rewrite (2.1) as

$$y''(t) + a(t)y(t) - a(t)G\big(y(t)\big) = \lambda b(t) f\big(\big(A^{-1}y\big)\big(t-\tau(t)\big)\big),$$

where

$$G\big(y(t)\big) = y(t) - \big(A^{-1}y\big)(t) = -c\big(A^{-1}y\big)(t-\delta).$$

Set $K = \{x \in X\colon\ x(t) \geqslant \alpha\|x\|_X\}$. Clearly, K is a cone in X. Note that $\Omega_r = \{x \in K\colon \|x\|_X < r\}$ and $\partial\Omega_r = \{x \in K\colon\ \|x\|_X = r\}$. Additionally, we let $C_\omega = \{x \in C(R, R_+)\colon x(t+\omega) = x(t)\}$.

By solving the inequality $|c| < \frac{l[m-|c|(M+m)]}{LM(1-|c|)}$, we can obtain the following result immediately.

Lemma 2.3. *If $|c| < \min\{k_1, \frac{m}{M+m}\}$, then $|c| < \alpha$.*

Lemma 2.4. *If $y \in K$ and $c \in (-\min\{k_1, \frac{m}{M+m}\}, 0]$, then*

(a) $\frac{\alpha-|c|}{1-c^2}\|y\|_X \leqslant (A^{-1}y)(t) \leqslant \frac{1}{1-|c|}\|y\|_X$;
(b) $\frac{|c|(\alpha-|c|)}{1-c^2}\|y\|_X \leqslant G(y(t)) \leqslant \frac{|c|}{1-|c|}\|y\|_X$, $t \in [0, \omega]$.

Proof. Part (a). For $y \in K$ and $c \in (-\min\{k_1, \frac{m}{M+m}\}, 0]$, by Lemma 1.1, we have

$$\big(A^{-1}y\big)(t) = \sum_{j\geqslant 0} c^j y(t-j\delta) = \sum_{j=2i} c^j y(t-j\delta) - \sum_{j=2i+1} |c|^j y(t-j\delta) \geqslant \frac{\alpha-|c|}{1-c^2}\|y\|_X,$$

$$\big(A^{-1}y\big)(t) \leqslant \frac{1}{1-|c|}\|y\|_X.$$

Part (b). From the definition of $G(y(t))$ and Part (a), we have

$$\frac{|c|(\alpha-|c|)}{1-c^2}\|y\|_X \leqslant G\big(y(t)\big) \leqslant \frac{|c|}{1-|c|}\|y\|_X.$$

The proof of Lemma 2.4 is completed. □

Firstly, we consider the following equation:

$$y''(t) + My(t) = \lambda h(t), \quad h \in C_\omega. \tag{2.2}$$

Define $G(t,s)$ by

$$G(t,s)=\frac{\cos\beta(t+\frac{\omega}{2}-s)}{2\beta\sin\frac{\beta\omega}{2}},\quad t\in R,\ t\leqslant s\leqslant t+\omega.$$

Thus,

$$\int_t^{t+\omega}G(t,s)\,ds=\int_t^{t+\omega}\frac{\cos\beta(t+\frac{\omega}{2}-s)}{2\beta\sin\frac{\beta\omega}{2}}\,ds=\frac{-\sin\beta(t+\frac{\omega}{s}-s)}{2\beta\sin\frac{\beta\omega}{2}}\bigg|_t^{t+\omega}$$
$$=\frac{1}{2\beta^2}+\frac{1}{2\beta^2}=\frac{1}{\beta^2}=\frac{1}{M},$$
$$0<l=\frac{\cos\frac{\beta\omega}{2}}{2\beta\sin\frac{\beta\omega}{2}}\leqslant G(t,s)\leqslant\frac{1}{2\beta\sin\frac{\beta\omega}{2}}=L$$

since $M<(\frac{\pi}{\omega})^2$. Let

$$T_\lambda h(t)=\lambda\int_t^{t+\omega}G(t,s)h(s)\,ds.$$

It is easy to show that $T_\lambda h(t)>0$ for $h(t)>0$. And by the properties of $G(t,s)$ and $h(t)$, T_λ is completely continuous. Also, by simple computations and the maximum principle, we establish the following lemma.

Lemma 2.5. *Suppose the assumptions* (A_1), (A_2) *hold and* $c\in(-\min\{k_1,\frac{m}{M+m}\},0]$. *For any* $h\in C_\omega$, $y(t)=T_\lambda h(t)$ *is the unique positive* ω*-periodic solution of* (2.2). *Meanwhile,* $\|T_\lambda\|=\frac{\lambda}{M}$.

