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An Approximation Method for a Class of Singularly Perturbed Second-Order Boundary Value Problems*

F. A. Howes

School of Mathematics, University of Minnesota, Minneapolis, Minnesota 55455

Submitted by K. L. Cooke

1. Introduction

In this note we present a method for generating successively better approximations to a solution of the second-order boundary value problem

$$\epsilon y'' = f(t, y, y', \epsilon), \qquad 0 < t < 1, \tag{1.1}$$

$$y(0, \epsilon) = A, \qquad y(1, \epsilon) = B, \tag{1.2}$$

for small, positive values of the perturbation parameter ϵ. The function f in (1.1) is assumed to be sufficiently smooth to permit the construction of an Nth-order outer solution, i.e., a solution of $\epsilon u'' = f(t, u, u', \epsilon) + \mathcal{O}(\epsilon^N)$,[1] $0 < t < 1$, $u(1, \epsilon) = B$, for N a positive integer. In addition, f is assumed to grow no faster than $(y')^2$ as $|y'| \to \infty$. The first assumption is used in defining a sequence of approximate problems whose solutions are shown to exist and to satisfy the desired asymptotic estimates as a consequence of the growth restriction placed on f. Another essential requirement is that the partial derivative $\partial f/\partial y' = f_{y'}$ be strictly negative in a certain region around the function $u_0 = u_0(t)$, which is assumed to be a solution of the reduced problem $0 = f(t, u_0, u_0', 0)$, $0 < t < 1$, $u_0(1) = B$. The function u_0 will turn out to be the first term in the asymptotic expansion of the outer solution.

The idea for the approximation scheme presented here originated from a similar discussion of a singularly perturbed second-order initial value problem in a paper by Nagumo [13]. This paper and an earlier one [12] formed the basis for Brish's [1] original study of the nonlinear problem (1.1), (1.2). Unfortunately the results in [1, 13] have not attracted much attention, although the author has employed the methodology behind them with some success in [9, 10]. More

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[1] Here and throughout the paper $\mathcal{O}$ denotes the standard-order symbol, i.e., $r(t, \epsilon) = \mathcal{O}(s(t, \epsilon))$ if there is a constant K (independent of t, ϵ) such that $|r(t, \epsilon)| \leqslant K|s(t, \epsilon)|$, for all (t, ϵ) in a certain domain.

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widely known are the techniques of matched expansions and two-timing, which are summarized and illustrated, for example, in the books by Cole [5] and O'Malley [14], respectively. However, these methods when applied to a boundary value problem like (1.1), (1.2) involve certain formal matching relations whose validity is usually justified by either heuristic or a posteriori arguments. Although the book by Eckhaus [7] is successful in providing rigorous justification for the method of matched expansions, the matching formulas in [7] are unnecessarily complicated when applied to the problem presented here. In contrast, the method developed below does not require the use of any form of matching. Further, the existence of solutions and the resulting asymptotic estimates are established rigorously at each step of the approximation procedure.

2. The Outer Solution

In this section the construction of an outer solution is outlined. A more complete discussion can be found, for instance, in [5] or [14]. Recall that the problem under consideration is

$$\epsilon y'' = f(t, y, y', \epsilon), \qquad 0 < t < 1, \tag{2.1}$$

$$y(0, \epsilon) = A, \qquad y(1, \epsilon) = B. \tag{2.2}$$

The parameter ϵ is assumed to be positive here and throughout the paper, and for simplicity, the boundary conditions are taken to be independent of ϵ. An Nth-order outer solution is then a function $u = u(t, \epsilon)$ satisfying

$$\epsilon u'' = f(t, u, u', \epsilon) + \mathcal{O}(\epsilon^N), \qquad 0 < t < 1,$$
$$u(1, \epsilon) = B.$$

To determine such a function we tentatively set $u(t, \epsilon) = \sum_{j=0}^{N} u_j(t)\, \epsilon^j$ and insert this expression into (2.1), (2.2). Equating coefficients of ϵ^j, $0 \leqslant j \leqslant N$, we have the following sequence of problems for the u_j on (0, 1):

$$0 = f(t, u_0, u_0', 0), \qquad u_0(1) = B, \qquad (2.3)_0$$

$$u_0'' = f_y(t, u_0, u_0', 0)\, u_1 + f_{y'}(t, u_0, u_0', 0)\, u_1', \qquad u_1(1) = 0, \qquad (2.3)_1$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$u_{j-1}'' = f_y(t, u_0, u_0', 0)\, u_j + f_{y'}(t, u_0, u_0', 0)\, u_j' + r_j(t), \qquad u_j(1) = 0, \qquad (2.3)_j$$

where r_j is a function depending on $u_0, u_1, \ldots, u_{j-1}$ and their derivatives. We have assumed, of course, that the function f is sufficiently differentiable to allow the above expansions. If we now assume that the reduced problem $(2.3)_0$ has a sufficiently differentiable solution $u_0 = u_0(t)$, then the resulting problems $(2.3)_j$,

$j \geqslant 1$, which are linear, can be solved provided $f_{y'}(t, u_0, u_0, 0) \neq 0$ for t in $(0, 1)$. This, however, is one of our basic assumptions; in fact, we assume $f_{y'} \leqslant -k < 0$ in a region around u_0 to be defined in the next section.