Secondly, we study the following equation corresponding to (2.2):

$$y''(t)+a(t)y(t)-a(t)G\big(y(t)\big)=\lambda h(t),\quad h\in C_\omega. \tag{2.3}$$

Let $By(t)=\frac{1}{\lambda}[(M-a(t))y(t)+a(t)G(y(t))]$. Clearly, $\|B\|\leqslant\frac{1}{\lambda}(M-m+M\frac{|c|}{1-|c|})$. Then, from Lemma 2.5, we have

$$y(t)=T_\lambda h(t)+T_\lambda By(t).$$

$|c|<\min\{k_1,\frac{m}{M+m}\}$ implies that $\frac{M-m+m|c|}{M(1-|c|)}<1$. So $\|T_\lambda B\|\leqslant\|T_\lambda\|\|B\|\leqslant\frac{M-m+m|c|}{M(1-|c|)}<1$. Thus we have

$$y(t)=(I-T_\lambda B)^{-1}T_\lambda h(t). \tag{2.4}$$

Let

$$P_\lambda h(t)=(I-T_\lambda B)^{-1}T_\lambda h(t).$$

Then we can make the following conclusion.

Lemma 2.6. *Suppose the assumptions* (A_1), (A_2) *hold and* $c\in(-\min\{k_1,\frac{m}{M+m}\},0]$. *For any* $h\in C_\omega$, $y(t)=P_\lambda h(t)$ *is the unique positive* ω*-periodic solution of* (2.3); P_λ *is completely continuous and satisfies*

$$T_\lambda h(t)\leqslant P_\lambda h(t)\leqslant\frac{M(1-|c|)}{m-(M+m)|c|}\|T_\lambda h\|_X,\quad h\in C_\omega.$$

Proof. By expansions of P_λ,

$$P_\lambda = (I - T_\lambda B)^{-1} T_\lambda = \big(I + T_\lambda B + (T_\lambda B)^2 + \cdots + (T_\lambda B)^n + \cdots\big) T_\lambda$$
$$= T_\lambda + T_\lambda B T_\lambda + (T_\lambda B)^2 T_\lambda + \cdots + (T_\lambda B)^n T_\lambda + \cdots, \tag{2.5}$$

P_λ is completely continuous since T_λ is completely continuous. From (2.5), we get

$$T_\lambda h(t) \leqslant P_\lambda h(t) \leqslant \frac{M(1-|c|)}{m-(M+m)|c|} \|T_\lambda h\|_X, \quad h \in C_\omega.$$

The proof is completed. □

Lemmas 2.5 and 2.6 are obtained similarly with Lemmas 1 and 2 in [15].

Let $Q_\lambda y(t) = P_\lambda(b(t) f((A^{-1}y)(t-\tau(t))))$. Since P_λ is completely continuous, Q_λ is completely continuous by the continuity of $b(\cdot)$ and $f(\cdot)$. Also, from the definition of T_λ and P_λ, it follows that Q_λ is continuous about λ.

From the above arguments, we can obtain the following lemma immediately.

Lemma 2.7. *Suppose the assumptions* (A_1), (A_2) *hold and* $c \in (-\min\{k_1, \frac{m}{M+m}\}, 0]$. *Then* $Q_\lambda(K) \subset K$.

Proof. From the above arguments, it is easy to verify that $Q_\lambda y(t+\omega) = Q_\lambda y(t)$. For $y \in K$, we have

$$Q_\lambda y(t) = P_\lambda\big(b(t) f\big(\big(A^{-1}y\big)\big(t-\tau(t)\big)\big)\big) \geqslant T_\lambda\big(b(t) f\big(\big(A^{-1}y\big)\big(t-\tau(t)\big)\big)\big)$$
$$= \lambda \int_t^{t+\omega} G(t,s) b(s) f\big(\big(A^{-1}y\big)\big(s-\tau(s)\big)\big)\, ds \geqslant \lambda l \int_0^\omega b(s) f\big(\big(A^{-1}y\big)\big(s-\tau(s)\big)\big)\, ds,$$

$$Q_\lambda y(t) = P_\lambda\big(b(t) f\big(\big(A^{-1}y\big)\big(t-\tau(t)\big)\big)\big)$$
$$\leqslant \frac{M(1-|c|)}{m-(M+m)|c|} \big\| T_\lambda\big(b(\cdot) f\big(\big(A^{-1}y\big)\big(\cdot-\tau(\cdot)\big)\big)\big)\big\|_X$$
$$= \lambda \frac{M(1-|c|)}{m-(M+m)|c|} \max_{t\in[0,\omega]} \int_t^{t+\omega} G(t,s) b(s) f\big(\big(A^{-1}y\big)\big(s-\tau(s)\big)\big)\, ds$$
$$\leqslant \lambda \frac{M(1-|c|)}{m-(M+m)|c|} L \int_0^\omega b(s) f\big(\big(A^{-1}y\big)\big(s-\tau(s)\big)\big)\, ds.$$

Therefore

$$Q_\lambda y(t) \geqslant \frac{l[m-(M+m)|c|]}{LM(1-|c|)} \|Q_\lambda y\|_X = \alpha \|Q_\lambda y\|_X.$$

So $Q_\lambda(K) \subset K$. This completes the proof. □

Lemma 2.8. *Suppose the assumptions* (A_1), (A_2) *hold and* $c \in (-\min\{k_1, \frac{m}{M+m}\}, 0]$. *Then* $y(t)$ *is a positive fixed point of* Q_λ *if and only if* $(A^{-1}y)(t)$ *is a positive* ω*-solution of* (1.1).

Proof. If $y(t)$ is a positive fixed point of Q_λ, then $y(t) = P_\lambda(b(t)f((A^{-1}y)(t-\tau(t))))$ and $y \in K$ from Lemma 2.7. By Lemma 2.6, $y(t)$ is a positive ω-periodic solution of the equation

$$y''(t) + a(t)y(t) - a(t)G\big(y(t)\big) = \lambda b(t) f\big(\big(A^{-1}y\big)\big(t-\tau(t)\big)\big).$$

That is, $y(t)$ is a positive ω-periodic solution of (2.1). Since $y \in K$, it follows from Lemma 2.4 that $(A^{-1}y)(t) > 0$. Therefore, $(A^{-1}y)(t)$ is a positive ω-periodic solution of (1.1) by Lemma 2.2.

Suppose that there exists $y \in X$ such that $(A^{-1}y)(t)$ is a positive ω-periodic solution of (1.1). Lemma 2.2 tells that $y(t)$ is an ω-periodic solution of (2.1), that is, $y(t)$ is an ω-periodic solution of the equation

$$y''(t) + a(t)y(t) - a(t)G\big(y(t)\big) = \lambda b(t) f\big(\big(A^{-1}y\big)\big(t-\tau(t)\big)\big).$$

Additionally, $y(t) = (A^{-1}y)(t) - c(A^{-1}y)(t-\delta) > 0$ since $c \in (-\min\{k_1, \frac{m}{M+m}\}, 0]$. It follows from Lemma 2.6 that $y(t) = P_\lambda(b(t)f((A^{-1}y)(t-\tau(t)))) = Q_\lambda y(t)$. Thus $y(t)$ is a positive fixed point of Q_λ. □

From Lemmas 2.2–2.8, in order to discuss existence of positive ω-periodic solutions of (1.1), it is sufficient to consider existence of positive fixed points of Q_λ. The following is about our prior estimations which play important roles in the proofs of our main results.

Lemma 2.9. *Suppose the assumptions* (A_1), (A_2) *hold and* $c \in (-\min\{k_1, \frac{m}{M+m}\}, 0]$, $\eta > 0$. *If* $f((A^{-1}y)(t-\tau(t))) \geqslant (A^{-1}y)(t-\tau(t))\eta$ *for* $t \in [0, \omega]$ *and* $y \in K$, *then*

$$\|Q_\lambda y\|_X \geqslant \lambda \bar{b} l \omega \eta \frac{\alpha - |c|}{1-c^2}\|y\|_X.$$

Proof. For $c \in (-\min\{k_1, \frac{m}{M+m}\}, 0]$ and $y \in K$, we have

$$\begin{aligned} Q_\lambda y(t) &= P_\lambda\big(b(t)f\big(\big(A^{-1}y\big)\big(t-\tau(t)\big)\big)\big) \geqslant T_\lambda\big(b(t)f\big(\big(A^{-1}y\big)\big(t-\tau(t)\big)\big)\big) \\ &= \lambda \int_t^{t+\omega} G(t,s)b(s)f\big(\big(A^{-1}y\big)\big(s-\tau(s)\big)\big)\,ds \geqslant l\lambda\eta \int_0^{\omega} b(s)\big(A^{-1}y\big)\big(s-\tau(s)\big)\,ds \\ &\geqslant l\lambda\bar{b}\omega\eta\frac{\alpha-|c|}{1-c^2}\|y\|_X. \end{aligned}$$

Hence

$$\|Q_\lambda y\|_X \geqslant \lambda \bar{b} l \omega \eta \frac{\alpha - |c|}{1-c^2}\|y\|_X. \qquad \square$$

Lemma 2.10. *Suppose the assumptions* (A_1), (A_2) *hold and* $c \in (-\min\{k_1, \frac{m}{M+m}\}, 0]$. *If there exists* $\varepsilon > 0$ *such that* $f((A^{-1}y)(t-\tau(t))) \leqslant (A^{-1}y)(t-\tau(t))\varepsilon$ *for* $t \in [0, \omega]$, *then*

$$\|Q_\lambda y\|_X \leqslant \lambda \bar{b} \omega \varepsilon \frac{LM}{m-(M+m)|c|}\|y\|_X.$$

Proof. In view of Lemmas 1.1, 2.4 and 2.6, we have

$$\begin{aligned}\|Q_\lambda y\|_X &\leqslant \lambda \frac{M(1-|c|)}{m-|c|(M+m)} L \int_0^\omega b(s) f\big(\big(A^{-1}y\big)\big(s-\tau(s)\big)\big)\,ds \\ &\leqslant \lambda \frac{M(1-|c|)}{m-|c|(M+m)} L\varepsilon \int_0^\omega b(s)\big(A^{-1}y\big)\big(s-\tau(s)\big)\,ds \\ &\leqslant \lambda \bar{b}\omega\varepsilon \frac{LM}{m-|c|(M+m)} \|y\|_X. \qquad \square\end{aligned}$$