The function $u(t, \epsilon) = \sum_{j=0}^{N} u_j(t)\, \epsilon^j$ is then easily seen to be an Nth-order outer solution of (2.1), (2.2). We remark that if we had assumed instead that $f_{y'} \geqslant k > 0$, then nonuniform (i.e., boundary layer) behavior would occur at $t = 1$. Consequently, the terms u_j in the expansion of the outer solution should satisfy the *initial* conditions $u_0(0) = A$, $u_j(0) = 0$, $j \geqslant 1$. Finally, if the boundary conditions A, B have asymptotic expansions in ϵ, i.e., $A(\epsilon)\ (B(\epsilon)) \sim \sum_{j=0}^{N} A_j \epsilon^j\ (\sum_{j=0}^{N} B_j \epsilon^j)$, then the functions u_j should be made to satisfy $u_j(1) = B_j$ or $u_j(0) = A_j$, $j \geqslant 0$, depending on the sign of $f_{y'}$.

3. The Approximation Sequence

Having indicated how to construct an Nth-order outer solution of the boundary value problem

$$\epsilon y'' = f(t, y, y', \epsilon), \qquad 0 < t < 1, \tag{3.1}$$

$$y(0, \epsilon) = A, \qquad y(1, \epsilon) = B, \tag{3.2}$$

we now present a method for obtaining both the existence of a solution $y = y(t, \epsilon)$ of (3.1), (3.2) for each $\epsilon > 0$, ϵ sufficiently small, and the existence of an Nth-order uniform approximation of the solution for such ϵ. To be specific, we construct a sequence of boundary value problems which effectively transforms an outer solution into a function $y_N = y_N(t, \epsilon)$ satisfying $y - y_N = \mathcal{O}(\epsilon^N)$. The precise result is contained in the following two theorems.

THEOREM 3.1. *Assume*

(1) *the reduced problem* $0 = f(t, u_0, u_0', 0)$, $u_0(1) = B$, *has a solution* $u_0 = u_0(t)$ *of class* $C^{(N+2)}[0, 1]$;

(2) *the function f is of class C with respect to t and of class $C^{(N+1)}$ with respect to y, y', ϵ in the region R:*

$$0 \leqslant t \leqslant 1, \qquad |y - u_0(t)| \leqslant d, \qquad |y'| < \infty,$$
$$0 \leqslant \epsilon < \epsilon_1 \qquad (d > 0, 0 < \epsilon_1 < 1);$$

(3) *the partial derivatives* $\{f_y, f_{yy}, f_{yy'}, f_{y'y'}\}$ *are of order* $\mathcal{O}(1)$ *in* R; *in particular,* $|f_y| \leqslant \ell$ *and* $f_y \not\equiv 0$;

(4) *there is a constant* $k > 0$ *such that the partial derivative* $f_{y'}$ *satisfies* $f_{y'} \leqslant -k$ *in* R.

Then the inductively-defined boundary value problems

$$\epsilon y_1'' = f(t, u(t, \epsilon), y_1', \epsilon), \quad 0 < t < 1, \quad y_1(0, \epsilon) = A, \quad y_1(1, \epsilon) = B; \tag{E$_1$}$$

$$\epsilon y_2'' = f(t, y_1(t, \epsilon), y_2', \epsilon), \quad 0 < t < 1, \quad y_2(0, \epsilon) = A, \quad y_2(1, \epsilon) = B; \tag{E$_2$}$$

$$\vdots$$

$$\epsilon y_N'' = f(t, y_{N-1}(t, \epsilon), y_N', \epsilon), \quad 0 < t < 1, \quad y_N(0, \epsilon) = A, \quad y_N(1, \epsilon) = B, \tag{E$_N$}$$

have solutions $y_1(t, \epsilon)$, $y_2(t, \epsilon),..., y_N(t, \epsilon)$, *respectively, for each* ϵ, $0 < \epsilon \leqslant \min\{\epsilon_1, k^2(4\ell)^{-1}\}$, *which satisfy the estimates*

$$y_j(t, \epsilon) - y_{j-1}(t, \epsilon) = \mathcal{O}(\epsilon^N) + \mathcal{O}(\epsilon^{j-1} \exp[-kt(2\epsilon)^{-1}]) \tag{*}$$

for t in $[0, 1]$ *and* $1 \leqslant j \leqslant N$. *Here* $y_0(t, \epsilon) = u(t, \epsilon)$ *is an* Nth-*order outer solution of* (3.1), (3.2).

Proof. We begin by noting that the assumptions of the theorem allow the construction of an Nth-order outer solution $u = u(t, \epsilon)$, for $0 < \epsilon \leqslant \epsilon_1$, as outlined in Section 2.