Lemma 2.11. *Suppose the assumptions* (A_1), (A_2) *hold and* $c \in (-\min\{k_1, \frac{m}{M+m}\}, 0]$. *If* $y \in \partial\Omega_r$, $r > 0$, *then*

$$\|Q_\lambda y\|_X \geqslant l\lambda\bar{b}\omega m(r).$$

Proof. Since $y \in \partial\Omega_r$ and $c \in (-\min\{k_1, \frac{m}{M+m}\}, 0]$, by Lemma 2.4, we obtain $\frac{\alpha-|c|}{1-c^2} r \leqslant (A^{-1}y)(t-\tau(t)) \leqslant \frac{r}{1-|c|}$. Thus $f((A^{-1}y)(t-\tau(t))) \geqslant m(r)$. Then it is easy to see that this lemma can be proved in a similar manner as in Lemma 2.9. □

Lemma 2.12. *Suppose the assumptions* (A_1), (A_2) *hold and* $c \in (-\min\{k_1, \frac{m}{M+m}\}, 0]$. *If* $y \in \partial\Omega_r$, $r > 0$, *then*

$$\|Q_\lambda y\|_X \leqslant L\lambda\bar{b}\omega \frac{M(1-|c|)}{m-(M+m)|c|} M(r).$$

Proof. Since $y \in \partial\Omega_r$ and $c \in (-\min\{k_1, \frac{m}{M+m}\}, 0]$, by Lemma 2.4, we obtain $0 \leqslant (A^{-1}y)(t-\tau(t)) \leqslant \frac{r}{1-|c|}$. Thus $f((A^{-1}y)(t-\tau(t))) \leqslant M(r)$. Then it is easy to see that this lemma can be shown in a similar manner as in Lemma 2.10. □

3. Proofs of main results

In this section, we give the proofs of main results based on lemmas in Section 2.

Proof of Theorem 1.1. Part (a). Take $r_1 = 1$ and $\lambda_0 = \frac{1}{m(r_1)\bar{b}l\omega} > 0$. By Lemma 2.11, for $y \in \partial\Omega_{r_1}$ and $\lambda > \lambda_0$,

$$\|Q_\lambda y\|_X > \|y\|_X.$$

From Lemma 2.1, $i(Q_\lambda, \Omega_{r_1}, K) = 0$.

Case I. If $f_0 = 0$, we can choose $0 < \bar{r}_2 < r_1$ so that $f(u) \leqslant \varepsilon u$ for $0 \leqslant u \leqslant \bar{r}_2$, where the constant $\varepsilon > 0$ satisfies

$$\lambda\bar{b}\omega\varepsilon \frac{LM}{m-(M+m)|c|} < 1. \tag{3.1}$$

Let $r_2 = (1-|c|)\bar{r}_2$. Since $0 \leqslant (A^{-1}y)(t-\tau(t)) \leqslant \frac{1}{1-|c|}\|y\|_X \leqslant \bar{r}_2$ for $y \in \partial\Omega_{r_2}$ by Lemma 2.4, we obtain $f((A^{-1}y)(t-\tau(t))) \leqslant \varepsilon(A^{-1}y)(t-\tau(t))$. Thus we have by Lemma 2.10 and (3.1) that, for $y \in \partial\Omega_{r_2}$,

$$\|Q_\lambda y\|_X \leqslant \lambda\bar{b}\omega\varepsilon \frac{LM}{m-(M+m)|c|} \|y\|_X < \|y\|_X.$$

It follows from Lemma 2.1 that $i(Q_\lambda, \Omega_{r_2}, K) = 1$. Thus $i(Q_\lambda, \Omega_{r_1} \setminus \bar{\Omega}_{r_2}, K) = -1$ and Q_λ has a fixed point y in $\Omega_{r_1} \setminus \bar{\Omega}_{r_2}$. By Lemma 2.8, we see that $(A^{-1}y)(t)$ is a positive ω-periodic solution of (1.1) for $\lambda > \lambda_0$.

Case II. If $f_\infty = 0$, there exists a constant $\tilde{H} > 0$ such that $f(u) \leqslant \varepsilon u$ for $u \geqslant \tilde{H}$, where the constant $\varepsilon > 0$ satisfies inequality (3.1).

Let $r_3 = \max\{2r_1, \frac{\tilde{H}(1-c^2)}{\alpha-|c|}\}$. Since $(A^{-1}y)(t - \tau(t)) \geqslant \frac{\alpha-|c|}{1-c^2}\|y\|_X \geqslant \tilde{H}$ for $y \in \partial\Omega_{r_3}$, we obtain $f((A^{-1}y)(t - \tau(t))) \leqslant \varepsilon(A^{-1}y)(t - \tau(t))$. Thus we have by Lemma 2.10 and (3.1) that, for $y \in \partial\Omega_{r_3}$,

$$\|Q_\lambda y\|_X \leqslant \lambda\bar{b}\omega\varepsilon\frac{LM}{m-(M+m)|c|}\|y\|_X < \|y\|_X.$$

It follows from Lemma 2.1 that $i(Q_\lambda, \Omega_{r_3}, K) = 1$. Thus $i(Q_\lambda, \Omega_{r_3} \setminus \bar{\Omega}_{r_1}, K) = 1$ and Q_λ has a fixed point y in $\Omega_{r_3} \setminus \bar{\Omega}_{r_1}$. By Lemma 2.8, we see that $(A^{-1}y)(t)$ is a positive ω-periodic solution of (1.1) for $\lambda > \lambda_0$.

Case III. If $f_\infty = f_0 = 0$, from the above arguments, there exist $0 < r_2 < r_1 < r_3$ such that Q_λ has a fixed point $y_1(t)$ in $\Omega_{r_1} \setminus \bar{\Omega}_{r_2}$ and a fixed point $y_2(t)$ in $\Omega_{r_3} \setminus \bar{\Omega}_{r_1}$. Consequently, $(A^{-1}y_1)(t)$ and $(A^{-1}y_2)(t)$ are two positive ω-periodic solutions of (1.1) for $\lambda > \lambda_0$.

Part (b). Let $r_1 = 1$. Take $\lambda_1 = \frac{m-(M+m)|c|}{L\bar{b}\omega(M-M|c|)M(r_1)} > 0$. By Lemma 2.12, for $y \in \partial\Omega_{r_1}$ and $0 < \lambda < \lambda_1$,

$$\|Q_\lambda y\|_X < \|y\|_X.$$

By Lemma 2.1, $i(Q_\lambda, \Omega_{r_1}, K) = 1$.

Case I. If $f_0 = \infty$, we can choose $0 < \bar{r}_2 < r_1$ so that $f(u) \geqslant \eta u$ for $0 \leqslant u \leqslant \bar{r}_2$, where the constant $\eta > 0$ satisfies

$$\lambda\bar{b}\omega\eta\frac{\alpha-|c|}{1-c^2} > 1. \tag{3.2}$$

Let $r_2 = (1-|c|)\bar{r}_2$. Since $0 \leqslant (A^{-1}y)(t-\tau(t)) \leqslant \frac{1}{1-|c|}\|y\|_X \leqslant \bar{r}_2$ for $y \in \partial\Omega_{r_2}$, we obtain $f((A^{-1}y)(t - \tau(t))) \geqslant \eta(A^{-1}y)(t - \tau(t))$. Thus we have by Lemma 2.9 and (3.2) that, for $y \in \partial\Omega_{r_2}$,

$$\|Q_\lambda y\|_X \geqslant \lambda\bar{b}l\omega\eta\frac{\alpha-|c|}{1-c^2}\|y\|_X > \|y\|_X.$$

It follows from Lemma 2.1 that $i(Q_\lambda, \Omega_{r_2}, K) = 0$. Thus $i(Q_\lambda, \Omega_{r_1} \setminus \bar{\Omega}_{r_2}, K) = 1$ and Q_λ has a fixed point y in $\Omega_{r_1} \setminus \bar{\Omega}_{r_2}$. By Lemma 2.8, we see that $(A^{-1}y)(t)$ is a positive ω-periodic solution of (1.1) for $\lambda \in (0, \lambda_1)$.

Case II. If $f_\infty = \infty$, there exists a constant $\tilde{H} > 0$ such that $f(u) \geqslant \eta u$ for $u \geqslant \tilde{H}$, where the constant $\eta > 0$ satisfies inequality (3.2).

Let $r_3 = \max\{2r_1, \frac{\tilde{H}(1-|c|^2)}{\alpha-|c|}\}$. Since $(A^{-1}y)(t - \tau(t)) \geqslant \frac{\alpha-|c|}{1-c^2}\|y\|_X \geqslant \tilde{H}$ for $y \in \partial\Omega_{r_3}$ by Lemma 2.4, we obtain $f((A^{-1}y)(t - \tau(t))) \geqslant \eta(A^{-1}y)(t - \tau(t))$. Thus we have by Lemma 2.9 and inequality (3.2) that, for $y \in \partial\Omega_{r_3}$,

$$\|Q_\lambda y\|_X \geqslant \lambda\bar{b}l\omega\eta\frac{\alpha-|c|}{1-c^2}\|y\|_X > \|y\|_X.$$

It follows from Lemma 2.1 that $i(Q_\lambda, \Omega_{r_3}, K) = 0$. Thus $i(Q_\lambda, \Omega_{r_3} \setminus \bar{\Omega}_{r_1}, K) = -1$ and Q_λ has a fixed point y in $\Omega_{r_3} \setminus \bar{\Omega}_{r_1}$. By Lemma 2.8, we see that $(A^{-1}y)(t)$ is a positive ω-periodic solution of (1.1) for $\lambda \in (0, \lambda_1)$.