To prove the existence of solutions of $(E_1),..., (E_N)$ and the corresponding estimates (*), we will use a differential inequality theorem originally proved by Nagumo [12] and later refined by Jackson [11]. In the present context, it asserts that if the function $F = F(t, x, x', \epsilon)$ is continuous in its variables and grows no faster than $(x')^2$ as $|x'| \to \infty$, and if there exist functions α, β of class $C^{(2)}[0, 1]$ satisfying $\alpha \leqslant \beta$, $\alpha(0) \leqslant A \leqslant \beta(0)$, $\alpha(1) \leqslant B \leqslant \beta(1)$, and $\epsilon\alpha'' \geqslant F(t, \alpha, \alpha', \epsilon)$, $\epsilon\beta'' \leqslant F(t, \beta, \beta', \epsilon)$ on $(0, 1)$, then the problem $\epsilon x'' = F(t, x, x', \epsilon)$, $0 < t < 1$, $x(0, \epsilon) = A$, $x(1, \epsilon) = B$, has a solution $x(t, \epsilon)$ of class $C^{(2)}[0, 1]$ with $\alpha(t, \epsilon) \leqslant x(t, \epsilon) \leqslant \beta(t, \epsilon)$, $0 \leqslant t \leqslant 1$. Since assumption (3) places the required growth restriction on f, to apply the Nagumo–Jackson theorem we must construct appropriate bounding solutions for each of the problems $(E_1),..., (E_N)$.

THE PROBLEM (E_1). Define for t in $[0, 1]$ and $0 < \epsilon \leqslant \epsilon_0 = \min\{\epsilon_1, k^2(4\ell)^{-1}\}$ the functions

$$\begin{aligned}
\alpha_1(t, \epsilon) &= u(t, \epsilon) - (u(0, \epsilon) - A) \exp[-kt\epsilon^{-1}] - \epsilon^N \gamma_1 \ell^{-1} \exp[\lambda(t-1)], \quad &&\text{if} \quad u(0, \epsilon) \geqslant A, \\
&= u(t, \epsilon) - \epsilon^N \gamma_1 \ell^{-1} \exp[\lambda(t-1)], &&\text{if} \quad u(0, \epsilon) < A; \\
\beta_1(t, \epsilon) &= u(t, \epsilon) + \epsilon^N \gamma_1 \ell^{-1} \exp[\lambda(t-1)], &&\text{if} \quad u(0, \epsilon) \geqslant A, \\
&= u(t, \epsilon) - (u(0, \epsilon) - A) \exp[-kt\epsilon^{-1}] + \epsilon^N \gamma_1 \ell^{-1} \exp[\lambda(t-1)], &&\text{if} \quad u(0, \epsilon) < A.
\end{aligned}$$

Here $\lambda = \lambda(\epsilon) < 0$ is the root of $\epsilon\lambda^2 + k\lambda + \ell = 0$ which satisfies $\lambda = -\ell k^{-1} + \mathcal{O}(\epsilon)$. (Such a λ exists by virtue of the restriction that $\epsilon \leqslant k^2(4\ell)^{-1}$.) The constant γ_1 is positive and will be determined below.

Clearly $\alpha_1 \leqslant \beta_1$, $\alpha_1(0, \epsilon) \leqslant A \leqslant \beta_1(0, \epsilon)$, and $\alpha_1(1, \epsilon) \leqslant B \leqslant \beta_1(1, \epsilon)$ (recall that $u(1, \epsilon) = B$). It is just as easy to verify that α_1, β_1 satisfy the required differential inequalities. For example, suppose $u(0, \epsilon) \geqslant A$ and consider $\alpha_1(t, \epsilon)$. Then, differentiating α_1, substituting into the equation in (E_1), and expanding by the Mean Value Theorem, we have

$$\begin{aligned}
&\epsilon\alpha_1'' - f(t, u(t, \epsilon), \alpha_1', \epsilon)\\
&\quad = \epsilon u'' - k^2\epsilon^{-1}(u(0, \epsilon) - A)\exp[-kt\epsilon^{-1}]\\
&\qquad - \epsilon\lambda^2\epsilon^N\gamma_1\ell^{-1}\exp[\lambda(t-1)] - f(t, u, u', \epsilon)\\
&\qquad - f_{u'}[t, \epsilon]\{k\epsilon^{-1}(u(0, \epsilon) - A)\exp[-kt\epsilon^{-1}] - \lambda\epsilon^N\gamma_1\ell^{-1}\exp[\lambda(t-1)]\},
\end{aligned}$$

where $[t, \epsilon]$ is the appropriate intermediate point. Since u is an Nth-order outer solution, there is a constant K_1 such that $|\epsilon u'' - f(t, u, u', \epsilon)| \leqslant \epsilon^N K_1$, $0 \leqslant t \leqslant 1$, $0 < \epsilon \leqslant \epsilon_1$. This, together with the assumption that $f_{u'}[t, \epsilon] \leqslant -k < 0$, allows us to continue with the inequality

$$\begin{aligned}
&\epsilon\alpha_1'' - f(t, u, \alpha_1', \epsilon)\\
&\quad \geqslant + k^2\epsilon^{-1}(u(0, \epsilon) - A)\exp[-kt\epsilon^{-1}] - \epsilon\lambda^2\epsilon^N\gamma_1\ell^{-1}\exp[\lambda(1-t)]\\
&\qquad - \epsilon^N K_1 - k\lambda\epsilon^N\gamma_1\ell^{-1}\exp[\lambda(1-t)]\\
&\quad \geqslant -\epsilon^N K_1 + \epsilon^N\gamma_1\exp[\lambda(t-1)],
\end{aligned}$$

since $\epsilon\lambda^2 + k\lambda + \ell = 0$. Thus by choosing $\gamma_1 \geqslant K_1$, we have the desired inequality. The other verifications follow similarly and we conclude by the Nagumo–Jackson theorem that for each ϵ, $0 < \epsilon \leqslant \epsilon_0$, the problem (E_1) has a solution $y_1 = y_1(t, \epsilon)$ with $\alpha_1(t, \epsilon) \leqslant y_1(t, \epsilon) \leqslant \beta_1(t, \epsilon)$, i.e., $y_1 - u = \mathcal{O}(\epsilon^N) + \mathcal{O}(\exp[-kt\epsilon^{-1}])$, $0 \leqslant t \leqslant 1$.