Case III. If $f_\infty = f_0 = \infty$, it is clear from the above proofs that Q_λ has a fixed point y_1 in $\Omega_{r_1} \setminus \bar{\Omega}_{r_2}$ and a fixed point y_2 in $\Omega_{r_3} \setminus \bar{\Omega}_{r_1}$. Consequently, $(A^{-1}y_1)(t)$ and $(A^{-1}y_2)(t)$ are two positive ω-periodic solutions of (1.1) for $\lambda \in (0, \lambda_1)$.

Part (c). By Lemma 2.4, $(A^{-1}y)(t - \tau(t)) \geqslant \frac{\alpha-|c|}{1-c^2}\|y\|_X \geqslant 0$ for $t \in [0, \omega]$ and $y \in K$.

Case I. If $i_0 = 0$, we have $f_0 > 0$ and $f_\infty > 0$. Let $c_1 = \min\{\frac{f(u)}{u}\colon u > 0\} > 0$, then we obtain

$$f(u) \geqslant c_1 u, \quad u \in [0, +\infty).$$

Assume $y(t)$ is a positive ω-periodic solution of (1.1) for $\lambda > \lambda_2$, where $\lambda_2 = \frac{1-c^2}{\bar{b}l\omega c_1(\alpha-|c|)}$. Since $Q_\lambda y(t) = y(t)$ for $t \in [0, \omega]$, it follows from Lemma 2.9 that, for $\lambda > \lambda_2$,

$$\|y\|_X = \|Q_\lambda y\|_X \geqslant \lambda \bar{b} l \omega c_1 \frac{\alpha - |c|}{1 - c^2}\|y\|_X > \|y\|_X,$$

which is a contradiction.

Case II. If $i_\infty = 0$, we have $f_0 < \infty$ and $f_\infty < \infty$. Let $c_2 = \max\{\frac{f(u)}{u}\colon u > 0\} > 0$, then we obtain

$$f(u) \leqslant c_2 u, \quad u \in [0, +\infty).$$

Assume $y(t)$ is a positive ω-periodic solution of (1.1) for $\lambda \in (0, \lambda_3)$, where $\lambda_3 = \frac{m-(M+m)|c|}{\bar{b}\omega c_2 LM}$. Since $Q_\lambda y(t) = y(t)$ for $t \in [0, \omega]$, it follows from Lemma 2.10 that, for $\lambda \in (0, \lambda_3)$,

$$\|y\|_X = \|Q_\lambda y\|_X \leqslant \lambda \bar{b} \omega c_2 \frac{LM}{m - (M+m)|c|}\|y\|_X < \|y\|_X,$$

which is a contradiction. □

Proof of Theorem 1.2. From the proof of part (c) in Theorem 1.1, we obtain this theorem immediately. □

Proof of Theorem 1.3. *Case* I. $f_0 \leqslant f_\infty$. In this case, we have

$$\frac{1 - c^2}{\bar{b}\omega l f_\infty(\alpha - |c|)} < \lambda < \frac{m - (M+m)|c|}{f_0 \bar{b}\omega LM}.$$

It is clear that there exists an $0 < \varepsilon < f_\infty$ such that

$$\frac{1 - c^2}{\bar{b}\omega l (f_\infty - \varepsilon)(\alpha - |c|)} < \lambda < \frac{m - (M+m)|c|}{(f_0 + \varepsilon)\bar{b}\omega LM}.$$

For the above ε, we choose $0 < \bar{r}_1$ so that $f(u) \leqslant (f_0 + \varepsilon)u$ for $0 \leqslant u \leqslant \bar{r}_1$. Let $r_1 = (1 - |c|)\bar{r}_1$. Since $0 \leqslant (A^{-1}y)(t - \tau(t)) \leqslant \frac{1}{1-|c|}\|y\|_X \leqslant \bar{r}_1$ for $y \in \partial\Omega_{r_1}$ by Lemma 2.4, we obtain $f((A^{-1}y)(t - \tau(t))) \leqslant (f_0 + \varepsilon)(A^{-1}y)(t - \tau(t))$. Thus we have by Lemma 2.10 that, for $y \in \partial\Omega_{r_1}$,

$$\|Q_\lambda y\|_X \leqslant \lambda \bar{b}\omega (f_0 + \varepsilon)\frac{LM}{m - (M+m)|c|}\|y\|_X < \|y\|_X.$$

On the other hand, there exists a constant $\tilde{H} > 0$ such that $f(u) \geqslant (f_\infty - \varepsilon)u$ for $u \geqslant \tilde{H}$. Let $r_2 = \max\{2r_1, \frac{\tilde{H}(1-c^2)}{\alpha-|c|}\}$. Since $(A^{-1}y)(t - \tau(t)) \geqslant \frac{\alpha-|c|}{1-c^2}\|y\|_X \geqslant \tilde{H}$ for $y \in \partial\Omega_{r_2}$ by Lemma 2.4, we obtain $f((A^{-1}y)(t-\tau(t))) \geqslant (f_\infty - \varepsilon)(A^{-1}y)(t-\tau(t))$. Thus we have by Lemma 2.9 that, for $y \in \partial\Omega_{r_2}$,

$$\|Q_\lambda y\|_X \geqslant \lambda\bar{b}\omega l(f_\infty - \varepsilon)\frac{\alpha - |c|}{1-c^2}\|y\|_X > \|y\|_X.$$

It follows from Lemma 2.1 that

$$i(Q_\lambda, \Omega_{r_1}, K) = 1, \quad i(Q_\lambda, \Omega_{r_2}, K) = 0.$$

Thus $i(Q_\lambda, \Omega_{r_2} \setminus \bar{\Omega}_{r_1}, K) = -1$ and Q_λ has a fixed point y in $\Omega_{r_2} \setminus \bar{\Omega}_{r_1}$. By Lemma 2.8, we see that $(A^{-1}y)(t)$ is a positive ω-periodic solution of (1.1).

Case II. $f_0 > f_\infty$. In this case, we have

$$\frac{1-c^2}{\bar{b}\omega l f_0(\alpha - |c|)} < \lambda < \frac{m-(M+m)|c|}{f_\infty \bar{b}\omega LM}.$$

It is clear that there exists an $0 < \varepsilon < f_0$ such that

$$\frac{1-c^2}{\bar{b}\omega l(f_0-\varepsilon)(\alpha - |c|)} < \lambda < \frac{m-(M+m)|c|}{(f_\infty+\varepsilon) \bar{b}\omega LM}.$$

For the above ε, we choose $0 < \bar{r}_1$ so that $f(u) \geqslant (f_0 - \varepsilon)u$ for $0 \leqslant u \leqslant \bar{r}_1$. Let $r_1 = (1-|c|)\bar{r}_1$. Since $0 \leqslant (A^{-1}y)(t-\tau(t)) \leqslant \frac{1}{1-|c|}\|y\|_X \leqslant \bar{r}_1$ for $y \in \partial\Omega_{r_1}$, we obtain $f((A^{-1}y)(t-\tau(t))) \geqslant (f_0-\varepsilon)(A^{-1}y)(t)$. Thus we have by Lemma 2.9 that, for $y \in \partial\Omega_{r_1}$,

$$\|Q_\lambda y\|_X \geqslant \lambda\bar{b}l\omega(f_0 - \varepsilon)\frac{\alpha - |c|}{1-c^2}\|y\|_X > \|y\|_X.$$

On the other hand, there exists a constant $\tilde{H} > 0$ such that $f(u) \leqslant (f_\infty + \varepsilon)u$ for $u \geqslant \tilde{H}$. Let $r_2 = \max\{2r_1, \frac{\tilde{H}(1-c^2)}{\alpha-|c|}\}$. Since $(A^{-1}y)(t-\tau(t)) \geqslant \frac{\alpha-|c|}{1-c^2}\|y\|_X \geqslant \tilde{H}$ for $y \in \partial\Omega_{r_2}$, we obtain $f((A^{-1}y)(t-\tau(t))) \leqslant (f_\infty+\varepsilon)(A^{-1}y)(t-\tau(t))$. Thus we have by Lemma 2.10 that, for $y \in \partial\Omega_{r_2}$,

$$\|Q_\lambda y\|_X \leqslant \lambda\bar{b}\omega(f_\infty + \varepsilon)\frac{LM}{m-(M+m)|c|}\|y\|_X < \|y\|_X.$$

It follows from Lemma 2.1 that

$$i(Q_\lambda, \Omega_{r_1}, K) = 0, \quad i(Q_\lambda, \Omega_{r_2}, K) = 1.$$

Thus $i(Q_\lambda, \Omega_{r_2} \setminus \bar{\Omega}_{r_1}, K) = 1$ and Q_λ has a fixed point y in $\Omega_{r_2} \setminus \bar{\Omega}_{r_1}$. By Lemma 2.8, we see that $(A^{-1}y)(t)$ is a positive ω-periodic solution of (1.1). □

Our results are applicable to consider existence problem of periodic solutions of many neutral differential systems.

Example 3.1. We consider the following neutral functional differential equation:

$$\left[u(t) + \frac{1}{3}u\left(t - \frac{\pi}{2}\right)\right]'' + \frac{1}{4}u(t) = \lambda[1-\sin 2t]u^a\big(t-\tau(t)\big)e^{-u(t-\tau(t))}, \tag{3.3}$$

where λ and a are positive parameters, $\tau(t+\pi) \equiv \tau(t)$. We see that $\delta = \frac{\pi}{2}$, $c = -\frac{1}{3}$, $a(t) \equiv \frac{1}{4}$, $b(t) = 1 - \sin 2t$, $f(u) = u^a e^{-u}$, $M = m = \frac{1}{4}$. Additionally, $\max_{u \in [0,\infty)} f(u) = f(a)$.