THE PROBLEM (E_2). We consider next the problem

$$\epsilon y_2'' = f(t, y_1(t, \epsilon), y_2', \epsilon), \quad 0 < t < 1, \qquad y_2(0, \epsilon) = A, \qquad y_2(1, \epsilon) = B,$$

where $y_1(t, \epsilon)$ is the solution of (E_1) whose existence was proved above. Define for t in $[0, 1]$ and $0 < \epsilon \leqslant \epsilon_0$ the functions

$$\alpha_2(t, \epsilon) = y_1(t, \epsilon) - \epsilon\Gamma_2\exp[-kt(2\epsilon)^{-1}] - \epsilon^N\gamma_2\ell^{-1}\exp[\lambda(t-1)],$$

$$\beta_2(t, \epsilon) = y_1(t, \epsilon) + \epsilon\Gamma_2\exp[-kt(2\epsilon)^{-1}] + \epsilon^N\gamma_2\ell^{-1}\exp[\lambda(t-1)].$$

Here Γ_2, γ_2 are positive constants whose magnitudes are determined below, and $\lambda = -\ell k^{-1} + \mathcal{O}(\epsilon)$ is the root of $\epsilon\lambda^2 + k\lambda + \ell = 0$, as before.

Then $\alpha_2 \leqslant \beta_2$, $\alpha_2(0, \epsilon) \leqslant A \leqslant \beta_2(0, \epsilon)$, and $\alpha_2(1, \epsilon) \leqslant B \leqslant \beta_2(1, \epsilon)$ (recall $y_1(0, \epsilon) = A$, $y_1(1, \epsilon) = B$). To verify that the differential inequalities are satisfied, consider β_2; the verification for α_2 follows by symmetry. Differentiating β_2, substituting into the equation in (E_2), and expanding by the Mean Value Theorem, we have

$$\begin{aligned}
&f(t, y_1(t, \epsilon), \beta_2', \epsilon) - \epsilon\beta_2'' \\
&\quad = f(t, y_1, y_1', \epsilon) + f_{y'}[t, \epsilon]\{-\tfrac{1}{2}k\Gamma_2 \exp[-kt(2\epsilon)^{-1}] + \lambda\epsilon^N\gamma_2\ell^{-1}\exp[\lambda(t-1)]\} \\
&\qquad - \epsilon y_1'' - \tfrac{1}{4}k^2\Gamma_2 \exp[-kt(2\epsilon)^{-1}] - \epsilon\lambda^2\epsilon^N\gamma_2\ell^{-1}\exp[\lambda(t-1)] \\
&\quad \geqslant f(t, y_1, y_1', \epsilon) - \epsilon y_1'' + (\tfrac{1}{2} - \tfrac{1}{4})\, k^2\Gamma_2 \exp[-kt(2\epsilon)^{-1}] \\
&\qquad - (\epsilon\lambda^2 + k\lambda)\,\epsilon^N\gamma_2\ell^{-1}\exp[\lambda(t-1)] \\
&\quad = f(t, y_1, y_1', \epsilon) - \epsilon y_1'' + \tfrac{1}{4}k^2\Gamma_2\exp[-kt(2\epsilon)^{-1}] + \epsilon^N\gamma_2\exp[\lambda(t-1)].
\end{aligned}$$

We now rewrite $f(t, y_1, y_1', \epsilon)$ as $f(t, u + (y_1 - u), y_1', \epsilon)$ and note that

$$\begin{aligned}
f(t, u + (y_1 - u), y_1', \epsilon) &= f(t, u, y_1', \epsilon) + \mathcal{O}(|y_1 - u|) \\
&= f(t, u, y_1', \epsilon) + \mathcal{O}(\epsilon^N) + \mathcal{O}(\exp[-kt(2\epsilon)^{-1}]),
\end{aligned}$$

since $f_y = \mathcal{O}(1)$ and $y_1 - u = \mathcal{O}(\epsilon^N) + \mathcal{O}(\exp[-kt(2\epsilon)^{-1}])$. More precisely, there are positive constants K_2, $\tilde{K}_2$ such that

$$f(t, y_1, y_1', \epsilon) \geqslant f(t, u, y_1', \epsilon) - \epsilon^N K_2 - \tilde{K}_2 \exp[-kt(2\epsilon)^{-1}].$$

We then have the desired inequality

$$\begin{aligned}
&f(t, y_1, \beta_2', \epsilon) - \epsilon\beta_2'' \\
&\qquad \geqslant f(t, u, y_1', \epsilon) - \epsilon y_1'' - \epsilon^N K_2 - \tilde{K}_2\exp[-kt(2\epsilon)^{-1}) \\
&\qquad\quad + \tfrac{1}{4}k^2\Gamma_2\exp[-kt(2\epsilon)^{-1}] + \epsilon^N\gamma_2\exp[\gamma(t-1)] \geqslant 0,
\end{aligned}$$

if we choose $\Gamma_2 \geqslant 4\tilde{K}_2k^{-2}$ and $\gamma_2 \geqslant K_2$, since $\epsilon y_1'' = f(t, u, y_1', \epsilon)$. Thus, applying the Nagumo–Jackson theorem, we conclude that (E_2) has a solution $y_2 = y_2(t, \epsilon)$ for each ϵ, $0 < \epsilon \leqslant \epsilon_0$, satisfying $y_2 - y_1 = \mathcal{O}(\epsilon^N) + \mathcal{O}(\epsilon\exp[-kt(2\epsilon)^{-1}])$, $0 \leqslant t \leqslant 1$.

THE PROBLEM (E_j), $j \geqslant 3$. The pattern is now clear. To deduce the existence of a solution $y_j = y_j(t, \epsilon)$ of the problem

$$\epsilon y_j'' = f(t, y_{j-1}(t, \epsilon), y_j', \epsilon), \qquad 0 < t < 1,$$

$$y_j(0, \epsilon) = A, \qquad y_j(1, \epsilon) = B,$$

satisfying $y_j - y_{j-1} = \mathcal{O}(\epsilon^N) + \mathcal{O}(\epsilon^{j-1} \exp[-kt(2\epsilon)^{-1}])$, we simply define, for t in $[0, 1]$ and $0 < \epsilon \leqslant \epsilon_0$,

$$\alpha_j(t, \epsilon) = y_{j-1}(t, \epsilon) - \epsilon^{j-1}\Gamma_j \exp[-kt(2\epsilon)^{-1}] - \epsilon^N \gamma_j \ell^{-1} \exp[\lambda(t - 1)],$$
$$\beta_j(t, \epsilon) = y_{j-1}(t, \epsilon) + \epsilon^{j-1}\Gamma_j \exp[-kt(2\epsilon)^{-1}] + \epsilon^N \gamma_j \ell^{-1} \exp[\lambda(t - 1)],$$

for Γ_j, γ_j sufficiently large, and proceed as in the case of (E_2). This concludes the proof of Theorem 3.1.

Remark 1. We have tacitly assumed that the constant d in the definition of the region R is large enough to permit the various expansions carried out above. Indeed, the parameter ϵ was allowed to be as large as $\min\{\epsilon_1, k^2(4l)^{-1}\}$. However, if one is interested in problems involving only very small values of ϵ, i.e., $0 < \epsilon \ll 1$, then it is sufficient for d to satisfy $|A - u_0(0)| < d$, where u_0 is the solution of the reduced problem. This follows by noting that $u(t, \epsilon) = u_0(t) + \mathcal{O}(\epsilon)$ and, consequently,

$$\begin{aligned} y_j(t, \epsilon) &= y_{j-1}(t, \epsilon) + \mathcal{O}(\epsilon^N) + \mathcal{O}(\epsilon^{j-1} \exp[-kt(2\epsilon)^{-1}]) \\ &= u_0(t) + |A - u_0(0)| \exp[-kt(2\epsilon)^{-1}] + \mathcal{O}(\epsilon), \qquad \text{for } j \geqslant 1. \end{aligned}$$

Remark 2. We can also make the following observation concerning the range of $\epsilon > 0$ for which Theorem 3.1 is valid. If the function f_y is positively bounded away from zero, i.e., if there are positive constants ν, $\ell(\nu < \ell)$ such that $\nu \leqslant f_y \leqslant \ell$ in R, then the conclusion of the theorem holds for each ϵ, $0 < \epsilon \leqslant \epsilon_1$. Of course, we assume that the constant d is sufficiently large (cf. Remark 1 above). This is easily seen by defining the following functions α_j, β_j for t in $[0, 1]$ and $0 < \epsilon \leqslant \epsilon_1$:

$$\begin{aligned} \alpha_1(t, \epsilon) &= u(t, \epsilon) - (u(0, \epsilon) - A) \exp[-kt\epsilon^{-1}] - \epsilon^N \gamma_1 \nu^{-1}, && \text{if} \quad u(0, \epsilon) \geqslant A, \\ &= u(t, \epsilon) - \epsilon^N \gamma_1 \nu^{-1}, && \text{if} \quad u(0, \epsilon) \leqslant A; \\ \beta_1(t, \epsilon) &= u(t, \epsilon) + \epsilon^N \gamma_1 \nu^{-1}, && \text{if} \quad u(0, \epsilon) \geqslant A, \\ &= u(t, \epsilon) - (u(0, \epsilon) - A) \exp[-kt\epsilon^{-1}] + \epsilon^N \gamma_1 \nu^{-1}, && \text{if} \quad u(0, \epsilon) < A; \end{aligned}$$

and for $j = 2,..., N$,

$$\alpha_j(t, \epsilon) = y_{j-1}(t, \epsilon) - \epsilon^{j-1}\Gamma_j \exp[-kt(2\epsilon)^{-1}] - \epsilon^N \gamma_j \nu^{-1},$$
$$\beta_j(t, \epsilon) = y_{j-1}(t, \epsilon) + \epsilon^{j-1}\Gamma_j \exp[-kt(2\epsilon)^{-1}] + \epsilon^N \gamma_j \nu^{-1},$$

where Γ_j, γ_j are sufficiently large positive constants. The verifications that these α_j, β_j satisfy the required inequalities are quite similar to those given above and are omitted.