Clearly, $M = \frac{1}{4} < (\frac{\pi}{\pi})^2 = 1$. The assumptions ($A_1$) and ($A_2$) are satisfied and $f_\infty = 0$. Then we conclude

Conclusion 3.2. (a) If $a \in (0, 1)$, then (3.3) has one positive π-periodic solution for $\lambda > \frac{1}{\pi r_0} > 0$ or $0 < \lambda < \frac{\sqrt{2}}{4\pi f(a)}$, where $r_0 = \min\{f(\frac{\sqrt{2}}{4}), f(\frac{3}{2})\}$;

(b) If $a = 1$, then (3.3) has one positive π-periodic solution for $\lambda > \frac{1}{\pi r_0} > 0$;

(c) If $a > 1$, then (3.3) has two positive π-periodic solutions for $\lambda > \frac{1}{\pi r_0} > 0$.

In fact, by simple computations, we have

$$\beta = \frac{1}{2}, \qquad L = \frac{1}{2\beta \sin\frac{\beta\pi}{2}} = \sqrt{2}, \qquad l = \frac{\cos\frac{\beta\pi}{2}}{2\beta \sin\frac{\beta\pi}{2}} = 1,$$

$$k = \frac{2+\sqrt{2}}{4}, \qquad \alpha = \frac{\sqrt{2}}{4}, \qquad k_1 = \frac{\sqrt{2}+1-\sqrt{3}}{2},$$

$$|c| = \frac{1}{3} < \min\left\{k_1, \frac{m}{m+M}\right\} = \frac{\sqrt{2}+1-\sqrt{3}}{2}, \qquad |c| = \frac{1}{3} < \frac{\sqrt{2}}{4} = \alpha.$$

Let $t_0 = \min\{a, \frac{3}{2}\}$ and $r_0 = \min\{f(\frac{\sqrt{2}}{4}), f(\frac{3}{2})\}$, we have

$$M(1) = \max\left\{f(t)\colon\ 0 \leqslant t \leqslant \frac{3}{2}\right\} = f(t_0),$$

$$m(1) = \min\left\{f(t)\colon\ \frac{\sqrt{2}}{4} \leqslant t \leqslant \frac{3}{2}\right\} = \min\left\{f\left(\frac{3}{2}\right), f\left(\frac{\sqrt{2}}{4}\right)\right\} = r_0,$$

$$\frac{1}{m(1)l\bar{b}\omega} = \frac{1}{\pi r_0}, \qquad \frac{m-(M+m)|c|}{L\bar{b}\omega(M-M|c|)M(1)} = \frac{\sqrt{2}}{4\pi f(t_0)}.$$

Additionally, if $a \in (0, 1)$, $f_0 = +\infty$; if $a = 1$, $f_0 = 1$ and $f_\infty = 0$; if $a > 1$, $f_0 = f_\infty = 0$. From Theorem 1.1, we can obtain Conclusion 3.2.

Acknowledgments

The authors thank the reviewers for their valuable suggestions.

References

[1] M. Krasnoselskii, Positive Solutions of Operators Equations, Noordhoff, Groningen, 1964.
[2] K. Deimling, Nonlinear Functional Analysis, Springer-Verlag, Berlin, 1985.
[3] D. Guo, V. Lakshmikantham, Nonlinear Problem in Abstract Cones, Academic Press, Orlando, FL, 1988.
[4] H.I. Freedman, J. Wu, Periodic solutions of single-species models with periodic delay, SIAM J. Math. Anal. 23 (1992) 689–701.
[5] J.K. Hale, Theory of Functional Differential Equations, Springer-Verlag, New York, 1977.
[6] Y. Kuang, Delay Differential Equations with Applications in Population Dynamics, Academic Press, New York, 1993.
[7] X. Haiyan Wang, Positive periodic solutions of functional differential equations, J. Differential Equations 202 (2004) 354–366.
[8] Y. Li, Positive periodic solutions for a periodic neutral differential equation with feedback control, Nonlinear Anal. Real World Appl. 6 (2005) 145–154.

[9] M. Zhang, Periodic solution of linear and quasilinear neutral functional differential equations, J. Math. Anal. Appl. 189 (1995) 378–392.
[10] S. Lu, W. Ge, Periodic solutions of neutral differential equation with multiple deviating arguments, Appl. Math. Comput. 156 (2004) 705–717.
[11] S. Lu, W. Ge, Z. Zheng, Periodic solutions to a kind of neutral functional differential equation in the critical case, J. Math. Anal. Appl. 293 (2004) 462–475.
[12] S. Tanaka, Existence of positive solutions for a class of first-order neutral functional differential equations, J. Math. Anal. Appl. 229 (1999) 501–518.
[13] F. Chen, Positive periodic solutions of neutral Lotka–Volterra system with feedback control, Appl. Math. Comput. 162 (2005) 1279–1302.
[14] M. Ait Babram, R. Benkhalti, Periodic solutions of functional differential equations of neutral type, J. Math. Anal. Appl. 204 (1996) 898–909.
[15] Y. Li, Positive periodic solutions of nonlinear second order ordinary differential equations, Acta Math. Sinica 45 (2002) 481–488.