Remark 3. If, in place of assumption (4), we assume that $f_{y'} \geqslant k > 0$ in R, then an analogous result can be proved, provided we assume that the Nth order outer solution $u(t, \epsilon)$ satisfies $u(0, \epsilon) = A = y(0, \epsilon)$. We simply make the change of independent variable $\tau = 1 - t$ and apply Theorem 3.1 to the transformed problem. The estimate (*) for $y_j(t, \epsilon) - y_{j-1}(t, \epsilon)$ then becomes

$$y_j(t, \epsilon) - y_{j-1}(t, \epsilon) = \mathcal{O}(\epsilon^N) + \mathcal{O}(\epsilon^{j-1} \exp[-k(2\epsilon)^{-1} (1 - t)]), \qquad 0 \leqslant t \leqslant 1.$$

Remark 4. It is possible to consider boundary conditions A, B which are sufficiently regular functions of ϵ to permit the construction of an Nth-order outer solution $u(t, \epsilon)$ satisfying $u(1, \epsilon) = B(\epsilon) + \mathcal{O}(\epsilon^N)$ (or $u(0, \epsilon) = A(\epsilon) + \mathcal{O}(\epsilon^N)$). The only change required in the proof of Theorem 3.1 is to make the constant γ_1 appearing in the definition of α_1, β_1 large enough so that, additionally, $\alpha_1(1, \epsilon) \leqslant B(\epsilon) \leqslant \beta_1(1, \epsilon)$.

Remark 5. The differential equations in (E_j) are of first order in y_j' and, as a result, it is generally possible to determine exact or numerical solutions for ϵ in the indicated range.

Remark 6. We note finally that if the function $f = f(t, y, y', \epsilon)$ is independent of y, i.e., $f_y \equiv 0$, then the sequence of problems (E_1),..., (E_N) reduces to the original problem (3.1), (3.2), and, consequently, the method presented above does not apply in this case.

We are now in a position to prove the existence of a solution $y = y(t, \epsilon)$ of the boundary value problem (3.1), (3.2). At the same time, we show that the solution $y_N = y_N(t, \epsilon)$ of (E_N) constructed above is a uniform-order $\mathcal{O}(\epsilon^N)$-approximation to y.

THEOREM 3.2. *Make the same assumptions as in Theorem* 3.1. *Then for each* ϵ, $0 < \epsilon < \min\{\epsilon_1, k^2(4\ell)^{-1}\}$, *there exists a solution* $y = y(t, \epsilon)$ *of* (3.1), (3.2). *In addition,*

(i) $y - y_N = \mathcal{O}(\epsilon^N)$, $0 \leqslant t \leqslant 1$;

(ii) $y' - y_N' = \mathcal{O}(\epsilon^N) + \mathcal{O}(\epsilon^{N-1} \exp[-kt(2\epsilon)^{-1}])$, $0 \leqslant t \leqslant 1$,

where $y_N = y_N(t, \epsilon)$ *is the solution of* (E_N) *considered in Theorem* 3.1.

Proof. The theorem is proved by noting that since

$$y_N - y_{N-1} = \mathcal{O}(\epsilon^N) + \mathcal{O}(\epsilon^{N-1} \exp[-kt(2\epsilon)^{-1}]), \qquad 0 \leqslant t \leqslant 1,$$

we may rewrite the function f in (E_N) as

$$\begin{aligned} &f(t, y_N + (y_{N-1} - y_N), y_N', \epsilon) \\ &\qquad = f(t, y_N, y_N', \epsilon) + \mathcal{O}(\epsilon^N) + \mathcal{O}(\epsilon^{N-1} \exp[-kt(2\epsilon)^{-1}]). \end{aligned}$$

That is, y_N is a solution of

$$\epsilon y_N'' = f(t, y_N, y_N', \epsilon) + \mathcal{O}(\epsilon^N) + \mathcal{O}(\epsilon^{N-1} \exp[-kt(2\epsilon)^{-1}]), \qquad 0 < t < 1,$$

$$y_N(0, \epsilon) = A, \qquad y_N(1, \epsilon) = B.$$

This is, however, precisely the type of approximate solution discussed by Willett [16], Erdélyi [8], Chang [3], and the present author [9, Chap. 6]. Indeed, to prove the existence of a solution of the original problem (3.1), (3.2) satisfying estimate (i), we need only define, for t in [0, 1] and $0 < \epsilon \leqslant \epsilon_0$,

$$\alpha(t, \epsilon) = y_N(t, \epsilon) - \epsilon^N \Gamma \exp[-kt(2\epsilon)^{-1}] - \epsilon^N \gamma \ell^{-1} \exp[\lambda(t-1)],$$
$$\beta(t, \epsilon) = y_N(t, \epsilon) + \epsilon^N \Gamma \exp[-kt(2\epsilon)^{-1}] + \epsilon^N \gamma \ell^{-1} \exp[\lambda(t-1)].$$

As in the proof of Theorem 3.1, these functions are seen to satisfy the required inequalities for Γ, γ sufficiently large and $\lambda = -\ell k^{-1} + \mathcal{O}(\epsilon)$, the root of $\epsilon\lambda^2 + k\lambda + \ell = 0$. The details of the argument, together with a proof of estimate (ii), can be found in [9, Chap. 6].

The comments made after the proof of Theorem 3.1 apply equally to Theorem 3.2. We remark, however, that the hypotheses in these theorems are only sufficient conditions for the existence of a uniform approximate solution of an actual solution of (3.1), (3.2). Another approach is presented very briefly in Section 6 of the paper by Dorr, Parter, and Shampine [6]. These authors consider the problem

$$\epsilon y'' - b(t)\, y' - g(t, y) = 0, \qquad 0 < t < 1, \tag{3.3}$$
$$y(0, \epsilon) = A, \qquad y(1, \epsilon) = B, \qquad A \leqslant B, \tag{3.4}$$

where $b > 0$ and $g_y \geqslant 0$, under assumptions which guarantee that the solution $u_0 = u_0(t)$ of the reduced problem, $b(t)\, u_0' + g(t, u_0) = 0$, $0 < t < 1$, $u_0(0) = A$, satisfies $u_0''(t) \geqslant 0$, $0 \leqslant t \leqslant 1$. They then use the solution $w = w(t, \epsilon)$ of the approximate problem (cf. (E_1))

$$\epsilon w'' - b(t)\, w' - g(t, u_0(t)) = 0, \qquad 0 < t < 1,$$
$$w(0, \epsilon) = A, \qquad w(1, \epsilon) = B$$

to obtain the estimate $u_0(t) \leqslant y(t, \epsilon) \leqslant w(t, \epsilon)$ for the solution $y = y(t, \epsilon)$ of (3.3), (3.4). In addition, the limits $\lim_{\epsilon\to0} w(t, \epsilon) = \lim_{\epsilon\to0} y(t, \epsilon) = u_0(t)$, $0 \leqslant t < 1$, are shown to hold by means of maximum principle arguments. This is as far as the approximation procedure is carried.

4. Discussion

In Theorem 3.2 we proved that the solution $y_N(t, \epsilon)$ of (E_N) is a uniform approximation of order $\mathcal{O}(\epsilon^N)$ of an actual solution of the original problem (3.1), (3.2). However, it is clear from the estimates (*) in Theorem 3.1 that each of the solutions $y_1(t, \epsilon),..., y_{N-1}(t, \epsilon)$ of (E_1),..., (E_{N-1}), respectively, could be used to construct a solution of (3.1), (3.2). For example, the function $y_2 = y_2(t, \epsilon)$ is a solution of $\epsilon y_2'' = f(t, y_1(t, \epsilon), y_2', \epsilon)$, $0 < t < 1$, $y_2(0, \epsilon) = A$, $y_2(1, \epsilon) = B$, which satisfies the estimate $y_2 - y_1(t, \epsilon) = \mathcal{O}(\epsilon^N) + \mathcal{O}(\epsilon \exp[-kt(2\epsilon)^{-1}])$. Consequently, y_2 is actually a solution of $\epsilon y_2'' = f(t, y_2, y_2', \epsilon) + \mathcal{O}(\epsilon^N) + \mathcal{O}(\epsilon \exp[-kt(2\epsilon)^{-1}])$. We may now proceed as in the proof of Theorem 3.2 to construct a solution $\tilde{y} = \tilde{y}(t, \epsilon)$ of (3.1), (3.2) which satisfies $\tilde{y} - y_2 = \mathcal{O}(\epsilon^N) + \mathcal{O}(\epsilon^2 \exp[-kt(2\epsilon)^{-1}])$, $0 \leqslant t \leqslant 1$. There arises then the question of whether the solution y constructed from $y_N(t, \epsilon)$, $N \geqslant 3$, say, is equal to this solution $\tilde{y}$. However, this is easily seen to be the case in view of the estimates (*), provided ϵ is sufficiently small. For, more generally, if $z_j = z_j(t, \epsilon)$ are solutions of (3.1), (3.2) constructed from the solutions $y_j(t, \epsilon)$ of (E_j), then the estimates

$$z_j - y_j = \mathcal{O}(\epsilon^N) + \mathcal{O}(\epsilon^j \exp[-kt(2\epsilon)^{-1}]),$$

$$y_j - y_{j-1} = \mathcal{O}(\epsilon^N) + \mathcal{O}(\epsilon^{j-1} \exp[-kt(2\epsilon)^{-1}])$$

imply that $z_j - z_{j-1} = \mathcal{O}(\epsilon)$ (or better), $j = 2,..., N$.

We note, on the other hand, that if an Nth-order outer solution *distinct* from $u = u(t, \epsilon)$ is used to generate the analogous sequence $(\tilde{E}_1),..., (\tilde{E}_N)$, then it is conceivable that another solution of (3.1), (3.2) could be constructed as in Theorem 3.2. It is understood that, in this case, the function f in (3.1) is sufficiently nonlinear so that (3.1), (3.2) possesses multiple solutions.

5. A Quasilinear Problem

In proving Theorems 3.1 and 3.2 we made essential use of the assumption that f_y was of order $\mathcal{O}(1)$ in R. However, consider the following quasilinear problem (treated, e.g., in [2, 4, 13, 15])

$$\epsilon y'' = g(t, y, \epsilon) y' + h(t, y, \epsilon), \qquad 0 < t < 1, \tag{5.1}$$

$$y(0, \epsilon) = A, \qquad y(1, \epsilon) = B, \tag{5.2}$$

where $g > 0$ and g, h are of class $C^{(1)}$ in R. If $g_y \neq \mathcal{O}(\epsilon)$, then the function $f(t, y, y', \epsilon) = g(t, y, \epsilon) y' + h(t, y, \epsilon)$ does *not* satisfy $f_y = \mathcal{O}(1)$ in R, since $f_y = g_y y' + h_y = \mathcal{O}(\epsilon^{-1})$ near $t = 0$. Consequently, the approximation method described above does not apply to such problems. As a specific example, consider the problem

$$\epsilon y'' = -(1 + 3y^2) y', \qquad 0 < t < 1, \tag{5.3}$$

$$y(0, \epsilon) = 3^{-1/2}, \qquad y(1, \epsilon) = 0. \tag{5.4}$$

In this simple case, the outer solution to all orders of ϵ is $u(t, \epsilon) \equiv 0$. The first approximate problem (E_1) is then

$$\epsilon y_1'' = -(1 + 3u^2) y_1' = -y_1', \qquad 0 < t < 1,$$
$$y_1(0, \epsilon) = 3^{-1/2}, \qquad y_1(1, \epsilon) = 0,$$

whose solution, to asymptotically small terms, is $y_1(t, \epsilon) = 3^{-1/2} \exp[-t\epsilon^{-1}]$. However, the solution of (5.3), (5.4), again to asymptotically small terms, is $y(t, \epsilon) = \frac{1}{2}\exp[-t\epsilon^{-1}] (1 - \frac{1}{4}\exp[-2t\epsilon^{-1}])^{-1/2}$. Consequently, $y(t, \epsilon) - y_1(t, \epsilon) \neq \mathcal{O}(\epsilon)$, $0 \leq t \leq 1$; for instance, $y(\epsilon, \epsilon) - y_1(\epsilon, \epsilon) = \frac{1}{2}e^{-1}(1 - \frac{1}{4}e^{-1})^{-1/2} - 3^{-1/2}e^{-1}$ is not small with respect to $\epsilon \to 0^+$. This failure of the first approximate solution y_1 is propagated throughout the sequence (E_2),..., (E_N), and as a result, at the Nth step, $y - y_N \neq \mathcal{O}(\epsilon^N)$. It is interesting to note, however, that in general the breakdown of the approximation sequence occurs only for t in the range $0 \leq t \leq \epsilon\sigma$, σ a positive constant depending on the problem.

In the case of the general quasilinear problem (5.1), (5.2), if g_y is identically zero or if $g_y = \mathcal{O}(\epsilon)$ in R, then Theorems 3.1 and 3.2 apply without difficulty. If, on the contrary, g_y is not small near $t = 0$, it is still possible to construct an approximation sequence like (E_j). We simply define $(E_j)'$, for $1 \leq j \leq N$, inductively as the problems

$$\epsilon y_j'' = g(t, y_j, \epsilon) y_j' + h(t, y_{j-1}(t, \epsilon), \epsilon), \qquad 0 < t < 1,$$
$$y_j(0, \epsilon) = A, \qquad y_j(1, \epsilon) = B,$$

where, as usual, $y_0(t, \epsilon) \equiv u(t, \epsilon)$ is an Nth order outer solution of (5.1), (5.2). Results corresponding to Theorems 3.1 and 3.2 then follow by the above construction applied to $(E_j)'$. Of course, this sequence is of value only if h_y is not identically zero in R.

6. Concluding Remarks

It is possible to apply the approximation method developed above to problems with more general boundary conditions, i.e., problems of the form

$$\epsilon y'' = f(t, y, y', \epsilon), \qquad 0 < t < 1, \tag{6.1}$$

$$a_1 y(0, \epsilon) + a_2 y'(0, \epsilon) = A, \qquad b_1 y(1, \epsilon) + b_2 y'(1, \epsilon) = B, \tag{6.2}$$

where $|a_1| + |a_2| > 0$, $|b_1| + |b_2| > 0$. Indeed, if $a_2 \neq 0$ in (6.2) and $f_{y'} \leq -k$, the original approximation technique of Nagumo [13] for initial value problems can be applied in conjunction with a type of shooting method to prove results similar to Theorems 3.1 and 3.2. On the other hand, if $a_2 = 0$, these theorems apply almost directly to (6.1), (6.2) under certain additional assumptions. These statements are most easily verified by consulting the discussion of (6.1), (6.2) in [1; 9, Chaps. 4 and 7] and then applying the techniques in [13] or Section 3 where appropriate. The details are straightforward and are omitted. Finally, we note that a very special problem of this type was treated briefly in [6, Sect. 6].

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